Spring 2018 CIS 610

Advanced Geometric Methods in Computer Science
Jean Gallier

Homework 2

February 13; Due March 13, 2018

Problem B1 (60). (a) Consider the map, $f\colon \mathbf{GL}(n,\mathbb{R}) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of B, for any matrix B (here, I is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n,\mathbb{R})$.

(b) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n,\mathbb{R})$ is a manifold of dimension $n^2 - 1$.

(c) Let J be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n,1)$ the group of real $(n+1) \times (n+1)$ matrices

$$\mathbf{SO}(n,1) = \{A \in \mathbf{GL}(n+1,\mathbb{R}) \mid A^\top JA = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n,1)$ is indeed a group with the inverse of A given by $A^{-1} = JA^\top J$ (this is the *special Lorentz group*.) Consider the function $f\colon \mathbf{GL}^+(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^\top JA - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1) \times (n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top JH + H^\top JA$$

for any matrix, H. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n,1)$ and that $\mathbf{SO}(n,1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

Problem B2 (30). (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2 \cosh \omega$.

Prove that the exponential map, $\exp\colon \mathfrak{sl}(2, \mathbb{C}) \to \mathbf{SL}(2, \mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2, \mathbb{C})$.

Problem B3 (60 pts). Given a group G, recall that its *center* is the subset

$$Z(G) = \{a \in G, \ ag = ga \quad \text{for all } g \in G\}.$$

(1) Check that $Z(G)$ is a commutative normal subgroup of G.

(2) Prove that a matrix $A \in \mathrm{M}_n(\mathbb{R})$ commutes with all matrices $B \in \mathbf{GL}(n, \mathbb{R})$ iff $A = \lambda I$ for some $\lambda \in \mathbb{R}$.

Hint. Remember the elementary matrices.

Prove that

$$Z(\mathbf{GL}(n, \mathbb{R})) = \{\lambda I \mid \lambda \in \mathbb{R}, \lambda \neq 0\}.$$

(3) Prove that for any $m \geq 1$,

$$\begin{aligned} Z(\mathbf{SO}(2(m+1))) &= \{I, -I\} \\ Z(\mathbf{SO}(2m-1)) &= \{I\} \\ Z(\mathbf{SL}(m, \mathbb{R})) &= \{\lambda I \mid \lambda \in \mathbb{R}, \lambda^m = 1\}. \end{aligned}$$

(4) Prove that a matrix $A \in \mathrm{M}_n(\mathbb{C})$ commutes with all matrices $B \in \mathbf{GL}(n, \mathbb{C})$ iff $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

(5) Prove that for any $n \geq 1$,

$$\begin{aligned} Z(\mathbf{GL}(n, \mathbb{C})) &= \{\lambda I \mid \lambda \in \mathbb{C}, \lambda \neq 0\} \\ Z(\mathbf{SL}(n, \mathbb{C})) &= \{e^{\frac{k2\pi}{n} i} I \mid k = 0, 1, \ldots, n-1\} \\ Z(\mathbf{U}(n)) &= \{e^{i\theta} I \mid 0 \leq \theta < 2\pi\} \\ Z(\mathbf{SU}(n)) &= \{e^{\frac{k2\pi}{n} i} I \mid k = 0, 1, \ldots, n-1\}. \end{aligned}$$

(6) Prove that the groups $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ are not isomorphic (although their Lie algebras *are* isomorphic).

Problem B4 (120 pts). Recall from Homework 1, Problem B6, the Cayley parametrization of rotation matrices in $\mathbf{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where B is any $n \times n$ skew symmetric matrix.

(a) Now, consider $n = 3$, i.e., $\mathbf{SO}(3)$. Let E_1, E_2 and E_3 be the rotations about the x-axis, y-axis, and z-axis, respectively, by the angle π, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$\begin{aligned} B &\mapsto C(B) \\ B &\mapsto E_1C(B) \\ B &\mapsto E_2C(B) \\ B &\mapsto E_3C(B) \end{aligned}$$

where B is skew symmetric, are parametrizations of $\mathbf{SO}(3)$ and that the union of the images of C, E_1C, E_2C and E_3C covers $\mathbf{SO}(3)$, so that $\mathbf{SO}(3)$ is a manifold.

(b) Let A be *any* matrix (not necessarily invertible). Prove that there is some diagonal matrix, E, with entries $+1$ or -1, so that $EA + I$ is invertible.

(c) Prove that every rotation matrix, $A \in \mathbf{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, B, and some diagonal matrix, E, with entries $+1$ and -1, and where the number of -1 is even. Moreover, prove that every orthogonal matrix $A \in \mathbf{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, B, and some diagonal matrix, E, with entries $+1$ and -1. The above provide parametrizations for $\mathbf{SO}(n)$ (resp. $\mathbf{O}(n)$) that show that $\mathbf{SO}(n)$ and $\mathbf{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with n.

Problem B5 (30). Consider the parametric surface given by

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

The trace of this surface is called a *crosscap*. In order to plot this surface, make the change of variables

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned}$$

Prove that we obtain the parametric definition

$$\begin{aligned} x &= \frac{4\rho^2}{(\rho^2+1)^2}\sin 2\theta, \\ y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\sin\theta, \\ z &= \frac{4\rho^2}{(\rho^2+1)^2}\cos 2\theta. \end{aligned}$$

Show that the entire trace of the surface is obtained for $\rho \in [0,1]$ and $\theta \in [-\pi,\pi]$.

Hint. What happens if you change ρ to $1/\rho$?

Plot the trace of the surface using the above parametrization. Show that there is a line of self-intersection along the portion of the z-axis corresponding to $0 \leq z \leq 1$. What can you say about the point corresponding to $\rho = 1$ and $\theta = 0$?

Plot the portion of the surface for $\rho \in [0,1]$ and $\theta \in [0,\pi]$.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$\begin{aligned} x &= \frac{16uv^2(1-u^2)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

Problem B6 (30). Consider the parametric surface given by

$$\begin{aligned} x(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

The trace of this surface is called the *Steiner Roman surface*. In order to plot this surface, make the change of variables

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned}$$

Prove that we obtain the parametric definition

$$\begin{aligned} x &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\sin\theta, \\ y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\cos\theta, \\ z &= \frac{4\rho^2}{(\rho^2+1)^2}\,\cos 2\theta. \end{aligned}$$

Show that the entire trace of the surface is obtained for $\rho \in [0,1]$ and $\theta \in [-\pi,\pi]$. Plot the trace of the surface using the above parametrization.

Plot the portion of the surface for $\rho \in [0,1]$ and $\theta \in [0,\pi]$.

Prove that this surface has five singular points.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$\begin{aligned} x &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{4v(1-u^4)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

Problem B7 (160). Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x,y,z) \mapsto (xy, yz, xz, x^2-y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

(a) Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$
\begin{aligned}
\psi_1(u, v) &= \left(\frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u}{u^2 + v^2 + 1}, \frac{u^2 - v^2}{u^2 + v^2 + 1}\right), \\
\psi_2(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{u^2 - 1}{u^2 + v^2 + 1}\right), \\
\psi_3(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{1 - u^2}{u^2 + v^2 + 1}\right).
\end{aligned}
$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian has rank 2).

(c) Prove that if $\psi_1(u, v) = (x, y, z, t)$, then

$$y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that if $\psi_1(u, v) = (x, y, z, t)$, then u and v satisfy the equations

$$
\begin{aligned}
(y^2 + z^2)u^2 - zu + z^2 = 0 \\
(y^2 + z^2)v^2 - yv + y^2 = 0.
\end{aligned}
$$

Prove that if $y^2 + z^2 \neq 0$, then

$$u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$

else

$$u = \frac{z(1+\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} \quad \text{if} \quad u^2+v^2 \geq 1,$$

and there are similar formulae for v. Prove that the expression giving u in terms of y and z is continuous everywhere in $\{(y,z) \mid y^2+z^2 \leq \frac{1}{4}\}$ and similarly for the expression giving v in terms of y and z. Conclude that $\psi_1 \colon \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)$ is a homeomorphism onto its image. Therefore, $U_1 = \psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_2(u,v) = (x,y,z,t)$, then

$$x^2+y^2 \leq \frac{1}{4} \quad \text{and} \quad x^2+y^2 = \frac{1}{4} \quad \text{iff} \quad u^2+v^2 = 1.$$

Prove that if $\psi_2(u,v) = (x,y,z,t)$, then u and v satisfy the equations

$$\begin{aligned}(x^2+y^2)u^2 - xu + x^2 &= 0\\ (x^2+y^2)v^2 - yv + y^2 &= 0.\end{aligned}$$

Conclude that $\psi_2 \colon \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_2 = \psi_2(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_3(u,v) = (x,y,z,t)$, then

$$x^2+z^2 \leq \frac{1}{4} \quad \text{and} \quad x^2+z^2 = \frac{1}{4} \quad \text{iff} \quad u^2+v^2 = 1.$$

Prove that if $\psi_3(u,v) = (x,y,z,t)$, then u and v satisfy the equations

$$\begin{aligned}(x^2+z^2)u^2 - xu + x^2 &= 0\\ (x^2+z^2)v^2 - zv + z^2 &= 0.\end{aligned}$$

Conclude that $\psi_3 \colon \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_3 = \psi_3(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that the union of the U_i's covers $\mathcal{H}(S^2)$. Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a smooth manifold in $\mathbb{R}^4$.

(d) Plot the surfaces obtained by dropping the fourth coordinate and the third coordinates, respectively (with $u, v \in [-1,1]$).

(e) Prove that if $(x,y,z,t) \in \mathcal{H}(S^2)$, then

$$\begin{aligned}x^2y^2 + x^2z^2 + y^2z^2 &= xyz\\ x(z^2-y^2) &= yzt.\end{aligned}$$

Prove that the zero locus of these equations strictly contains $\mathcal{H}(S^2)$. This is a "famous mistake" of Hilbert and Cohn-Vossen in *Geometry and the Immagination*!

Finding a set of equations defining exactly $\mathcal{H}(S^2)$ appears to be an open problem.

Problem B8 (80). Recall that $\mathrm{ad}_A = L_A - R_A$, and that L_A and R_A commute. Prove that

$$d(\exp)_A = e^{L_A} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!}(L_A - R_A)^j.$$

Hint. Recall from Homework 1 Problem B3 that

$$d(\exp)_A = \sum_{h,k\geq 0} \frac{L_A^h R_A^k}{(h+k+1)!}.$$

To simplify notation, write a for L_A and b for L_B. Then, the problem is to prove that

$$e^a \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!}(a-b)^j = \sum_{h,k\geq 0} \frac{a^h b^k}{(h+k+1)!}, \tag{$*$}$$

assuming that $ab = ba$.

Expand the expression on the left and equate the coefficients of the monomial $a^h b^k$. To conclude, you will need to prove the following identity:

$$\sum_{i=0}^{h} (-1)^{h-i} \binom{h+k+1}{i} \binom{h+k-i}{k} = 1.$$

TOTAL: 570 **points.**