Summer 1, 2009 CIS 610

Advanced Geometric Methods in Computer Science

Jean Gallier

Homework 2

June 9, 2009; Due June 18, 2009

"B problems" must be turned in.

Problem B1 (30 pts). The purpose of this problem is to study certain affine maps of $\mathbb{A}^2$.

(1) Consider affine maps of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Prove that such maps have a unique fixed point c if $\theta \neq 2k\pi$, for all integers k. Show that these are rotations of center c, which means that with respect to a frame with origin c (the unique fixed point), these affine maps are represented by rotation matrices.

(2) Consider affine maps of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda\cos\theta & -\lambda\sin\theta \\ \mu\sin\theta & \mu\cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Prove that such maps have a unique fixed point iff $(\lambda + \mu)\cos\theta \neq 1 + \lambda\mu$. Prove that if $\lambda\mu = 1$ and $\lambda > 0$, there is some angle θ for which either there is no fixed point, or there are infinitely many fixed points.

(3) Prove that the affine map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 8/5 & -6/5 \\ 3/10 & 2/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has no fixed point.

(4) Prove that an arbitrary affine map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has a unique fixed point iff the matrix

$$\begin{pmatrix} a_1 - 1 & a_2 \\ a_3 & a_4 - 1 \end{pmatrix}$$

is invertible.

Problem B2 (30 pts). Prove Proposition 2.3 of the notes on *Convex Sets, Polytopes, Polyhedra, ...*, that is:

If K is any compact subset of $\mathbb{A}^m$, then the convex hull, $\mathrm{conv}(K)$, of K is also compact.

Problem B3 (30 pts). Prove the version of Carathéodory's theorem for cones (Theorem 2.4 of notes on *Convex Sets, Polytopes, Polyhedra, ...*), that is:

Given any vector space, E, of dimension m, for any (nonvoid) family $S = (v_i)_{i \in L}$ of vectors in E, the cone, $\mathrm{cone}(S)$, spanned by S is equal to the set of positive combinations of families of m vectors in S.

Problem B4 (30 pts). (i) Show that if E is an affine space of dimension m and S is a finite subset of E with n elements, if either $n \geq m + 3$ or $n = m + 2$ and some family of $m + 1$ points of S is affinely dependent, then S has at least two Radon partitions.

(ii) Prove the version of Radon's theorem for cones (Theorem 2.11 of *Convex Sets, Polytopes, Polyhedra, ...*), namely:

Given any vector space E of dimension m, for every subset X of E, if $\mathrm{cone}(X)$ is a pointed cone such that X has at least $m+1$ nonzero vectors, then there is a partition of X into two nonempty disjoint subsets, X_1 and X_2, such that the cones, $\mathrm{cone}(X_1)$ and $\mathrm{cone}(X_2)$, have a nonempty intersection not reduced to $\{0\}$.

(iii) (**Extra Credit (30 pts)**) Does the converse of (i) hold?

Problem B5 (30 pts). Let S be any nonempty subset of an affine space E. Given some point $a \in S$, we say that S is *star-shaped with respect to* a iff the line segment $[a, x]$ is contained in S for every $x \in S$, i.e. $(1 - \lambda)a + \lambda x \in S$ for all λ such that $0 \leq \lambda \leq 1$. We say that S is *star-shaped* iff it is star-shaped w.r.t. to some point $a \in S$.

(1) Prove that every nonempty convex set is star-shaped.

(2) Show that there are star-shaped subsets that are not convex. Show that there are nonempty subsets that are not star-shaped (give an example in $\mathbb{A}^n$, $n = 1, 2, 3$).

(3) Given a star-shaped subset S of E, let $N(S)$ be the set of all points $a \in S$ such that S is star-shaped with respect to a. Prove that $N(S)$ is convex.

Problem B6 (50 pts). (a) Let E be a vector space, and let U and V be two subspaces of E so that they form a direct sum $E = U \oplus V$. Recall that this means that every vector $x \in E$ can be written as $x = u + v$, for some unique $u \in U$ and some unique $v \in V$. Define the function $p_U \colon E \to U$ (resp. $p_V \colon E \to V$) so that $p_U(x) = u$ (resp. $p_V(x) = v$), where $x = u + v$, as explained above. Check that that p_U and p_V are linear.

(b) Now assume that E is an affine space (nontrivial), and let U and V be affine subspaces such that $\overrightarrow{E} = \overrightarrow{U} \oplus \overrightarrow{V}$. Pick any $\Omega \in V$, and define $q_U \colon E \to \overrightarrow{U}$ (resp. $q_V \colon E \to \overrightarrow{V}$, with $\Omega \in U$) so that

$$q_U(a) = p_{\overrightarrow{U}}(\mathbf{\Omega a}) \quad \text{(resp.} \quad q_V(a) = p_{\overrightarrow{V}}(\mathbf{\Omega a})), \quad \text{for every } a \in E.$$

Prove that q_U does not depend on the choice of $\Omega \in V$ (resp. q_V does not depend on the choice of $\Omega \in U$). Define the map $p_U \colon E \to U$ (resp. $p_V \colon E \to V$) so that

$$p_U(a) = a - q_V(a) \quad \text{(resp.} \quad p_V(a) = a - q_U(a)), \quad \text{for every } a \in E.$$

Prove that p_U (resp. p_V) is affine.

The map p_U (resp. p_V) is called the *projection onto U parallel to V (resp. projection onto V parallel to U)*.

(c) Let $(a_0, \ldots, a_n)$ be $n+1$ affinely independent points in $\mathbb{A}^n$, and let $\Delta(a_0, \ldots, a_n)$ denote the convex hull of $(a_0, \ldots, a_n)$ (an n-simplex). Prove that if $f \colon \mathbb{A}^n \to \mathbb{A}^n$ is an affine map sending $\Delta(a_0, \ldots, a_n)$ inside itself, i.e.,

$$f(\Delta(a_0, \ldots, a_n)) \subseteq \Delta(a_0, \ldots, a_n),$$

then, f has some fixed point $b \in \Delta(a_0, \ldots, a_n)$, i.e.,

$$f(b) = b.$$

Hint: Proceed by induction on n. First, treat the case $n = 1$. The affine map is determined by $f(a_0)$ and $f(a_1)$, which are affine combinations of a_0 and a_1. There is an explicit formula for some fixed point of f. For the induction step, compose f with some suitable projections.

TOTAL: 200 (+ 30) points.