Spring, 2011 CIS 610

Advanced Geometric Methods in Computer Science
Jean Gallier

Homework 1

February 22, 2011; Due March 17, 2011

"A problems" are for practice only, and should not be turned in.

Problem A1. Given a finite dimensional Euclidean space, E, if U and V are two orthogonal subspaces that span E, i.e., $E = U \oplus V$, we have the linear projections $p_U : E \to U$ and $p_V : E \to V$. Recall: since every $w \in E$ can be written uniquely as $w = u + v$, with $u \in U$ and $v \in V$, we have $p_U(w) = u$, $p_V(w) = v$ and $p_U(w) + p_V(w) = w$, for all $w \in E$. We define the *orthogonal reflection with respect to U and parallel to V* as the linear map, s, given by

$$s(w) = 2p_U(w) - w = w - 2p_V(w),$$

for all $w \in E$. Observe that $s \circ s = \mathrm{id}$, that s is the identity on U and $s = -\mathrm{id}$ on V. When $U = H$ is a hyperplane, s is called a *hyperplane reflection* (about H).

(a) If w is any nonzero vector orthogonal to the hyperplane H, prove that s is given by

$$s(x) = x - 2\frac{\langle x, w\rangle}{\|w\|^2} w,$$

for all $x \in E$. (Here, $\|w\|^2 = \langle w, w\rangle$.)

(b) In matrix form, if the vector w is represented by the column vector W, show that the matrix of the hyperplane reflection about the hyperplane $K = \{w\}^\perp$ is

$$I - 2\frac{WW^\top}{W^\top W}.$$

Such matrices are called *Householder matrices*.

Problem A2. Given an $m \times n$ matrix, A, prove that its Frobenius norm,

$$\|A\|_F = \sqrt{\sum_{ij} |a_{ij}|^2}$$

satisfies

$$\|A\|_F = \sqrt{\mathrm{tr}(A^* A)} = \sqrt{\mathrm{tr}(AA^*)}$$

where $\mathrm{tr}(B)$ is the trace of the square matrix B (the sum of its diagonal elements).

Problem A3. (a) Find two symmetric matrices, A and B, such that AB is not symmetric.

(b) Find two matrices, A and B, such that

$$e^A e^B \neq e^{A+B}.$$

Try

$$A = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

"B problems" must be turned in.

Problem B1 (40 pts). (a) Given a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$, prove that there is a skew symmetric matrix B such that

$$R = (I - B)(I + B)^{-1}.$$

(b) If B is a skew symmetric $n \times n$ matrix, prove that $\lambda I_n - B$ and $\lambda I_n + B$ are invertible for all $\lambda \neq 0$, and that they commute.

(c) Prove that

$$R = (\lambda I_n - B)(\lambda I_n + B)^{-1}$$

is a rotation matrix that does not admit -1 as an eigenvalue. (Recall, a rotation is an orthogonal matrix R with positive determinant, i.e., $\det(R) = 1$.)

(d) Given any rotation matrix R that does not admit -1 as an eigenvalue, prove that there is a skew symmetric matrix B such that

$$R = (I_n - B)(I_n + B)^{-1} = (I_n + B)^{-1}(I_n - B).$$

This is known as the *Cayley representation* of rotations (Cayley, 1846).

(e) Given any rotation matrix R, prove that there is a skew symmetric matrix B such that

$$R = \left((I_n - B)(I_n + B)^{-1}\right)^2.$$

Problem B2 (40). (a) Consider the map, $f\colon \mathbf{GL}^+(n) \to \mathbf{S}(n)$, given by

$$f(A) = A^\top A - I.$$

Check that

$$df(A)(H) = A^\top H + H^\top A,$$

for any matrix, H.

(b) Consider the map, $f\colon \mathbf{GL}(n) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of B, for any matrix B (here, I is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n)$.

(c) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n)$ is a manifold of dimension $n^2 - 1$.

(d) Let J be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n, 1)$ the group of real $(n+1) \times (n+1)$ matrices

$$\mathbf{SO}(n, 1) = \{A \in \mathbf{GL}(n+1) \mid A^\top J A = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n, 1)$ is indeed a group with the inverse of A given by $A^{-1} = JA^\top J$ (this is the *special Lorentz group*.) Consider the function $f\colon \mathbf{GL}^+(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^\top J A - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1) \times (n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top J H + H^\top J A$$

for any matrix, H. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n, 1)$ and that $\mathbf{SO}(n, 1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

Problem B3 (40 pts). (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2\cosh \omega$.

Prove that the exponential map, $\exp\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2,\mathbb{C})$.

(b) Recall that a matrix, N, is *nilpotent* iff there is some $m \geq 0$ so that $N^m = 0$. Let A be any $n \times n$ matrix of the form $A = I - N$, where N is nilpotent. Why is A invertible? prove that there is some B so that $e^B = I - N$ as follows: Recall that for any $y \in \mathbb{R}$ so that $|y - 1|$ is small enough, we have

$$\log(y) = -(1-y) - \frac{(1-y)^2}{2} - \cdots - \frac{(1-y)^k}{k} - \cdots.$$

As N is nilpotent, we have $N^m = 0$, where m is the smallest integer with this propery. Then, the expression

$$B = \log(I - N) = -N - \frac{N^2}{2} - \cdots - \frac{N^{m-1}}{m-1}$$

is well defined. Use a formal power series argument to show that

$$e^B = A.$$

We denote B by $\log(A)$.

(c) Let $A \in \mathbf{GL}(n,\mathbb{C})$. Prove that there is some matrix, B, so that $e^B = A$. Thus, the exponential map, $\exp\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathbf{GL}(n,\mathbb{C})$, is surjective.

First, use the fact that A has a Jordan form, PJP^{-1}. Then, show that finding a log of A reduces to finding a log of every Jordan block of J. As every Jordan block, J, has a fixed nonzero constant, λ, on the diagonal, with 1's immediately above each diagonal entry and zero's everywhere else, we can write J as $(\lambda I)(I - N)$, where N is nilpotent. Find B_1 and B_2 so that $\lambda I = e^{B_1}$, $I - N = e^{B_2}$, and $B_1 B_2 = B_2 B_1$. Conclude that $J = e^{B_1 + B_2}$.

Problem B4 (60 pts). Recall from Homework 1, Problem B1, the Cayley parametrization of rotation matrices in $\mathbf{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where B is any $n \times n$ skew symmetric matrix. In that problem, it was shown that $C(B)$ is a rotation matrix that does not admit -1 as an eigenvalue and that every such rotation matrix is of the form $C(B)$.

(a) If you have not already done so, prove that the map $B \mapsto C(B)$ is injective.

(b) Prove that

$$dC(B)(A) = D_A((I - B)(I + B)^{-1}) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1}.$$

Hint. First, show that $D_A(B^{-1}) = -B^{-1}AB^{-1}$ (where B is invertible) and that $D_A(f(B)g(B)) = (D_A f(B))g(B) + f(B)(D_A g(B))$, where f and g are differentiable matrix functions.

Deduce that $dC(B)$ is injective, for every skew-symmetric matrix, B. If we identify the space of $n \times n$ skew symmetric matrices with $\mathbb{R}^{n(n-1)/2}$, show that the Cayley map, $C\colon \mathbb{R}^{n(n-1)/2} \to \mathbf{SO}(n)$, is a parametrization of $\mathbf{SO}(n)$.

(c) Now, consider $n = 3$, i.e., $\mathbf{SO}(3)$. Let E_1, E_2 and E_3 be the rotations about the x-axis, y-axis, and z-axis, respectively, by the angle π, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$\begin{aligned} B &\mapsto C(B) \\ B &\mapsto E_1 C(B) \\ B &\mapsto E_2 C(B) \\ B &\mapsto E_3 C(B) \end{aligned}$$

where B is skew symmetric, are parametrizations of $\mathbf{SO}(3)$ and that the union of the images of C, E_1C, E_2C and E_3C covers $\mathbf{SO}(3)$, so that $\mathbf{SO}(3)$ is a manifold.

(d) Let A be *any* matrix (not necessarily invertible). Prove that there is some diagonal matrix, E, with entries $+1$ or -1, so that $EA + I$ is invertible.

(e) Prove that every rotation matrix, $A \in \mathbf{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, B, and some diagonal matrix, E, with entries $+1$ and -1, and where the number of -1 is even. Moreover, prove that every orthogonal matrix $A \in \mathbf{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, B, and some diagonal matrix, E, with entries $+1$ and -1. The above provide parametrizations for $\mathbf{SO}(n)$ (resp. $\mathbf{O}(n)$) that show that $\mathbf{SO}(n)$ and $\mathbf{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with n.

TOTAL: 180 points.