Summary 1, 2013 CIS 610

Advanced Geometric Methods in Computer Science
Jean Gallier & Dan Guralnik

Homework 2

June 4; Due June 14, 2013

Do Exercise 2.I.1, 2.II.1, 2.II.3, 2.II.4, 2.III.1, 2.III.2, 2.IV.3, from the handout on the web, and the two problems below.

Problem B1 (40 pts). Given a field K and any nonempty set I, let $K^{(I)}$ be the subset of the cartesian product K^I consisting of all functions $\lambda\colon I \to K$ with *finite support*, which means that $\lambda(i) = 0$ for all but finitely many $i \in I$. We usually denote the function defined by λ as $(\lambda_i)_{i\in I}$, and call is a *family indexed by* I. We define addition and multiplication by a scalar as follows:

$$(\lambda_i)_{i\in I} + (\mu_i)_{i\in I} = (\lambda_i + \mu_i)_{i\in I},$$

and

$$\alpha \cdot (\mu_i)_{i\in I} = (\alpha\mu_i)_{i\in I}.$$

(1) Check that $K^{(I)}$ is a vector space.

(2) If I is any nonempty subset, for any $i \in I$, we denote by e_i the family $(e_j)_{j\in I}$ defined so that

$$e_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

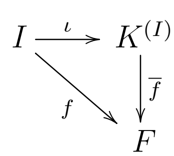
Prove that the family $(e_i)_{i\in I}$ is linearly independent and spans $K^{(I)}$, so that it is a basis of $K^{(I)}$ called the *canonical basis* of $K^{(I)}$. When I is finite, say of cardinality n, then prove that $K^{(I)}$ is isomorphic to K^n.

(3) The function $\iota\colon I \to K^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$, is clearly an injection.

For any other vector space F, for any function $f\colon I \to F$, prove that there is a *unique linear map* $\overline{f}\colon K^{(I)} \to F$, such that

$$f = \overline{f} \circ \iota,$$

as in the following commutative diagram:



We call the vector space $K^{(I)}$ the vector space *freely generated* by the set I.

Problem B2 (100 pts). (Some pitfalls of infinite dimension) Let E be the vector space freely generated by the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and let $(e_0, e_1, e_2, \ldots, e_n, \ldots)$ be its canonical basis. We define the function φ such that

$$\varphi(e_i, e_j) = \begin{cases} \delta_{ij} & \text{if } i, j \geq 1, \\ 1 & \text{if } i = j = 0, \\ 1/2^j & \text{if } i = 0, j \geq 1, \\ 1/2^i & \text{if } i \geq 1, j = 0, \end{cases}$$

and we extend φ by bilinearity to a function $\varphi\colon E \times E \to K$. This means that if $u = \sum_{i \in \mathbb{N}} \lambda_i e_i$ and $v = \sum_{j \in \mathbb{N}} \mu_j e_j$, then

$$\varphi\left(\sum_{i \in \mathbb{N}} \lambda_i e_i, \sum_{j \in \mathbb{N}} \mu_j e_j\right) = \sum_{i,j \in \mathbb{N}} \lambda_i \mu_j \varphi(e_i, e_j),$$

but remember that $\lambda_i \neq 0$ and $\mu_j \neq 0$ *only for finitely many indices* i, j.

(1) Prove that φ is positive definite, so that it is an inner product on E.

What would happen if we changed $1/2^j$ to 1 (or any constant)?

(2) Let H be the subspace of E spanned by the family $(e_i)_{i \geq 1}$, a hyperplane in E. Find $H^\perp$ and $H^{\perp\perp}$, and prove that

$$H \neq H^{\perp\perp}.$$

(3) Let U be the subspace of E spanned by the family $(e_{2i})_{i \geq 1}$, and let V be the subspace of E spanned by the family $(e_{2i-1})_{i \geq 1}$. Prove that

$$\begin{aligned} U^\perp &= V \\ V^\perp &= U \\ U^{\perp\perp} &= U \\ V^{\perp\perp} &= V, \end{aligned}$$

yet

$$(U \cap V)^\perp \neq U^\perp + V^\perp$$

and

$$(U + V)^{\perp\perp} \neq U + V.$$

If W is the subspace spanned by e_0 and e_1, prove that

$$(W \cap H)^\perp \neq W^\perp + H^\perp.$$

(4) Consider the dual space E^* of E, and let $(e_i^*)_{i\in\mathbb{N}}$ be the family of dual forms of the basis $(e_i)_{i\in N}$. Check that the family $(e_i^*)_{i\in\mathbb{N}}$ is linearly independent.

(5) Let $f \in E^*$ be the linear form defined by

$$f(e_i) = 1 \quad \text{for all } i \in \mathbb{N}.$$

Prove that f is not in the subspace spanned by the e_i^*. If F is the subspace of E^* spanned by the e_i^* and f, find F^0 and F^{00}, and prove that

$$F \neq F^{00}.$$

TOTAL: 120 + 40 points.