Homework V (due March 31), Math 603, Spring 2003. (GJZ)

B I(a). First, assume that A is a Dedekind domain. We proved in class that every ideal, $\mathfrak{A}$, can be expressed uniquely (up to the order of the factors) as a product of powers of prime ideals:

$$\mathfrak{A} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}.$$

In fact, since A is a Dedekind domain, every nonzero prime is maximal, so, we may assume that the $\mathfrak{p}_i$ are maximal. We will prove that $\mathfrak{A}$ is generated by at most two elements.

The first step is to prove that for every nonzero ideal, $\mathfrak{A}$, the ring $A/\mathfrak{A}$ is a P.I.D. Since,

$$\mathfrak{A} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t},$$

where the $\mathfrak{p}_i$ are maximal, we have $\mathfrak{p}_i + \mathfrak{p}_j = A$, whenever $i \neq j$. This implies that $\mathfrak{p}_i^k + \mathfrak{p}_j^l = A$, for all $k, l \geq 1$. This is because if $\mathfrak{A}$ and $\mathfrak{B}$ are any two ideals, then it is easy to prove that $\mathfrak{A} + \mathfrak{B} = A$ iff $\sqrt{\mathfrak{A}} + \sqrt{\mathfrak{B}} = A$. Since $\sqrt{\mathfrak{p}_i^k} = \mathfrak{p}_i$ and $\sqrt{\mathfrak{p}_j^l} = \mathfrak{p}_j$, we have $\mathfrak{p}_i^k + \mathfrak{p}_j^l = A$, for all $k, l \geq 1$, as claimed. Thus, the Chinese remainder's theorem applies, and we deduce that

$$A/\mathfrak{A} \cong \prod_{i=1}^{t} A/\mathfrak{p}_i^{e_i}.$$

If every $A/\mathfrak{p}_i^{e_i}$ is a P.I.D., then, A itself is a P.I.D., since every ideal of A is a product $\prod_{i=1}^t \mathfrak{A}_i$, where $\mathfrak{A}_i$ is an ideal of $A/\mathfrak{p}_i^{e_i}$. Therefore, we are reduced to the case where $\mathfrak{A} = \mathfrak{m}^d$, for some maximal ideal, $\mathfrak{m}$. Since A is a Dedekind domain, it is noetherian; by the Krull intersection theorem, since $\mathfrak{m} \neq A$, we have $\bigcap_{j \geq 0} \mathfrak{m}^j = (0)$. Thus, we can find some $t \in \mathfrak{m} - \mathfrak{m}^2$.

The second step is to prove that for any $n \geq 1$,

$$\mathfrak{m}^s = At^s + \mathfrak{m}^n, \quad \text{for every } s = 1, \ldots, n.$$

First, we prove that $\mathfrak{m} = At + \mathfrak{m}^n$, for every $n \geq 1$. Of course, something needs to be proved only if $n > 1$. The ideal At has a unique decomposition

$$At = \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_r^{e_r},$$

and since $t \in \mathfrak{m}$, one of the $\mathfrak{m}_i$ must be equal to $\mathfrak{m}$. Say $\mathfrak{m}_1 = \mathfrak{m}$. Now, $t \in \mathfrak{m}_1^{e_1}$ and $t \notin \mathfrak{m}_1^2$ implies $e_1 = 1$. If follows that

$$At + \mathfrak{m}^n = \mathfrak{m}(\mathfrak{m}_2^{e_2} \cdots \mathfrak{m}_r^{e_r} + \mathfrak{m}^{n-1}).$$

On the other hand, we have $\mathfrak{m}_i + \mathfrak{m}_j = A$, whenever $i \neq j$, and we know that this implies that $\mathfrak{m}_i^k + \mathfrak{m}_j^l = A$, for all $k, l \geq 1$. Consequently, we have

$$A = \mathfrak{m}_i^{e_i} + \mathfrak{m}^{n-1}, \quad \text{for } i = 2, \ldots, r,$$

so there exist some $x_i \in \mathfrak{m}_i$ and $y_i \in \mathfrak{m}^{n-1}$ so that $x_i + y_i = 1$. Thus,

$$1 = \prod_{i=2}^{r} (x_i + y_i) = x_2 \cdots x_r + z,$$

where $z \in \mathfrak{m}^{n-1}$, and so,

$$A = \prod_{i=2}^{r} \mathfrak{m}_i^{e_i} + \mathfrak{m}^{n-1},$$

and as a consequence,

$$At + \mathfrak{m}^n = \mathfrak{m}.$$

We now proceed by induction on s. We just proved the base case, $s = 1$. For the induction step, as $s \leq n$, observe that

$$At^s + \mathfrak{m}^n \subseteq \mathfrak{m}^s = \mathfrak{m}^{s-1}\mathfrak{m} = (At^{s-1} + \mathfrak{m}^n)(At + \mathfrak{m}^n) \subseteq At^s + \mathfrak{m}^n,$$

which shows that $\mathfrak{m}^s = At^s + \mathfrak{m}^n$, as desired.

Now, any ideal, $\overline{\mathfrak{A}}$, of $A/\mathfrak{m}^d$ is of the form $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_t^{e_t}/\mathfrak{m}^d$, where $\mathfrak{m}^d \subseteq \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_t^{e_t}$. As $\mathfrak{m}$ and the $\mathfrak{m}_i$ are maximal, we must have $t = 1$, $\mathfrak{m} = \mathfrak{m}_1$ and $e_1 \leq d$. Since $\mathfrak{m}^{e_1} = At^{e_1} + \mathfrak{m}^d$, we deduce that $\overline{\mathfrak{A}} = (\overline{t}^{e_1})$ is a principal ideal. Therefore, we proved that $A/\mathfrak{A}$ is a P.I.D. for every nonzero ideal, $\mathfrak{A}$, of A.

Finally, let $\mathfrak{A}$ be a nonprincipal ideal of a Dedekind domain. If we pick any $a \neq 0$ in $\mathfrak{A}$, we know that the ideal, $\mathfrak{A}/(a)$, of $A/(a)$ is a P.I.D. Say $\mathfrak{A}/(a)$ is generated by $\overline{b} \in A/(a)$, where $b \in \mathfrak{A}$. Then, for every $x \in \mathfrak{A}$, we have $\overline{x} = \overline{\alpha}\overline{b}$, with $\overline{\alpha} \in A/(a)$; thus, $x - \alpha b \in (a)$, which shows that $\mathfrak{A} = (a, b)$.

For the converse, assume that A is an integral domain so that every ideal, $\mathfrak{A}$, of A, for every $a \in \mathfrak{A}$ ($a \neq 0$), there is some $b \in \mathfrak{A}$ so that $\mathfrak{A} = (a, b)$. Then, A is noetherian. Now, it can be shown that a noetherian domain is a Dedekind domain iff for every maximal ideal, $\mathfrak{m}$, the local ring $A_{\mathfrak{m}}$ is a P.I.D. and in turn, a local noetherian domain is a P.I.D. iff its maximal ideal is principal. Thus, it would be enough to prove that for every maximal ideal, $\mathfrak{m}$, the ideal $\mathfrak{m}^e$ is principal in $A_{\mathfrak{m}}$. Now, as A is noetherian, $A_{\mathfrak{m}}$ is also noetherian. Moreover, as ideals in $A_{\mathfrak{m}}$ are in one-to-one correspondence with ideals of A contained in $\mathfrak{m}$, the ideals of $A_{\mathfrak{m}}$ also have the property assumed for ideals of A. Thus, we are reduced to the case of a noetherian local ring, B, with maximal ideal, $\mathfrak{m}$. Since $\mathfrak{m} \neq B$, we have $\mathfrak{m}^2 < \mathfrak{m}$. If we pick any $a \in \mathfrak{m}^2$, we can find some $t \in \mathfrak{m}$ so that $\mathfrak{m} = Bt + Ba = Bt + \mathfrak{m}^2$. As $\mathfrak{m}^2 < \mathfrak{m}$, we must have $t \in \mathfrak{m} - \mathfrak{m}^2$. Then, observe that the vector space $\mathfrak{m}/\mathfrak{m}^2$ over the field $B/\mathfrak{m}$ has dimension 1, since $\mathfrak{m}/\mathfrak{m}^2 \cong \overline{t}(B/\mathfrak{m})$. Now, by a corollary to Nakayama's lemma applied to the module $M = \mathfrak{m}$ and to the Jacobson ideal $\mathcal{J} = \mathfrak{m}$ (since we are in a local ring), as $M/\mathcal{J}M = \mathfrak{m}/\mathfrak{m}^2$ is generated by a single element, we deduce that $M = \mathfrak{m}$ is generated by a single element. Thus, $\mathfrak{m}$ is indeed principal in B, as desired. Therefore, $\mathfrak{m}^e$ is indeed a principal ideal in $A_{\mathfrak{m}}$, which concludes our proof.

B IV. a) First, note that a homomorphism, $f\colon A \to B$, is continuous (where the local rings A and B are given their $\mathfrak{m}$-adic topology, which makes them metric spaces). Since $\widehat{A}$ and $\widehat{B}$ are the usual completion of a metric space and since $B \subseteq \widehat{B}$, the continuous map, $f\colon A \to \widehat{B}$, extends uniquely to a map $\widehat{f}\colon \widehat{A} \to \widehat{B}$. Namely, since every $a \in \widehat{A}$ is the limit of some Cauchy sequence, $\{a_n\}$, we must have

$$\widehat{f}(a) = \widehat{f}(\lim_{n\mapsto\infty} a_n) = \lim_{n\mapsto\infty} f(a_n).$$

By continuity, it is easy to show that $\widehat{f}$ is a homomorphism and that $\mathfrak{m}_{\widehat{A}}$ goes to $\mathfrak{m}_{\widehat{B}}$.

B IV. b) We have a homomorphism $f\colon A \to B$ of local rings such that $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ and we assume that

(i) B is flat over A.

(ii) $f(\mathfrak{m}_A)B = \mathfrak{m}_B$.

(iii) $A/\mathfrak{m}_A \longrightarrow B/\mathfrak{m}_B$ is an isomorphism.

We will prove that $\widehat{f}\colon \widehat{A} \to \widehat{B}$ is an isomorphism.

First, we prove that if A and B are local ring and $f\colon A \to B$ is a flat homomorphism satisfying (ii), then B is faithfully flat over A. For this, we prove

Proposition 1.1 *A flat A-module, M, is faithfully flat iff $\mathfrak{m}M \neq M$ for every maximal ideal, $\mathfrak{m}$, of A.*

Proof. If M is faithfully flat, as

$$M/\mathfrak{m}M \cong (A/\mathfrak{m}) \otimes_A M,$$

the fact that $A/\mathfrak{m} \neq (0)$ implies that $\mathfrak{m}M \neq M$.

Conversely, let $N \neq (0)$ be an A-module. For any $x \in N$ so that $x \neq 0$, we have $Ax \cong A/\mathrm{Ann}(x)$. Now, there is some maximal ideal, $\mathfrak{m}$, with $\mathrm{Ann}(x) \subseteq \mathfrak{m}$, and since $\mathfrak{m}M \neq M$, we get $M \neq \mathrm{Ann}(x)M$. Consequently,

$$Ax \otimes_A M = (A/\mathrm{Ann}(x)) \otimes_A M \cong M/(\mathrm{Ann}(x)M) \neq (0).$$

As M is flat and $Ax \hookrightarrow N$ is an injection, the map $Ax \otimes_A M \hookrightarrow N \otimes_A M$ is also an injection and since $Ax \otimes_A M \neq (0)$, we have $N \otimes_A M \neq (0)$, which shows that M is faithfully flat. □

Let us apply the above to the flat morphism of local rings $f\colon A \to B$. The ring B is an A-module *via* f, means which that the action of A on B is given by $a \cdot b = f(a)b$, for all $a \in A$ and $b \in B$. Since $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, we see that $\mathfrak{m}_A \cdot B = f(\mathfrak{m}_A)B = \mathfrak{m}_B$, and as B is a local ring, by Nakayama, $B \neq \mathfrak{m}_B B$. By Proposition 1.1, the module B is faithfully flat over A.

We also need the fact that if $f\colon A \to B$ is a faithfully flat ring homomorphism, then it is injective. This is because for every A-module, M, the map $M \longrightarrow M \otimes_A B$ defined by $m \mapsto m \otimes 1$ is injective. Setting $M = A$, as $A \otimes_A B = B$, we get the desired result.

It remains to prove that for every A-module, M, the map $M \longrightarrow M \otimes_A B$ defined by $m \mapsto m \otimes 1$ is injective.

Pick any $m \neq 0$ in M. Then, the module $(Am) \otimes_A B$ is a B-submodule of $M \otimes_A B$ and it is clear that it is isomorphic to $(m \otimes 1)B$. Since B is faithfully flat over A, we have $(Am) \otimes_A B \neq (0)$, which proves that $m \otimes 1 \neq 0$, and the map $m \mapsto m \otimes 1$ is injective.

The key point is that the functor $A \rightsquigarrow \widehat{A}$ is exact. This can be proved using the characterization of $\widehat{A}$ as a left limit and using the Artin/Rees lemma. Sorry, we ran out of time to give a proof. The proof also shows that $\widehat{A}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}}_A$ and that

$$\widehat{A}/\widehat{\mathfrak{m}}_A^k \cong A/\mathfrak{m}_A^k \quad \text{for all} \quad k \geq 1.$$

In particular, for $k = 1$, we get $\widehat{A}/\widehat{\mathfrak{m}}_A \cong A/\mathfrak{m}_A$. We have the exact sequences

$$0 \longrightarrow \mathfrak{m}_A \longrightarrow A \longrightarrow A/\mathfrak{m}_A \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{m}_B \longrightarrow B \longrightarrow B/\mathfrak{m}_B \longrightarrow 0$$

and by hypothesis, $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B$ and $f(\mathfrak{m}_A)B = \mathfrak{m}_B$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}_A & \longrightarrow & A & \longrightarrow & A/\mathfrak{m}_A & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & \mathfrak{m}_B & \longrightarrow & B & \longrightarrow & B/\mathfrak{m}_B & \longrightarrow & 0.
\end{array}$$

Applying completion to the above diagram, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathfrak{m}}_A & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{A}/\widehat{\mathfrak{m}}_A & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \widehat{f}} & & \| & & \\
0 & \longrightarrow & \widehat{\mathfrak{m}}_B & \longrightarrow & \widehat{B} & \longrightarrow & \widehat{B}/\widehat{\mathfrak{m}}_B & \longrightarrow & 0.
\end{array}$$

Since f is injective and since $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, we have $\widehat{\mathfrak{m}}_A \cong \widehat{\mathfrak{m}}_B$. We deduce from the commutative diagram and the 5-lemma that $\widehat{f}$ is an isomorphism.