Chapter 19
Basics of the Differential Geometry of Curves

19.1 Introduction: Parametrized Curves

In this chapter we consider parametric curves, and we introduce two important invariants, curvature and torsion (in the case of a 3D curve).

Properties of curves can be classified into *local properties* and *global properties*. Local properties are the properties that hold in a small neighborhood of a point on a curve. Curvature is a local property. Local properties can be studied more conveniently by assuming that the curve is parametrized locally. Thus, it is important and useful to study parametrized curves. In order to study the global properties of a curve, such as the number of points where the curvature is extremal, the number of times that a curve wraps around a point, or convexity properties, topological tools are needed. A proper study of global properties of curves really requires the introduction of the notion of a manifold, a concept beyond the scope of this book. In this chapter we study only local properties of parametrized curves. Readers interested in learning about curves as manifolds and about global properties of curves are referred to do Carmo [7] and Berger and Gostiaux [2]. Kreyszig [15] is also an excellent source, which does a great job at tracing the origin of concepts. It turns out that it is easier to study the notions of curvature and torsion if a curve is parametrized by arc length, and thus we will discuss briefly the notion of arc length.

Let $\mathcal{E}$ be some normed affine space of finite dimension, for the sake of simplicity the Euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$. Recall that the Euclidean space $\mathbb{E}^m$ is obtained from the affine space $\mathbb{A}^m$ by defining on the vector space $\mathbb{R}^m$ the standard inner product

$$(x_1, \ldots, x_m) \cdot (y_1, \ldots, y_m) = x_1 y_1 + \cdots + x_m y_m.$$

The corresponding Euclidean norm is

$$\|(x_1, \ldots, x_m)\| = \sqrt{x_1^2 + \cdots + x_m^2}.$$

Inspired by a kinematic view, we can define a curve as a continuous map $f\colon\ ]a,b[\to \mathcal{E}$ from an open interval $I =]a,b[$ of $\mathbb{R}$ to the affine space $\mathcal{E}$. From this point of view we can think of the parameter $t \in]a,b[$ as time, and the function f gives the position $f(t)$ at time t of a moving particle. The image $f(I) \subseteq \mathcal{E}$ of the interval I is the trajectory of the particle. In fact, asking only that f be continuous turns out to be too liberal, as rather strange curves turn out to be definable, such as "square-filling curves," due to Peano, Hilbert, Sierpiński, and others (see the problems).

Example 19.1. A very pretty square-filling curve due to Hilbert is defined by a sequence (h_n) of polygonal lines $h_n\colon [0,1] \to [0,1] \times [0,1]$ starting from the simple pattern h_0 (a "square cap" $\sqcap$) shown on the left in Figure 19.1.

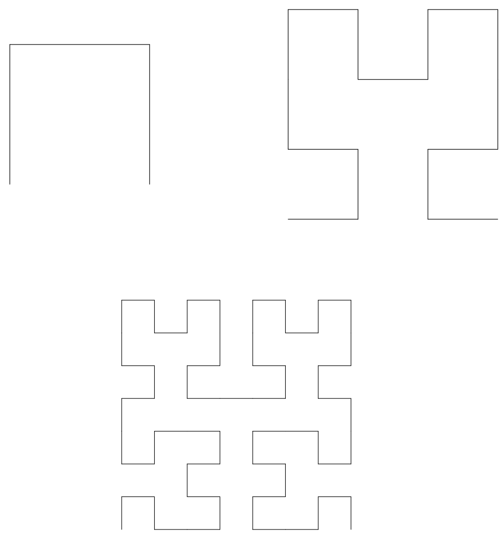

Fig. 19.1 A sequence of Hilbert curves h_0, h_1, h_2.

The curve h_{n+1} is obtained by scaling down h_n by a factor of $\frac{1}{2}$, and connecting the four copies of this scaled–down version of h_n obtained by rotating by $\pi/2$ (left lower part), rotating by $-\pi/2$ and translating right (right lower part), translating up (left upper part), and translating diagonally (right upper part), as illustrated in Figure 19.1.

It can be shown that the sequence (h_n) converges (pointwise) to a continuous curve $h\colon [0,1] \to [0,1] \times [0,1]$ whose trace is the entire square $[0,1] \times [0,1]$. The Hilbert curve h is nowhere differentiable. It also has infinite length! The curve h_5 is shown in Figure 19.2.

Actually, there are many fascinating curves that are only continuous, fractal curves being a major example (see Edgar [8]), but for our purposes we need the existence of the tangent at every point of the curve (except perhaps for finitely many

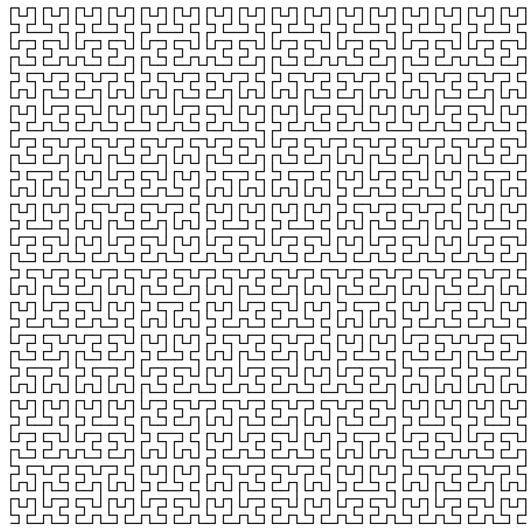

Fig. 19.2 The Hilbert curve h_5.

points). This leads us to require that $f\colon\ ]a,b[\to \mathcal{E}$ be at least continuously differentiable. Recall that a function $f\colon\ ]a,b[\to \mathbb{A}^n$ is *of class C^p, or is C^p-continuous*, if all the derivatives $f^{(k)}$ exist and are continuous for all k, $0 \leq k \leq p$ (when $p = 0$, $f^{(0)} = f$). Thus, we require f to be at least a C^1-function. However, asking that $f\colon\ ]a,b[\to \mathcal{E}$ be a C^p-function for $p \geq 1$ still allows unwanted curves.

Example 19.2. The plane curve defined such that

$$f(t) = \begin{cases} (0, e^{1/t}) & \text{if } t < 0; \\ (0, 0) & \text{if } t = 0; \\ (e^{-1/t}, 0) & \text{if } t > 0; \end{cases}$$

is a C^∞-function, but $f'(0) = 0$, and thus the tangent at the origin is undefined. What happens is that the curve has a sharp "corner" at the origin.

Example 19.3. Similarly, the plane curve defined such that

$$f(t) = \begin{cases} (-e^{1/t}, e^{1/t}\sin(e^{-1/t})) & \text{if } t < 0; \\ (0, 0) & \text{if } t = 0; \\ (e^{-1/t}, e^{-1/t}\sin(e^{1/t})) & \text{if } t > 0; \end{cases}$$

shown in Figure 19.3 is a C^∞-function, but $f'(0) = 0$. In this case, the curve oscillates more and more rapidly as it approaches the origin.

The problem with the above examples is that the origin is a singular point for which $f'(0) = 0$ (a stationary point).

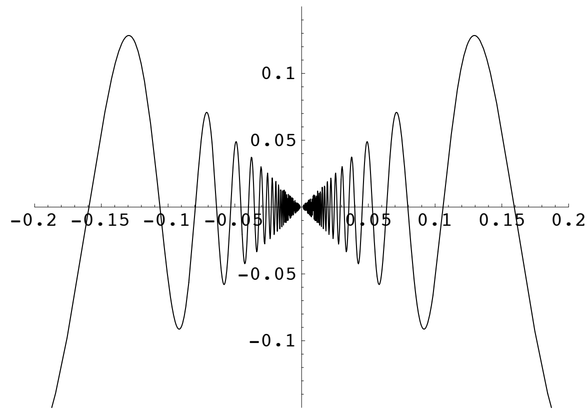

Fig. 19.3 Stationary point at the origin.

Although it is possible to define the tangent when f is sufficiently differentiable and when for every $t \in]a,b[$, $f^{(p)}(t) \neq 0$ for some $p \geq 1$ (where $f^{(p)}$ denotes the pth derivative of f), a systematic study is rather cumbersome. Thus, we will restrict our attention to curves having only regular points, that is, for which $f'(t) \neq 0$ for every $t \in]a,b[$. However, we will allow functions $f\colon\ ]a,b[\to \mathscr{E}$ that are not necessarily injective, unless stated otherwise.

Definition 19.1. An *open curve (or open arc) of class* C^p is a map $f\colon\ ]a,b[\to \mathscr{E}$ of class C^p, with $p \geq 1$, where $]a,b[$ is an open interval (allowing $a = -\infty$ or $b = +\infty$). The set of points $f(]a,b[)$ in $\mathscr{E}$ is called the *trace of the curve* f. A point $f(t)$ is *regular at* $t \in]a,b[$ if $f'(t)$ exists and $f'(t) \neq 0$, and *stationary* otherwise. A *regular open curve (or regular open arc) of class* C^p is an open curve of class C^p, with $p \geq 1$, such that every point is regular, i.e., $f'(t) \neq 0$ for every $t \in]a,b[$.

Note that Definition 19.1 is stated for an open interval $]a,b[$, and thus f may not be defined at a or b. If we want to include the boundary points at a and b in the curve (when $a \neq -\infty$ and $b \neq +\infty$), we use the following definition.

Definition 19.2. A *curve (or arc) of class* C^p is a map $f\colon [a,b] \to \mathscr{E}$, with $p \geq 1$, such that the restriction of f to $]a,b[$ is of class C^p, and where $f^{(i)}(a) = \lim_{t\to a, t>a} f^{(i)}(t)$ and $f^{(i)}(b) = \lim_{t\to b, t<b} f^{(i)}(t)$ exist, where $0 \leq i \leq p$. A *regular curve (or regular arc) of class* C^p is a curve of class C^p, with $p \geq 1$, such that every point is regular, i.e., $f'(t) \neq 0$ for every $t \in [a,b]$. The set of points $f([a,b])$ in $\mathscr{E}$ is called the *trace of the curve* f.

It should be noted that even if f is injective, the trace $f(I)$ of f may be self-intersecting.

Example 19.4. Consider the curve $f\colon \mathbb{R} \to \mathbb{E}^2$ defined such that

$$f_1(t) = \frac{t(1+t^2)}{1+t^4},$$
$$f_2(t) = \frac{t(1-t^2)}{1+t^4}.$$

The trace of this curve, shown in Figure 19.4, is called the "lemniscate of Bernoulli" and it has a self-intersection at the origin. The map f is continuous, and in fact bijective, but its inverse f^{-1} is not continuous. Self-intersection is due to the fact that

$$\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) = f(0).$$

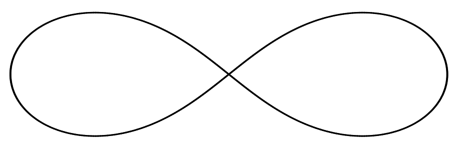

Fig. 19.4 Lemniscate of Bernoulli.

If we consider a curve $f\colon [a,b] \to \mathscr{E}$ and we assume that f is injective on the entire *closed* interval $[a,b]$, then the trace $f([a,b])$ of f has no self-intersection. Such curves are usually called *Jordan arcs*, or *simple arcs*. The theory of Jordan arcs $f\colon [a,b] \to \mathscr{E}$ where f is only required to be continuous is quite rich. Because $[a,b]$ is compact, f is in fact a homeomorphism between $[a,b]$ and $f([a,b])$. Many fractal curves are only continuous Jordan arcs that are not differentiable.

We can also define closed curves. A simple way to do so is to say that a closed curve is a curve $f\colon [a,b] \to \mathscr{E}$ such that $f(a) = f(b)$. However, this does not ensure that the derivatives at a and b agree, a situation that is quite undesirable. A better solution is to define a closed curve as an open curve $f\colon \mathbb{R} \to \mathscr{E}$, where f is periodic.

Definition 19.3. A *closed curve (or closed arc) of class C^p* is a map $f\colon \mathbb{R} \to \mathscr{E}$ such that f is of class C^p, with $p \geq 1$, and such that f is *periodic*, which means that there is some $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. A *regular closed curve (or regular closed arc) of class C^p* is a closed curve of class C^p, with $p \geq 1$, such that every point is regular, i.e., $f'(t) \neq 0$ for every $t \in \mathbb{R}$. The set of points $f([0,T])$ (or $f(\mathbb{R})$) in $\mathscr{E}$ is called the *trace of the curve f*.

A closed curve is a *Jordan curve (or a simple closed curve)* if f is injective on the interval $[0,T[$. A Jordan curve has no self-intersection. The ellipse defined by the map $t \mapsto (a\cos t, b\sin t)$ is an example of a closed curve of type C^∞ that is a Jordan curve. In this example, the period is $T = 2\pi$. Again, the theory of Jordan curves $f\colon [0,T] \to \mathcal{E}$ where f is only required to be continuous is quite rich.

An observant reader may have noticed that a curve has been defined as a map $f\colon\ ]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$), rather than as a certain set of points. In fact, it is possible for the trace of a curve to be defined by many parametrizations, as illustrated by the unit circle, which is the trace of the parametrized curves $f_k\colon\ ]0,2\pi[\to \mathcal{E}$ (or $f_k\colon [0,2\pi] \to \mathcal{E}$), where $f_k(t) = (\cos kt, \sin kt)$, with $k \geq 1$. A clean way to handle this phenomenon is to define a notion of *geometric curve (or arc)*. Such a treatment is given in Berger and Gostiaux [2]. For our purposes it will be sufficient to define a notion of change of parameter that does not change the "geometric shape" of the trace. Recall that a *diffeomorphism* $g\colon\ ]a,b[\to]c,d[$ *of class* C^p from an open interval $]a,b[$ to another open interval $]c,d[$ is a bijection such that both $g\colon\ ]a,b[\to]c,d[$ and its inverse $g^{-1}\colon\ ]c,d[\to]a,b[$ are C^p-functions. This implies that $g'(t) \neq 0$ for every $t \in]a,b[$.

Definition 19.4. Two regular curves $f\colon\ ]a,b[\to \mathcal{E}$ and $g\colon\ ]c,d[\to \mathcal{E}$ of class C^p, with $p \geq 1$, are *C^p-equivalent* if there is a diffeomorphism $\theta\colon\ ]a,b[\to]c,d[$ of class C^p such that $f = g \circ \theta$.

It is immediately verified that Definition 19.4 yields an equivalence relation on open curves. Definition 19.4 is adapted to curves, by extending the notion of C^p-diffeomorphism to closed intervals in the obvious way.

Remark: Using Definition 19.4, we could define a *geometric curve (or arc) of class C^p* as an equivalence class of (parametrized) curves. This is done in Berger and Gostiaux [2].

From now on, in most cases we will drop the word "regular" when referring to regular curves, and simply say "curves." Also, when we refer to a point $f(t)$ on a curve, we mean that $t \in]a,b[$ for an open curve $f\colon\ ]a,b[\to \mathcal{E}$, and $t \in [a,b]$ for a curve $f\colon [a,b] \to \mathcal{E}$. In the case of a closed curve $f\colon \mathbb{R} \to \mathcal{E}$, we can assume that $t \in [0,T]$, where T is the period of f, and thus closed curves will be treated simply as curves in the sequel. We now define tangent lines and osculating planes. According to Kreyszig [15], the term osculating plane was apparently first introduced by Tinseau in 1780.

19.2 Tangent Lines and Osculating Planes

We begin with the definition of a tangent line.

Definition 19.5. For any open curve $f\colon\ ]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if f is locally injective at M_0 and if for any point $M_1 = f(t+h)$ near M_0 the line $T_{t,h}$ determined by the points M_0 and M_1 has a limit T_t when $h \neq 0$ approaches 0, we say that T_t is the *tangent line to* f *in* $M_0 = f(t)$ *at* t.

More precisely, if there is an open interval $]t-\eta, t+\eta[\subseteq]a,b[$ (with $\eta > 0$) such that $M_1 = f(t+h) \neq f(t) = M_0$ for all $h \neq 0$ with $h \in]-\eta, \eta[$ and the line $T_{t,h}$ determined by the points M_0 and M_1 has a limit T_t when $h \neq 0$ approaches 0 (with $h \in]-\eta, \eta[$), then T_t is the tangent line to f in M_0 at t.

For simplicity we will often say "tangent," instead of "tangent line." The definition is simpler when f is a simple curve (there is no danger that $M_1 = M_0$ when $h \neq 0$). In this chapter there will be situations where it is notationally more convenient to denote the vector $\overrightarrow{ab}$ by $b-a$. The following lemma shows why regular points are important.

Lemma 19.1. *For any open curve $f\colon\ ]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if M_0 is a regular point at t, then the tangent line to f in M_0 at t exists and is determined by the derivative $f'(t)$ of f at t.*

Proof. Provided that $M_0 \neq M_1$, the line $T_{t,h}$ is determined by the point M_0 and the vector $M_1 - M_0 = f(t+h) - f(t)$. By the definition of $f'(t)$, we have

$$f(t+h) - f(t) = hf'(t) + h\varepsilon(h),$$

where $\lim_{h\to 0, h\neq 0} \varepsilon(h) = 0$. We claim that there must be an open interval $]t-\eta, t+\eta[\subseteq]a,b[$ (with $\eta > 0$) such that $f(t+h) \neq f(t)$ for all $h \neq 0$ with $-\eta < h < \eta$. Otherwise, since $f'(t)$ exists, for every $\alpha > 0$ there is some $\eta > 0$ such that

$$\left\| \frac{f(t+h) - f(t)}{h} - f'(t) \right\| \leq \alpha$$

for all h, with $-\eta < h < \eta$, and since $f(t+h) - f(t) = 0$ for some $h \neq 0$ with $h \in]-\eta, \eta[$, we would have $\|f'(t)\| \leq \alpha$. Since this holds for every $\alpha > 0$, we would have $f'(t) = 0$, a contradiction. Thus, the line $T_{t,h}$ is determined by the point M_0 and the vector $f'(t) + \varepsilon(h)$, which has the limit $f'(t)$ when $h \neq 0$ tends to 0, with $h \in]-\eta, +\eta[$. Thus, the line $T_{t,h}$ has for limit the line determined by M_0 and the derivative $f'(t)$ of f at t. $\square$

Remark: If $f'(t) = 0$, the above argument breaks down. However, if f is a C^p-function and $f^{(p)}(t) \neq 0$ for some $p \geq 2$, where p is the smallest integer with that property, we can show that the line $T_{t,h}$ has the limit determined by M_0 and the derivative $f^{(p)}(t)$. Thus, the tangent line may still exist at a stationary point. For example, the curve f defined by the map $t \mapsto (t^2, t^3)$ is a C^∞-function, but $f'(0) = 0$. Nevertheless, the tangent at the origin is defined for $t = 0$ (it is the x-axis). However, some strange things can happen at a stationary point. Assuming that a curve is of class C^p for p large enough, using Taylor's formula it is possible to study precisely the behavior of the curve at a stationary point.

Note that the tangent at a point can exist, even when the derivative f' is not continuous at this point.

Example 19.5. The C^0-curve f defined such that

$$f(t) = \begin{cases} (t, t^2 \sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0; \end{cases}$$

and shown in Figure 19.5 has a tangent at $t = 0$.

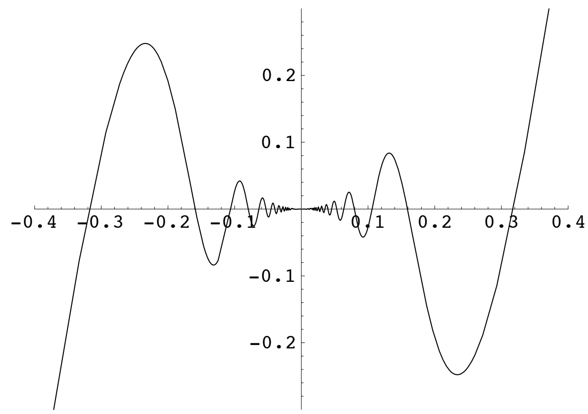

Fig. 19.5 Curve with tangent at O and yet f' discontinuous at O.

Indeed, $f(0) = (0,0)$, and $\lim_{t\to 0} t \sin(1/t) = 0$, and the derivative at $t = 0$ is the vector $(1,0)$. For $t \neq 0$,

$$f'(t) = (1, 2t \sin(1/t) - \cos(1/t)),$$

which has no limit as t tends to 0. Thus, f' is discontinuous at 0. What happens is that f oscillates more and more near the origin, but the amplitude of the oscillations decreases.

If $g = f \circ \theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, the tangent at $\theta(t)$ to f exists iff the tangent at t to g exists, and the two tangents are identical. Indeed, $g'(t) = f'(u)\theta'(t)$, where $u = \theta(t)$, and since $\theta'(t) \neq 0$ because θ is a diffeomorphism, the result is clear. Thus, the notion of tangent is intrinsic to the geometric curve defined by f. We now consider osculating planes.

Definition 19.6. For any open curve $f\colon\ ]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 2$, given any point $M_0 = f(t)$ on the curve, if the tangent T_t at M_0 exists, the point $M_1 = f(t+h)$ is not on T_t for $h \neq 0$ small enough, and the plane $P_{t,h}$ determined by the tangent T_t and the point M_1 has a limit P_t as $h \neq 0$ approaches 0, we say that P_t is the *osculating plane to f in $M_0 = f(t)$ at t*.

More precisely, if the tangent T_t at M_0 exists, there is an open interval $]t-\eta, t+\eta[\subseteq]a,b[$ (with $\eta > 0$) such that the point $M_1 = f(t+h)$ is not on T_t for every $h \neq 0$ with $h \in]-\eta, +\eta[$, and the plane $P_{t,h}$ determined by the tangent T_t and the point M_1 has a limit P_t when $h \neq 0$ approaches 0 (with $h \in]-\eta, +\eta[$), we say that P_t is the osculating plane to f in $M_0 = f(t)$ at t.

Again, the definition is simpler when f is a simple curve. The following lemma gives a simple condition for the existence of the osculating plane at a point.

Lemma 19.2. *For any open curve $f\colon]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b]\to\mathcal{E}$ of class C^p), with $p \geq 2$, given any point $M_0 = f(t)$ on the curve, if $f'(t)$ and $f''(t)$ are linearly independent (which implies that M_0 is a regular point at t), then the osculating plane to f in M_0 at t exists and is determined by the first and second derivatives $f'(t)$ and $f''(t)$ of f at t.*

Proof. The plane $P_{t,h}$ is determined by the point M_0, the vector $f'(t)$, and the vector $M_1 - M_0 = f(t+h) - f(t)$, provided that $M_1 - M_0$ and $f'(t)$ are linearly independent. By Taylor's formula, for $h > 0$ small enough we have

$$f(t+h) - f(t) = hf'(t) + \frac{h^2}{2}f''(t) + \frac{h^2}{2}\varepsilon(h),$$

where $\lim_{h\to 0, h\neq 0}\varepsilon(h) = 0$. By an argument similar to that used in Lemma 19.1, we can show that there is some open interval $]t-\eta, t+\eta[\subseteq]a,b[$ (with $\eta > 0$) such that for every $h \neq 0$ with $-\eta < h < \eta$, the point $M_1 = f(t+h)$ is not on the tangent T_t (otherwise, we could prove that $f''(t)$ is the limit of a sequence of vectors proportional to $f'(t)$, and thus that $f'(t)$ and $f''(t)$ are linearly dependent, a contradiction). Thus, for $h \neq 0$ with $h \in]-\eta, +\eta[$, the plane $P_{t,h}$ is determined by the point M_0, the vector $f'(t)$, and the vector $f''(t) + \varepsilon(h)$, which has the limit $f''(t)$ as $h \neq 0$ tends to 0, with $h \in]-\eta, +\eta[$. Thus, the plane $P_{t,h}$ has for limit the plane determined by M_0 and the derivatives $f'(t)$ and $f''(t)$ of f at t, since $f'(t)$ and $f''(t)$ are assumed to be linearly independent. $\square$

When $f'(t)$ and $f''(t)$ exist and are linearly independent, it is sometimes said that f is *biregular at t*, and that $f(t)$ *is a biregular point at t*. From the kinematic point of view, the osculating plane at time t is determined by the position of the moving particle $f(t)$, the velocity vector $f'(t)$, and the acceleration vector $f''(t)$.

Remark: If the curve f is a plane curve, then the osculating plane at every regular point is the plane containing the curve. Even when $f'(t)$ and $f''(t)$ are linearly dependent, the osculating plane may still exist, for instance, if there are two derivatives $f^{(p)}(t) \neq 0$ and $f^{(q)}(t) \neq 0$ that are linearly independent, with $p < q$, the smallest integers with that property.

In general, the curve crosses its osculating plane at the point of contact t.

If $g = f\circ\theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, the osculating plane at $\theta(t)$ to f exists iff the osculating plane at t to g exists, and these two planes are identical. Indeed, $g'(t) = f'(u)\theta'(t)$, and

$$g''(t) = f''(u)\theta'(t)^2 + f'(u)\theta''(t),$$

where $u = \theta(t)$. Since $\theta'(t) \neq 0$ because θ is a diffeomorphism, the planes defined by $(f'(u), f''(u))$ and $(g'(t), g''(t))$ are identical. Thus, the notion of osculating plane is intrinsic to the geometric curve defined by f.

It should also be noted that the notions of tangent and osculating plane are affine notions, that is, preserved under affine bijections.

We now consider the notion of arc length. For this, we assume that the affine space $\mathscr{E}$ is a normed affine space of finite dimension with norm $\| \, \|$. For simplicity, we can assume that $\mathscr{E} = \mathbb{E}^n$.

19.3 Arc Length

Given an interval $[a,b]$ (where $a \neq -\infty$ and $b \neq +\infty$), a *subdivision* of $[a,b]$ is any finite increasing sequence $t_0, \ldots, t_n$ such that $t_0 = a$, $t_n = b$, and $t_i < t_{i+1}$, for all i, $0 \leq i \leq n-1$, where $n \geq 1$. Given any curve $f\colon [a,b] \to \mathscr{E}$ of class C^p, with $p \geq 0$, for any subdivision $\sigma = t_0, \ldots, t_n$ of $[a,b]$ we obtain a polygonal line $f(t_0), f(t_1), \ldots, f(t_n)$ with endpoints $f(a)$ and $f(b)$, and we define the length of this polygonal line as

$$l(\sigma) = \sum_{i=0}^{n-1} \|f(t_{i+1}) - f(t_i)\|.$$

Definition 19.7. For any curve $f\colon [a,b] \to \mathscr{E}$ of class C^p, with $p \geq 0$, if the set $\mathscr{L}(f)$ of the lengths $l(\sigma)$ of the polygonal lines induced by all subdivisions $\sigma = t_0, \ldots, t_n$ of $[a,b]$ is bounded, we say that f is *rectifiable*, and we call the least upper bound $l(f)$ of the set $\mathscr{L}(f)$ the *length of* f.

It is obvious that $\|f(b) - f(a)\| \leq l(f)$. If $g = f \circ \theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, since $\theta'(t) \neq 0$, θ is a strictly increasing or decreasing function, and thus the set of sums of the form $l(\sigma)$ is the same for both f and g. Thus, the notion of length is intrinsic to the geometric curve defined by f. This is false if θ is not strictly increasing or decreasing. The following lemma can be shown.

Lemma 19.3. *For any curve $f\colon [a,b] \to \mathscr{E}$ of class C^p, with $p \geq 1$, f is rectifiable.*

Remark: In fact, Lemma 19.3 can be shown under the hypothesis that f is of class C^0, and that $f'(t)$ exists and $\|f'(t)\| \leq M$ for some $M \geq 0$, for all $t \in [a,b]$.

Definition 19.8. For any open curve $f\colon \,]a,b[\, \to \mathscr{E}$ of class C^p (or curve $f\colon [a,b] \to \mathscr{E}$ of class C^p), with $p \geq 1$, for any closed interval $[t_0,t] \subseteq \,]a,b[$ (or $[t_0,t] \subseteq [a,b]$, in the case of a curve), letting $f_{[t_0,t]}$ be the restriction of f to $[t_0,t]$, the length $l(f_{[t_0,t]})$ (which exists, by Lemma 19.3) is called the *arc length of* $f_{[t_0,t]}$. For any fixed $t_0 \in \,]a,b[$ (or any fixed $t_0 \in [a,b]$, in the case of a curve), we define the function $s\colon \,]a,b[\, \to \mathbb{R}$ (or $s\colon [a,b] \to \mathbb{R}$, in the case of a curve), called *algebraic arc* *length w.r.t.* t_0, as follows:

$$s(t) = \begin{cases} l(f_{[t_0,t]}) & \text{if } [t_0,t] \subseteq \,]a,b[\,; \\ -l(f_{[t_0,t]}) & \text{if } [t,t_0] \subseteq \,]a,b[\,; \end{cases}$$

(and similarly in the case of a curve, except that $[t_0,t] \subseteq [a,b]$ or $[t,t_0] \subseteq [a,b]$).

For the sake of brevity, we will often call s the arc length, rather than algebraic arc length w.r.t. t_0.

Lemma 19.4. *For any open curve $f\colon\,]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, for any fixed $t_0 \in \,]a,b[$ (or $t_0 \in [a,b]$, in the case of a curve), the algebraic arc length $s(t)$ w.r.t. t_0 is of class C^p, and furthermore, $s'(t) = \|f'(t)\|$.*

Thus, the arc length is given by the integral

$$s(t) = \int_{t_0}^{t} \|f'(u)\|\, du.$$

In particular, when $\mathcal{E} = \mathbb{E}^n$ and the norm is the Euclidean norm, we have

$$s(t) = \int_{t_0}^{t} \sqrt{f_1'(u)^2 + \cdots + f_n'(u)^2}\, du.$$

where $f = (f_1, \ldots, f_n)$. The number $\|f'(t)\|$ is often called the *speed* of $f(t)$ at time t. For every regular point at t, the unit vector

$$\mathbf{t} = \frac{f'(t)}{\|f'(t)\|}$$

is called the *unit tangent (vector) at t*.

Now, if $f\colon\,]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) is a regular curve of class C^p, with $p \geq 1$, since $s'(t) = \|f'(t)\|$, and $f'(t) \neq 0$ for all $t \in \,]a,b[$ (or $t \in [a,b]$), we have $s'(t) > 0$ for all $t \in \,]a,b[$ (or $t \in [a,b]$). The mean value theorem implies that s is injective, and that $s\colon\,]a,b[\to \,]s(a),s(b)[$ (or $s\colon [a,b] \to [s(a),s(b)]$) is a diffeomorphism of class C^p. In particular, the curve $f \circ \varphi\colon\,]s(a),s(b)[\to \mathcal{E}$ (or $f \circ \varphi\colon [s(a),s(b)] \to \mathcal{E}$), with $\varphi = s^{-1}$, is C^p-equivalent to the original curve f, but it is parametrized by the arc length $s \in \,]s(a),s(b)[$ (or $s \in [s(a),s(b)]$). As a consequence, since $\varphi = s^{-1}$, we have

$$\varphi'(s(t)) = (s'(t))^{-1},$$

and letting $g = f \circ \varphi$, by the chain rule

$$g'(s(t)) = f'(\varphi(s(t)))\varphi'(s(t)) = f'(t)(s'(t))^{-1} = \frac{f'(t)}{\|f'(t)\|}.$$

This shows that $\|g'(s)\| = 1$, i.e., that when a regular curve is parametrized by arc length, its velocity vector has unit length. From a kinematic point of view, when a curve is parametrized by arc length, the moving particle travels at constant unit speed.

Remark: If a curve f (or a closed curve) is of class C^p, for $p \geq 1$, and it is a Jordan arc, then the algebraic arc length $s\colon [a,b] \to \mathbb{R}$ w.r.t. t_0 is strictly increasing, and thus injective. Thus, s^{-1} exists, and the curve can still be parametrized by arc length as $g = f \circ s^{-1}$. However, $g'(s)$ exists only when $s(t)$ corresponds to a regular point at t. Thus, it still seems necessary to restrict our attention to regular curves, in order to avoid complications.

We now consider the notion of curvature. In order to do so, we assume that the affine space $\mathcal{E}$ has a Euclidean structure (an inner product), and that the norm on $\mathcal{E}$ is the norm induced by this inner product. For simplicity, we assume that $\mathcal{E} = \mathbb{E}^n$.

19.4 Curvature and Osculating Circles (Plane Curves)

In a Euclidean space, orthogonality makes sense, and we can define normal lines and normal planes. We begin with plane curves, i.e., the case where $\mathcal{E} = \mathbb{E}^2$.

Definition 19.9. Given a regular plane curve $f\colon\ ]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p, with $p \geq 1$, the *normal line N_t to f at t* is the line through $f(t)$ and orthogonal to the tangent line T_t to f at t. Any nonnull vector defining the direction of the normal line N_t is called a *normal vector to f at t*.

From now on, we also assume that we are dealing with curves f that are biregular for all t. This means that $f'(t)$ and $f''(t)$ always exist and are linearly independent. A fairly intuitive way to introduce the notion of curvature is to study the variation of the normal line N_t to a curve f at t, in a small neighborhood of t. The intuition is that the normal N_t to f at t rotates around a certain point, and that the "speed" of rotation of the normal measures how much the curve bends around t. In other words, the rate at which the normal turns corresponds to the curvature of the curve at t. Another way to look at it is to focus on the point around which the normal turns, the center of curvature C at t, and to consider the radius $\mathcal{R}$ of the circle centered at C and tangent to the curve at $f(t)$ (i.e., tangent to the tangent line to f at t). Intuitively, the smaller $\mathcal{R}$ is, the faster the curve bends, and thus the curvature can be defined as $1/\mathcal{R}$.

Let us assume that some origin O is chosen in the affine plane, and to simplify the notation, for any curve f let us denote $f(t) - O$ by $\mathbf{M}(t)$ or $\mathbf{M}$, for any point P denote $P - O$ by $\mathbf{P}$, denote $P - M$ by $\overrightarrow{MP}$, and denote $f'(t)$ by $\mathbf{M}'(t)$ or $\mathbf{M}'$. The normal line N_t to f at t is the set of points P such that

$$\mathbf{M}' \cdot \overrightarrow{MP} = 0,$$

or equivalently

$$\mathbf{M}' \cdot \mathbf{P} = \mathbf{M}' \cdot \mathbf{M}.$$

Similarly, for any small $\delta \neq 0$ such that $f(t+\delta)$ is defined, the normal line $N_{t+\delta}$ to f at $t+\delta$ is the set of points Q such that

$$\mathbf{M}'(t+\delta) \cdot \mathbf{Q} = \mathbf{M}'(t+\delta) \cdot \mathbf{M}(t+\delta).$$

Thus, the intersection point P of N_t and $N_{t+\delta}$, if it exists, is given by the equations

$$\begin{aligned} \mathbf{M}' \cdot \mathbf{P} &= \mathbf{M}' \cdot \mathbf{M}, \\ \mathbf{M}'(t+\delta) \cdot \mathbf{P} &= \mathbf{M}'(t+\delta) \cdot \mathbf{M}(t+\delta). \end{aligned}$$

Thus, P would also satisfy the equation obtained by subtracting the first one from the second, that is,

$$(\mathbf{M}'(t+\delta) - \mathbf{M}') \cdot \mathbf{P} = \mathbf{M}'(t+\delta) \cdot \mathbf{M}(t+\delta) - \mathbf{M}' \cdot \mathbf{M}.$$

This equation can be written as

$$\begin{aligned} \left(\frac{\mathbf{M}'(t+\delta) - \mathbf{M}'}{\delta}\right) \cdot \mathbf{P} = \left(\frac{\mathbf{M}'(t+\delta) - \mathbf{M}'}{\delta}\right) \cdot \mathbf{M}(t+\delta) \\ + \mathbf{M}' \cdot \left(\frac{\mathbf{M}(t+\delta) - \mathbf{M}}{\delta}\right), \end{aligned}$$

and as $\delta \neq 0$ tends to 0, it has the following equation for limit:

$$\mathbf{M}'' \cdot \mathbf{P} = \mathbf{M}'' \cdot \mathbf{M} + \mathbf{M}' \cdot \mathbf{M}',$$

that is,

$$\mathbf{M}'' \cdot \overrightarrow{MP} = \|\mathbf{M}'\|^2.$$

Consequently, if it exists, P is the intersection of the two lines of equations

$$\begin{aligned} \mathbf{M}' \cdot \overrightarrow{MP} &= 0 \\ \mathbf{M}'' \cdot \overrightarrow{MP} &= \|\mathbf{M}'\|^2. \end{aligned}$$

Thus, if $\mathbf{M}'$ and $\mathbf{M}''$ are linearly independent, which is equivalent to saying that $f'(t)$ and $f''(t)$ are linearly independent, i.e., f is biregular at t, the above two equations have a unique solution P. Also, the above analysis shows that the intersection of the two normals N_t and $N_{t+\delta}$, for $\delta \neq 0$ small enough, has a limit C (really, $C(t)$). This limit is called the *center of curvature of f at t*. It is possible to compute the distance $\mathscr{R} = \|\overrightarrow{MC}\|$, the *radius of curvature at t*, and the coordinates of C, given any affine frame for the plane. It is worth noting that the equation

$$\mathbf{M}'' \cdot \mathbf{P} = \mathbf{M}'' \cdot \mathbf{M} + \mathbf{M}' \cdot \mathbf{M}'$$

is obtained by taking the derivative of the equation

$$\mathbf{M}' \cdot \mathbf{P} = \mathbf{M}' \cdot \mathbf{M}$$

with respect to t. This observation can be used to compute the coordinates of the center of curvature, but first we show that the radius of curvature has a very simple expression when the curve is parametrized by arc length. Indeed, in this case, $\|f'(s)\| = \|\mathbf{M}'\| = 1$, that is, $f'(s) \cdot f'(s) = 1$, and by taking the derivatives of both sides, we get

$$f''(s) \cdot f'(s) = 0,$$

which shows that $f''(s) = \mathbf{M}''$ and $f'(s) = \mathbf{M}'$ are orthogonal, and since the center of curvature C is determined by the equations

$$\begin{aligned} \mathbf{M}' \cdot \overrightarrow{MC} &= 0, \\ \mathbf{M}'' \cdot \overrightarrow{MC} &= \|\mathbf{M}'\|^2, \end{aligned}$$

the vector $\overrightarrow{MC}$ must be collinear with $\mathbf{M}''$ (since it is orthogonal to $\mathbf{M}'$, which itself is orthogonal to $\mathbf{M}''$). Then, letting

$$\mathbf{n} = \frac{\mathbf{M}''}{\|\mathbf{M}''\|}$$

be the unit vector associated with the acceleration vector $\mathbf{M}''$, we have $\overrightarrow{MC} = \mathscr{R}\mathbf{n}$, and since $\|\mathbf{M}'\| = 1$, from $\mathbf{M}'' \cdot \overrightarrow{MC} = \|\mathbf{M}'\|^2$ we get

$$\mathbf{M}'' \cdot \overrightarrow{MC} = \mathbf{M}'' \cdot \mathscr{R}\,\frac{\mathbf{M}''}{\|\mathbf{M}''\|} = \mathscr{R}\,\frac{(\mathbf{M}'' \cdot \mathbf{M}'')}{\|\mathbf{M}''\|} = \mathscr{R}\,\frac{\|\mathbf{M}''\|^2}{\|\mathbf{M}''\|} = \mathscr{R}\,\|\mathbf{M}''\| = 1,$$

that is,

$$\mathscr{R} = \frac{1}{\|\mathbf{M}''\|} = \frac{1}{\|f''(s)\|}.$$

Thus, the radius of curvature is the inverse of the norm of the acceleration vector $f''(s)$. We define the *curvature* κ as the inverse of the radius of curvature $\mathscr{R}$, that is, as

$$\kappa = \|f''(s)\|.$$

In summary, when the curve f is parametrized by arc length, we found that the curvature κ and the radius of curvature $\mathscr{R}$ are defined by the equations

$$\boxed{\kappa = \|f''(s)\|, \quad \mathscr{R} = \frac{1}{\kappa}.}$$

We now come back to the general case. Assuming that $\mathbf{M}'$ and $\mathbf{M}''$ are linearly independent, we can write $\overrightarrow{MC} = \alpha\mathbf{M}' + \beta\mathbf{M}''$, for some unique α, β. Since C is determined by the equations

$$\mathbf{M}' \cdot \overrightarrow{MC} = 0,$$
$$\mathbf{M}'' \cdot \overrightarrow{MC} = \|\mathbf{M}'\|^2,$$

we get the system

$$(\mathbf{M}' \cdot \mathbf{M}')\,\alpha + (\mathbf{M}' \cdot \mathbf{M}'')\,\beta = 0,$$
$$(\mathbf{M}' \cdot \mathbf{M}'')\,\alpha + (\mathbf{M}'' \cdot \mathbf{M}'')\,\beta = \|\mathbf{M}'\|^2,$$

and we also note that

$$\mathscr{R}^2 = \overrightarrow{MC} \cdot \overrightarrow{MC} = \overrightarrow{MC} \cdot (\alpha\mathbf{M}' + \beta\mathbf{M}'') = \beta\,\|\mathbf{M}'\|^2.$$

The reader can verify that we obtain

$$\beta = \frac{\|\mathbf{M}'\|^4}{\|\mathbf{M}'\|^2\|\mathbf{M}''\|^2 - (\mathbf{M}' \cdot \mathbf{M}'')^2},$$

and thus

$$\mathscr{R}^2 = \frac{\|\mathbf{M}'\|^6}{\|\mathbf{M}'\|^2\|\mathbf{M}''\|^2 - (\mathbf{M}' \cdot \mathbf{M}'')^2}.$$

However, if we remember about the cross product of vectors and the Lagrange identity, we have

$$\|\mathbf{M}'\|^2\|\mathbf{M}''\|^2 - (\mathbf{M}' \cdot \mathbf{M}'')^2 = \|\mathbf{M}' \times \mathbf{M}''\|^2,$$

and thus

$$\mathscr{R} = \frac{\|\mathbf{M}'\|^3}{\|\mathbf{M}' \times \mathbf{M}''\|} = \frac{\|f'(t)\|^3}{\|f'(t) \times f''(t)\|},$$

and the curvature is given by

$$\kappa = \frac{\|\mathbf{M}' \times \mathbf{M}''\|}{\|\mathbf{M}'\|^3} = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}.$$

In summary, when the curve f is not necessarily parametrized by arc length, we found that the curvature κ and the radius of curvature $\mathscr{R}$ are defined by the equations

$$\boxed{\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}, \quad \mathscr{R} = \frac{1}{\kappa}.}$$

Note that from an analytical point of view, the curvature has the advantage of being defined at every regular point, since $\kappa = 0$ when either $f''(t) = 0$ or $f''(t)$ is collinear to $f'(t)$, whereas at such points, the radius of curvature goes to $+\infty$.

We leave as an exercise to show that if $g = f \circ \theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, then

$$\kappa = \frac{\|f'(u) \times f''(u)\|}{\|f'(u)\|^3} = \frac{\|g'(t) \times g''(t)\|}{\|g'(t)\|^3},$$

where $u = \theta(t)$, i.e., κ has the same value for both f and g. Thus, the curvature is an *invariant* intrinsic to the geometric curve defined by f. In view of the above considerations, we give the following definition of the curvature, which is more intrinsic.

Definition 19.10. For any regular plane curve $f\colon\ ]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p parametrized by arc length, with $p \geq 2$, the *curvature* κ *at* s is defined as the nonnegative real number $\kappa = \|f''(s)\|$. For every s such that $f''(s) \neq 0$, letting $\mathbf{n} = f''(s)/\|f''(s)\|$ be the unit vector associated with $f''(s)$, we have $f''(s) = \kappa\mathbf{n}$, the point C defined such that $C - f(s) = \mathbf{n}/\kappa$ is the *center of curvature at* s, and $\mathscr{R} = 1/\kappa$ is the *radius of curvature at* s. The circle of center C and of radius $\mathscr{R}$ is called the *osculating circle to* f *at* s. When $f''(s) = 0$, by convention $\mathscr{R} = \infty$, and the center of curvature is undefined.

The locus of the center of curvature is a curve that is regular, except at points for which $\mathscr{R}' = 0$. Properties of this curve, called the *evolute*, will be given in Lemma 19.5.

Example 19.6. The evolute of an ellipse, the center of curvature corresponding to a specific point on the ellipse, and the osculating circle at that point are shown in Figure 19.6.

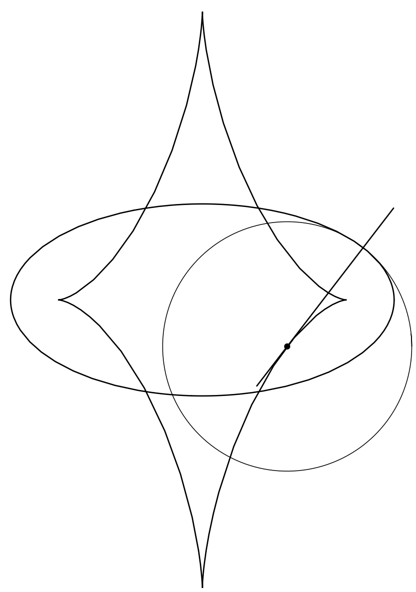

Fig. 19.6 The evolute of an ellipse, and an osculating circle.

It is also possible to define the notion of osculating circle more geometrically as a limit, in the spirit of the definition of a tangent.

Definition 19.11. Given any plane curve $f\colon\ ]a,b[\to\mathscr{E}$ (or $f\colon [a,b]\to\mathscr{E}$) of class C^p, with $p\geq 1$, and given any point $M_0 = f(t)$ on the curve, if f is locally injective at M_0, the tangent T_t to f at t exists, and the circle $\Sigma_{t,h}$ tangent to T_t and passing through M_1 has a limit Σ_t as $h\neq 0$ approaches 0, we say that Σ_t is the *osculating circle to f in $M_0 = f(t)$ at t*.

More precisely, if there is an open interval $]t-\eta, t+\eta[\,\subseteq\,]a,b[$ (with $\eta > 0$) such that, $M_1 = f(t+h)\neq f(t) = M_0$ for all $h\neq 0$ with $h\in]-\eta,\eta[$, the tangent T_t to f at t exists, and the circle $\Sigma_{t,h}$ tangent to T_t and passing through M_1 has a limit Σ_t as $h\neq 0$ approaches 0 (with $h\in]-\eta,\eta[$), we say that Σ_t is the osculating circle to f in $M_0 = f(t)$ at t.

It is not hard to show that if the center of curvature C (and thus the radius of curvature $\mathscr{R}$) exists at t, then the osculating circle at t is indeed the circle of center C and radius $\mathscr{R}$ (also called *circle of curvature at t*).

Remark: It is possible that the osculating circle exists at a point t but that the center of curvature at t is undefined.

Example 19.7. Consider the curve defined such that

$$f(t) = \begin{cases} (t, t^2 + t^3\sin(1/t)) & \text{if } t\neq 0;\\ (0,0) & \text{if } t = 0,\end{cases}$$

and shown in Figure 19.7.

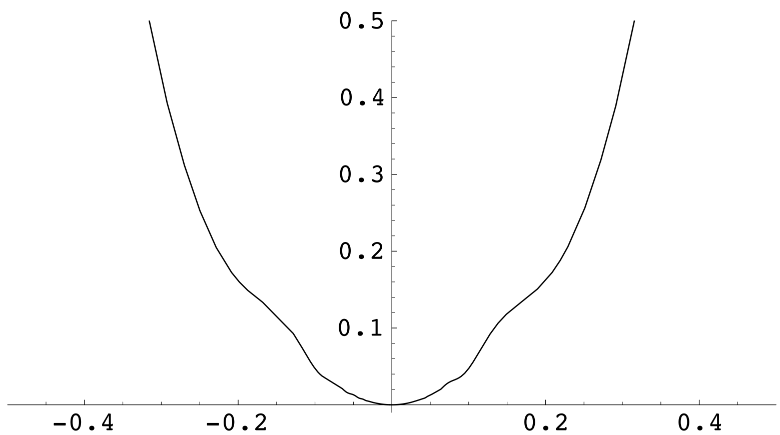

Fig. 19.7 Osculating circle at O exists and yet $f''(0)$ is undefined.

We leave as an exercise to show that the osculating circle for $t = 0$ is the circle of center $\left(0,\frac{1}{2}\right)$, but $f''(0)$ is undefined, so that the center of curvature is undefined at $t = 0$. This is because the intersection point of the normal line N_0 at $t = 0$ (the y-axis) and the normal N_δ for δ small oscillates forever as δ goes to zero.

In general, the osculating circle intersects the curve in another point besides the point of contact, which means that near the point of contact, one of the two branches of the curve is outside the osculating circle, and the other branch is inside. This property fails for points on an axis of symmetry for the curve, such as the points on the axes of an ellipse.

Osculating circles give a very good approximation of the curve around each (biregular) point. We will see in later examples that plotting enough osculating circles gives the illusion that the curve is plotted, when in fact it is not!

Recalling that we denoted the (unit) tangent vector $f'(s)$ at s by $\mathbf{t}$, and the unit normal vector $f''(s)/\|f''(s)\|$ by $\mathbf{n}$, since

$$\mathbf{t}' = f''(s) = \kappa\mathbf{n},$$

we have

$$\mathbf{t}' = \kappa\mathbf{n}.$$

Since $\mathbf{t}\cdot\mathbf{n} = 0$ and $\mathbf{n}\cdot\mathbf{n} = 1$, by taking derivatives of these equations we get $\mathbf{n}\cdot\mathbf{n}' = 0$ and $\mathbf{t}'\cdot\mathbf{n}+\mathbf{t}\cdot\mathbf{n}' = 0$. Since $\mathbf{n}'$ is orthognal to $\mathbf{n}$, it is collinear to $\mathbf{t}$, and from the second equation, since $\mathbf{t}' = \kappa\mathbf{n}$, we get

$$\kappa\mathbf{n}\cdot\mathbf{n}+\mathbf{t}\cdot\mathbf{n}' = \kappa+\mathbf{t}\cdot\mathbf{n}' = 0,$$

and thus

$$\mathbf{n}' = -\kappa\mathbf{t}.$$

Using the identity $\mathbf{n}' = -\kappa\mathbf{t}$, we can also show the following lemma, confirming the geometric characterization of the center of curvature.

Lemma 19.5. *For any regular plane curve $f\colon\ ]a,b[\to\mathscr{E}$ (or $f\colon [a,b]\to\mathscr{E}$) of class C^p parametrized by arc length, with $p\geq 2$, assuming that $f''(s)\neq 0$, the center of curvature is on a curve c of class C^0 defined such that $c(s)-f(s) = \mathscr{R}\mathbf{n}$, where $\mathscr{R} = 1/\|f''(s)\|$ and $\mathbf{n} = f''(s)/\|f''(s)\|$, and whenever $\mathscr{R}'(s)\neq 0$, c is regular at s and $c'(s) = \mathscr{R}'\mathbf{n}$, which means that the normal to f at s is the tangent to c at s.*

Proof. Fixing any origin O in the plane, from $c(s)-f(s) = \mathscr{R}\mathbf{n}$ we have

$$c(s)-O = f(s)-O+\mathscr{R}\mathbf{n},$$

and thus

$$c'(s) = f'(s)+\mathscr{R}'\mathbf{n}+\mathscr{R}\mathbf{n}',$$

and since $\mathbf{n}' = -\kappa\mathbf{t}$, with $\kappa = 1/\mathscr{R}$, we get

$$c'(s) = \mathbf{t}+\mathscr{R}'\mathbf{n}-\mathbf{t} = \mathscr{R}'\mathbf{n}.$$

□

In other words, for every s where κ'/κ^2 is defined and not equal to zero, the point $c(s)$ is regular. This is not the case for points for which the curvature is minimal or maximal. The example of an ellipse is typical (see below). The curve c defined in Lemma 19.5 is called the *evolute* of the curve f. Conversely, f is called the *involute* of c.

Summarizing the discussion before Definition 19.10, we also have the following lemma.

Lemma 19.6. *For any regular plane curve $f\colon\ ]a,b[\to \mathscr{E}$ (or $f\colon [a,b]\to\mathscr{E}$) of class C^p, with $p\geq 2$, the curvature at t is given by the expression*

$$\kappa = \frac{\|f'(t)\times f''(t)\|}{\|f'(t)\|^3}.$$

Furthermore, whenever $f'(t)\times f''(t)\neq 0$, the center of curvature C defined such that $C-f(t)=\mathbf{n}/\kappa$ is the limit of the intersection of any normal $N_{t+\delta}$ and the normal N_t at t as $\delta\neq 0$ small enough approaches 0.

Lemma 19.6 gives us a way of calculating the curvature at any point, for any (regular) parametrization of a curve. Let us now determine the coordinates of the center of curvature (when defined). Let $(O,\mathbf{i},\mathbf{j})$ be an orthonormal frame for the plane, and let the curve be defined by the map $f(t)=O+u(t)\mathbf{i}+v(t)\mathbf{j}$. The equation of the normal to f at t is $(x-u)u'+(y-v)v'=0$, or

$$u'x+v'y=uu'+vv'.$$

As we noted earlier, the center of curvature is obtained by intersecting this normal with the line whose equation is obtained by taking the derivative of the equation of the normal w.r.t. t. Thus, the center of curvature is the solution of the system

$$\begin{aligned} u'x+v'y &= uu'+vv',\\ u''x+v''y &= uu''+vv''+u'^2+v'^2. \end{aligned}$$

We leave as an exercise to verify that the solution is given by

$$\begin{aligned} x &= u-\frac{v'(u'^2+v'^2)}{u'v''-v'u''},\\ y &= v+\frac{u'(u'^2+v'^2)}{u'v''-v'u''}, \end{aligned}$$

provided that $u'v''-v'u''\neq 0$. One will also check that the radius of curvature is given by

$$\mathscr{R}=\frac{(u'^2+v'^2)^{3/2}}{|u'v''-v'u''|}.$$

This result can also be obtained from Lemma 19.6, by calculating the coordinates of the cross product $f'(t)\times f''(t)$.

We now give a few examples.

Example 19.8. If f is a straight line, then $f''(t) = 0$, and thus the curvature is null for every point of a line.

Example 19.9. A circle of radius a can be defined by

$$x = a\cos t,$$
$$y = a\sin t.$$

We have $u' = -a\sin t$, $v' = a\cos t$, $u'' = -a\cos t$, $v'' = -a\sin t$, and thus

$$u'v'' - v'u'' = (-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t) = a^2$$

and

$$u'^2 + v'^2 = a^2(\sin^2 t + \cos^2 t) = a^2,$$

and thus

$$\mathscr{R} = \frac{(u'^2 + v'^2)^{3/2}}{|u'v'' - v'u''|} = a$$

and $\kappa = 1/a$. Thus, as expected, the circle has constant curvature $1/a$, where a is its radius, and the center of curvature is reduced to a single point, the center of the circle. Indeed, every normal to the circle goes through it!

Example 19.10. An ellipse is defined by

$$x = a\,\cos\theta,$$
$$y = b\,\sin\theta.$$

The equation of the normal to the ellipse at θ is

$$(x - a\,\cos\theta)(-a\sin\theta) + (y - b\,\sin\theta)(b\cos\theta) = 0,$$

or

$$a\sin\theta\, x - b\cos\theta\, y = \sin\theta\cos\theta(a^2 - b^2).$$

Assuming that $a \geq b$ (the other case being similar), and letting $c^2 = a^2 - b^2$, the above equation is

$$a\sin\theta\, x - b\cos\theta\, y = c^2\sin\theta\cos\theta.$$

We leave as an exercise to show that the radius of curvature is

$$\mathscr{R} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab},$$

and, that the center of curvature is on the curve defined by

$$x = \frac{c^2}{a}\cos^3\theta, \quad y = -\frac{c^2}{b}\sin^3\theta.$$

This curve has four cusps, corresponding to the two maxima and minima of the curvature. Letting

$$N = \left(\frac{c^2}{a}\cos\theta, 0\right)$$

be the intersection of the normal to the point M on the ellipse with Ox, and $d = \|MN\|$ be the distance between M and N, we leave as an exercise to show that the radius of curvature is given by

$$\mathscr{R} = \frac{a^2}{b^4} d^3.$$

It is fun to plot the locus of the center of curvature and enough osculating circles to the ellipse. Figure 19.8 shows 64 osculating circles of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(with $a \geq b$), for $a = 4$, $b = 2$, and the locus of the center of curvature. Although the ellipse is not explicitly plotted, it seems to be present!

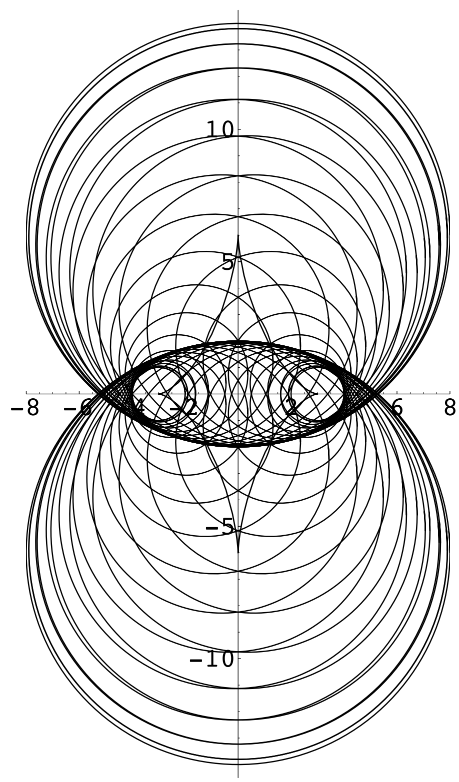

Fig. 19.8 Osculating circles of an ellipse.

Example 19.11. The logarithmic spiral given in polar coordinates by $r = a\mathrm{e}^{m\theta}$, or by

$$x = a\mathrm{e}^{m\theta}\cos\theta,$$
$$y = a\mathrm{e}^{m\theta}\sin\theta$$

(with $a > 0$), is particularly interesting. We leave as an exercise to show that the radius of curvature is

$$\mathscr{R} = a\sqrt{1+m^2}\,\mathrm{e}^{m\theta},$$

and that the center of curvature is on the spiral (in fact, equal to the original spiral) defined by

$$x = -ma\,\mathrm{e}^{m\theta}\sin\theta,$$
$$y = ma\,\mathrm{e}^{m\theta}\cos\theta.$$

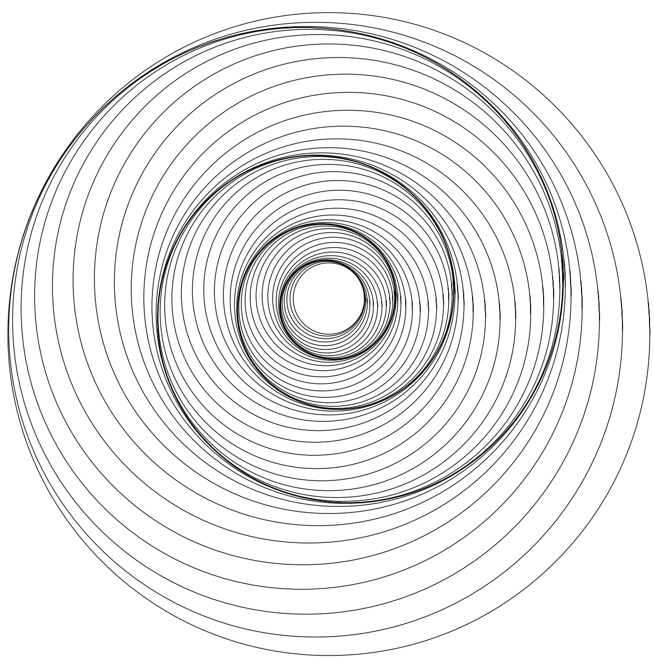

Fig. 19.9 Osculating circles of a logarithmic spiral.

Figure 19.9 shows 50 osculating circles of the logarithmic spiral given in polar coordinates by $r = a\mathrm{e}^{m\theta}$, for $a = 0.6$ and $m = 0.1$. The spiral definitely shows up very clearly, even though it is not explicitly plotted. Also, note that since the radius of curvature is increasing, no two osculating circles intersect!

Example 19.12. The cardioid given in polar coordinates by $r = a(1+\cos\theta)$, or by

$$\begin{aligned} x &= a(1+\cos\theta)\cos\theta, \\ y &= a(1+\cos\theta)\sin\theta, \end{aligned}$$

is also a neat example. Figure 19.10 shows 50 osculating circles of the cardioid given in polar coordinates by $r = a(1+\cos\theta)$, for $a = 2$, and the locus of the center of curvature.

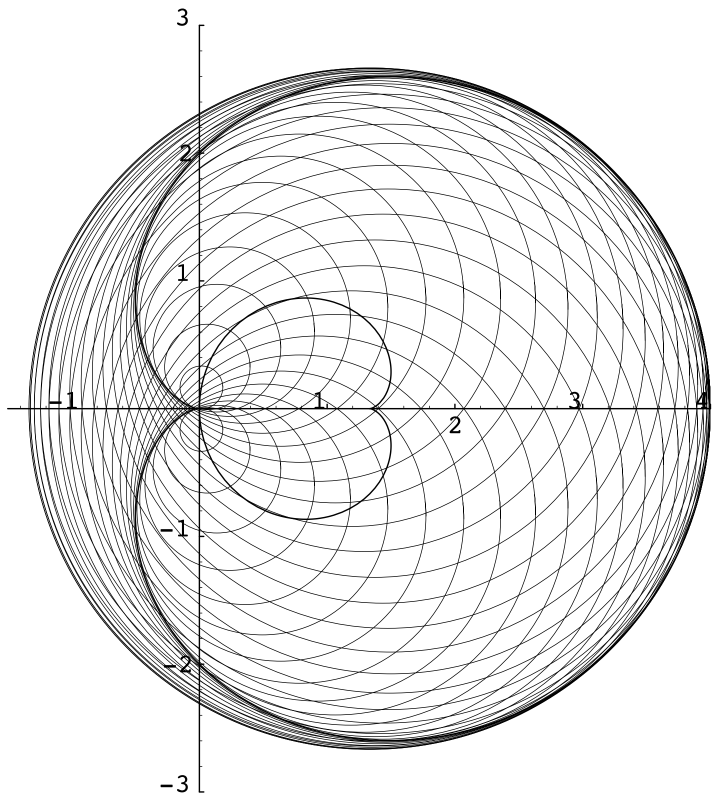

Fig. 19.10 Osculating circles of a cardioid.

We leave as an exercise to show that the radius of curvature is

$$\mathscr{R} = \left|\frac{2a}{3}\cos(\theta/2)\right|,$$

and that the center of curvature is on the cardioid defined by

$$\begin{aligned} x &= \frac{2a}{3} + \frac{a}{3}(1-\cos\theta)\cos\theta, \\ y &= \frac{a}{3}(1-\cos\theta)\sin\theta. \end{aligned}$$

We conclude our discussion of the curvature of plane curves with a brief look at the algebraic curvature. Since a plane can be oriented, it is possible to give a sign to the curvature. Let us assume that the plane is oriented by an othonormal frame $(O,\mathbf{i},\mathbf{j})$, assumed to have a positive orientation, and that the curve f is parametrized by arc length. Then, given any unit tangent vector $\mathbf{t}$ at s to a curve f, there exists a unit normal vector ν such that $(O,\mathbf{t},\nu)$ has positive orientation. In fact, if θ is the angle (mod 2π) between $\mathbf{i}$ and $\mathbf{t}$, so that

$$\mathbf{t} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$

we have

$$\nu = \cos(\theta+\pi/2)\,\mathbf{i} + \sin(\theta+\pi/2)\,\mathbf{j} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Note that this normal vector ν is not necessarily equal to the unit normal vector $\mathbf{n} = f''(s)/\|f''(s)\|$: It can be of the opposite direction. Furthermore, ν exists for every regular point, even when $f''(s) = 0$, which is not true of $\mathbf{n}$. We define the *algebraic curvature k at s* as the real number (negative, null, or positive) such that

$$f''(s) = k\nu.$$

We also define the *algebraic radius of curvature R* as $R = 1/k$. Clearly, $\kappa = |k|$ and $\mathscr{R} = |R|$. Thus, we also have

$$\mathbf{t}' = k\nu,$$

and it is immediately verified that the center of curvature is still given by $C - f(s) = R\nu$, and that

$$\nu' = -k\mathbf{t}.$$

The algebraic curvature plays an important role in some global theorems of differential geometry. It is also possible to prove that if $c\colon\ ]a,b[\,\to \mathbb{R}$ is a continuous function and $s_0 \in]a,b[$, then there is a unique curve $f\colon\ ]a,b[\,\to \mathscr{E}$ such that $f(s_0)$ is any given point, $f'(s_0)$ is any given vector, and such that $c(s)$ is the algebraic curvature of f. Roughly speaking, the algebraic curvature k determines the curve completely, up to rigid motion.

One should be careful to note that this result fails if we consider the curvature κ instead of the algebraic curvature k. Indeed, it is possible that $k(s) = c(s) = 0$, and thus that $\kappa(s) = 0$. Such points may be inflection points, and counterexamples to the above result with κ instead of k are easily found. However, if we require $c(s) > 0$ for all s, the above result holds for the curvature κ.

We now consider curves in affine Euclidean 3D spaces (i.e. $\mathscr{E} = \mathbb{E}^3$).

19.5 Normal Planes and Curvature (3D Curves)

The first thing to do is to define the notion of a normal plane.

Definition 19.12. Given any regular 3D curve $f\colon\ ]a,b[\ \to \mathscr{E}$ (or $f\colon [a,b] \to \mathscr{E}$) of class C^p, with $p \geq 2$, the *normal plane N_t to f at t* is the plane through $f(t)$ and orthogonal to the tangent line T_t to f at t. The intersection of the normal plane and the osculating plane (if it exists) is called the *principal normal line to f at t*.

In order to get an intuitive idea of the notion of curvature, we need to look at the variation of the normal plane around t, since there are infinitely many normal lines to a given line in 3-space. This time, we will see that the normal plane rotates around a line perpendicular to the osculating plane (called the *polar axis at t*). The intersection of this line with the osculating plane is the center of curvature. But now, not only does the normal plane rotate around an axis, so do the osculating plane and the plane containing the tangent line and normal to the osculating plane, called the rectifying plane. Thus, a second quantity, called the torsion, will make its appearance. But let us go back to the intersection of normal planes around t.

Actually, the treatment that we gave for the plane extends immediately to space (in 3D). Indeed, the normal plane N_t to f at t is the set of points P such that

$$\mathbf{M}' \cdot \overrightarrow{MP} = 0,$$

or equivalently

$$\mathbf{M}' \cdot \mathbf{P} = \mathbf{M}' \cdot \mathbf{M}.$$

Similarly, for any small $\delta \neq 0$ such that $f(t+\delta)$ is defined, the normal plane $N_{t+\delta}$ to f at $t+\delta$ is the set of points Q such that

$$\mathbf{M}'(t+\delta) \cdot \mathbf{Q} = \mathbf{M}'(t+\delta) \cdot \mathbf{M}(t+\delta).$$

Thus, the intersection points P of N_t and $N_{t+\delta}$, if they exist, are given by the equations

$$\begin{aligned} \mathbf{M}' \cdot \mathbf{P} &= \mathbf{M}' \cdot \mathbf{M}, \\ \mathbf{M}'(t+\delta) \cdot \mathbf{P} &= \mathbf{M}'(t+\delta) \cdot \mathbf{M}(t+\delta). \end{aligned}$$

As in the planar case, for δ very small, the intersection of the two planes N_t and $N_{t+\delta}$ is given by the equations

$$\begin{aligned} \mathbf{M}' \cdot \overrightarrow{MP} &= 0, \\ \mathbf{M}'' \cdot \overrightarrow{MP} &= \|\mathbf{M}'\|^2. \end{aligned}$$

Thus, if $\mathbf{M}'$ and $\mathbf{M}''$ are linearly independent, which is equivalent to saying that $f'(t)$ and $f''(t)$ are linearly independent, i.e., f is biregular at t, the above two equations define a unique line Δ orthogonal to the osculating plane. This line is called the *polar axis at* t. Also, the above analysis shows that the intersection of the two normal planes N_t and $N_{t+\delta}$, for $\delta \neq 0$ small enough, has the limit Δ. Since the line Δ is perpendicular to the osculating plane, it intersects the osculating plane in a single point C (really, $C(t)$), the *center of curvature of* f *at* t. The distance $\mathscr{R} = \|\overrightarrow{MC}\|$ is the *radius of curvature at* t, and its inverse $\kappa = 1/\mathscr{R}$ is the *curvature at* t. Note that C is on the normal line to the curve f at t contained in the osculating plane, i.e., on the principal normal at t.

19.6 The Frenet Frame (3D Curves)

When $f'(t)$ and $f''(t)$ are linearly independent, we can find a unit vector in the plane spanned by $f'(t)$ and $f''(t)$ and orthogonal to the unit tangent vector $\mathbf{t} = f'(t)/\|f'(t)\|$ at t, and equal to the unit vector $f''(t)/\|f''(t)\|$ when $f'(t)$ and $f''(t)$ are orthogonal, namely the unit vector

$$\mathbf{n} = \frac{-(f'(t)\cdot f''(t))\,f'(t) + \|f'(t)\|^2 f''(t)}{\|-(f'(t)\cdot f''(t))\,f'(t) + \|f'(t)\|^2 f''(t)\|}.$$

The unit vector $\mathbf{n}$ is called the *principal normal vector to* f *at* t. Note that $\mathbf{n}$ defines the direction of the principal normal at t. We define the *binormal vector* $\mathbf{b}$ *at* t as $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Thus, the triple $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is a basis of orthogonal unit vectors. It is usually called the *Frenet (or Frenet–Serret) frame at t* (this concept was introduced independently by Frenet, in 1847, and Serret, in 1850). This concept is sufficiently important to warrant the following definition.

Definition 19.13. Given a biregular 3D curve $f\colon\ ]a,b[\to \mathscr{E}$ (or $f\colon [a,b] \to \mathscr{E}$) of class C^p, with $p \geq 2$, the *Frenet frame (or Frenet trihedron) at t* is the triple $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ consisting of the three orthogonal unit vectors such that $\mathbf{t} = f'(t)/\|f'(t)\|$ is the unit tangent vector at t,

$$\mathbf{n} = \frac{-(f'(t)\cdot f''(t))\,f'(t) + \|f'(t)\|^2 f''(t)}{\|-(f'(t)\cdot f''(t))\,f'(t) + \|f'(t)\|^2 f''(t)\|}$$

is a unit vector orthogonal to $\mathbf{t}$ called the *principal normal vector to* f *at* t, and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the *binormal vector at* t. The plane containing $\mathbf{t}$ and $\mathbf{b}$ is called the *rectifying plane at* t.

As we will see shortly, the principal normal $\mathbf{n}$ has a simpler expression when the curve is parametrized by arc length. The calculations of $\mathscr{R}$ are still valid, since the cross product $\mathbf{M}' \times \mathbf{M}''$ makes sense in 3-space, and thus we have

$$\mathscr{R} = \frac{\|\mathbf{M}'\|^3}{\|\mathbf{M}' \times \mathbf{M}''\|} = \frac{\|f'(t)\|^3}{\|f'(t) \times f''(t)\|},$$

and the curvature is given by

$$\kappa = \frac{\|\mathbf{M}' \times \mathbf{M}''\|}{\|\mathbf{M}'\|^3} = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}.$$

Example 19.13. Consider the curve given by

$$f(t) = (t, t^2, t^3),$$

known as the *twisted cubic*. We have $f'(t) = (1, 2t, 3t^2)$ and $f''(t) = (0, 2, 6t)$, and thus at $t = 0$ (the origin), the vectors

$$f'(t) = (1,0,0) \quad \text{and} \quad f''(t) = (0,2,0)$$

are orthogonal, and the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ consists of the three unit vectors $i = (1,0,0)$, $j = (0,1,0)$, and $k = (0,0,1)$. Thus, the osculating plane is the xy-plane, the normal plane is the yz-plane, and the rectifying plane is the xz-plane. It is easily checked that

$$f' \times f'' = (6t^2, -6t, 2),$$

and the curvature at t is given by

$$\kappa(t) = \frac{2(9t^4 + 9t^2 + 1)^{1/2}}{(9t^4 + 4t^2 + 1)^{3/2}}.$$

In particular, $\kappa(0) = 2$, and the polar line is the vertical line in the yz-plane passing through the point $C = \left(0, \frac{1}{2}, 0\right)$, the center of curvature.

When the curve is parametrized by arc length, $\mathbf{t} = f'(s)$, and we obtain the same results as in the planar case, namely,

$$\mathscr{R} = \frac{1}{\|\mathbf{M}''\|} = \frac{1}{\|f''(s)\|}.$$

The radius of curvature is the inverse of the norm of the acceleration vector $f''(s)$, and the curvature κ is

$$\kappa = \|f''(s)\|.$$

Again, as in the planar case, the curvature is an invariant intrinsic to the geometric curve defined by f.

We now consider how the rectifying plane varies. This will uncover the torsion. According to Kreyszig [15], the term torsion was first used by de la Vallée in 1825. We leave as an easy exercise to show that the osculating plane rotates around the tangent line for points $t + \delta$ close enough to t.

19.7 Torsion (3D Curves)

Recall that the rectifying plane is the plane orthogonal to the principal normal at t passing through $f(t)$. Thus, its equation is

$$\mathbf{n} \cdot \overrightarrow{MP} = 0,$$

where $\mathbf{n}$ is the principal normal vector. However, things get a bit messy when we take the derivative of $\mathbf{n}$, because of the denominator, and it is easier to use the vector

$$\mathbf{N} = -(f'(t) \cdot f''(t))\, f'(t) + \|f'(t)\|^2 f''(t),$$

which is collinear to $\mathbf{n}$, but not necessarily a unit vector. Still, we have $\mathbf{N} \cdot \mathbf{M}' = 0$, which is the important fact. Since the equation of the rectifying plane is $\mathbf{N} \cdot \overrightarrow{MP} = 0$ or

$$\mathbf{N} \cdot \mathbf{P} = \mathbf{N} \cdot \mathbf{M},$$

by familiar reasoning, the equation of a rectifying plane for $\delta \neq 0$ small enough is

$$\mathbf{N}(t+\delta) \cdot \mathbf{P} = \mathbf{N}(t+\delta) \cdot \mathbf{M}(t+\delta),$$

and we can easily prove that the intersection of these two planes is given by the equations

$$\begin{aligned} \mathbf{N} \cdot \overrightarrow{MP} &= 0, \\ \mathbf{N}' \cdot \overrightarrow{MP} &= \mathbf{N} \cdot \mathbf{M}' = 0, \end{aligned}$$

since $\mathbf{N} \cdot \mathbf{M}' = 0$. Thus, if $\mathbf{N}$ and $\mathbf{N}'$ are linearly independent, the intersection of these two planes is a line in the rectifying plane, passing through the point $M = f(t)$. We now have to take a closer look at $\mathbf{N}'$. It is easily seen that

$$\mathbf{N}' = -(\|\mathbf{M}''\|^2 + \mathbf{M}' \cdot \mathbf{M}''')\mathbf{M}' + (\mathbf{M}' \cdot \mathbf{M}'')\mathbf{M}'' + \|\mathbf{M}'\|^2 \mathbf{M}'''.$$

Thus, $\mathbf{N}$ and $\mathbf{N}'$ are linearly independent iff $\mathbf{M}'$, $\mathbf{M}''$, and $\mathbf{M}'''$ are linearly independent. Now, since the line in question is in the rectifying plane, every point P on this line can be expressed as

$$\overrightarrow{MP} = \alpha \mathbf{b} + \beta \mathbf{t},$$

where α and β are related by the equation

$$(\mathbf{N}' \cdot \mathbf{b})\alpha + (\mathbf{N}' \cdot \mathbf{t})\beta = 0,$$

obtained from $\mathbf{N}' \cdot \overrightarrow{MP} = 0$. However, $\mathbf{t} = \mathbf{M}'/\|\mathbf{M}'\|$, and it is immediate that

$$\mathbf{b} = \frac{\mathbf{M}' \times \mathbf{M}''}{\|\mathbf{M}' \times \mathbf{M}''\|}.$$

Recalling that the radius of curvature is given by $\mathscr{R} = \|\mathbf{M}'\|^3/\|\mathbf{M}' \times \mathbf{M}''\|$, it is tempting to investigate the value of α when $\beta = \mathscr{R}$. Then the equation

$$(\mathbf{N}' \cdot \mathbf{b})\alpha + (\mathbf{N}' \cdot \mathbf{t})\beta = 0$$

becomes

$$(\mathbf{N}' \cdot (\mathbf{M}' \times \mathbf{M}''))\alpha + \|\mathbf{M}'\|^2(\mathbf{N}' \cdot \mathbf{M}') = 0.$$

Since

$$\mathbf{N}' = -(\|\mathbf{M}''\|^2 + \mathbf{M}' \cdot \mathbf{M}''')\mathbf{M}' + (\mathbf{M}' \cdot \mathbf{M}'')\mathbf{M}'' + \|\mathbf{M}'\|^2\mathbf{M}''',$$

we get

$$\mathbf{N}' \cdot (\mathbf{M}' \times \mathbf{M}'') = \|\mathbf{M}'\|^2(\mathbf{M}', \mathbf{M}'', \mathbf{M}'''),$$

where $(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')$ is the mixed product of the three vectors, i.e., their determinant, and since $\mathbf{N} \cdot \mathbf{M}' = 0$, we get $\mathbf{N}' \cdot \mathbf{M}' + \mathbf{N} \cdot \mathbf{M}'' = 0$. Thus,

$$\mathbf{N}' \cdot \mathbf{M}' = -\mathbf{N} \cdot \mathbf{M}'' = (\mathbf{M}' \cdot \mathbf{M}'')^2 - \|\mathbf{M}'\|^2\|\mathbf{M}''\|^2 = -\|\mathbf{M}' \times \mathbf{M}''\|^2,$$

and finally, we get

$$\|\mathbf{M}'\|^2(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')\alpha - \|\mathbf{M}'\|^2\|\mathbf{M}' \times \mathbf{M}''\|^2 = 0,$$

which yields

$$\alpha = \frac{\|\mathbf{M}' \times \mathbf{M}''\|^2}{(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')}.$$

So finally, we have shown that the axis of rotation of the rectifying planes for $t + \delta$ close to t is determined by the vector

$$\overrightarrow{MP} = \alpha\mathbf{b} + \mathscr{R}\mathbf{t},$$

or equivalently, that

$$(\kappa\mathbf{t} + \tau\mathbf{b}) \cdot \overrightarrow{MP} = 0,$$

where κ is the curvature and $\tau = -1/\alpha$ is called the *torsion at* t, and is given by

$$\tau = -\frac{(\mathbf{M}', \mathbf{M}'', \mathbf{M}''')}{\|\mathbf{M}' \times \mathbf{M}''\|^2}.$$

Its inverse $\mathscr{T} = 1/\tau$ is called the *radius of torsion at* t. The vector $-\tau\mathbf{t} + \kappa\mathbf{b}$ giving the direction of the axis or rotation of the rectifying plane is called the *Darboux vector*. In summary, we have obtained the following formulae for the curvature and the torsion of a 3D-curve:

$$\boxed{\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}, \quad \tau = -\frac{(f'(t), f''(t), f'''(t))}{\|f'(t) \times f''(t)\|^2}.}$$

Example 19.14. Returning to the example of the twisted cubic

$$f(t) = (t, t^2, t^3),$$

since $f'(t) = (1, 2t, 3t^2)$, $f''(t) = (0, 2, 6t)$, and $f'''(t) = (0, 0, 6)$, we get

$$(f', f'', f''') = 12,$$

and since

$$f' \times f'' = (6t^2, -6t, 2),$$

the torsion at t is given by

$$\tau(t) = -\frac{3}{9t^4 + 9t^2 + 1}.$$

In particular, $\tau(0) = -3$, and the rectifying plane rotates around the line through the origin and of direction

$$-\tau\mathbf{t} + \kappa\mathbf{b} = (3, 2, 0).$$

The twisted cubic, the locus of the centers of curvature, the Frenet frame, the polar line (D), and the Darboux vector (Db) corresponding to $t = 0$ are shown in Figure 19.11.

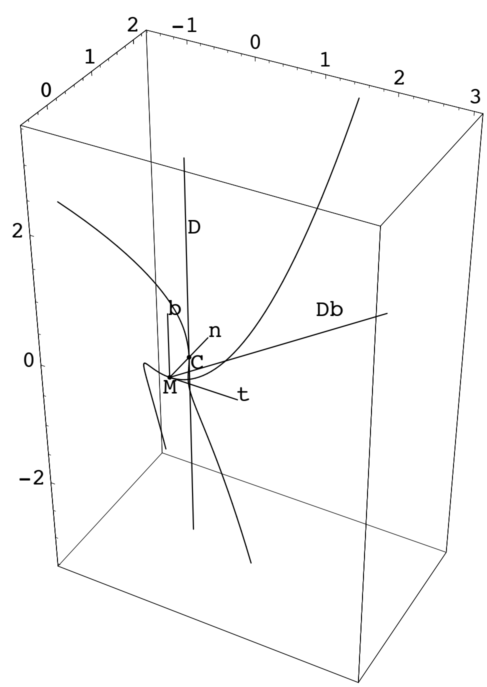

Fig. 19.11 The twisted cubic, the centers of curvature, a Frenet frame, a polar line, and a Darboux vector.

If $g = f \circ \theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, we leave as an exercise to prove that

$$\tau = -\frac{(f'(u), f''(u), f'''(u))}{\|f'(u) \times f''(u)\|^2} = -\frac{(g'(t), g''(t), g'''(t))}{\|g'(t) \times g''(t)\|^2},$$

where $u = \theta(t)$, i.e., τ has the same value for both f and g. Thus, the torsion is an *invariant* intrinsic to the geometric curve defined by f.

19.8 The Frenet Equations (3D Curves)

Assuming that curves are parametrized by arc length, we are now going to see how κ and τ reappear naturally when we determine how the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ varies with s, and more specifically, in expressing $(\mathbf{t}', \mathbf{n}', \mathbf{b}')$ over the basis $(\mathbf{t}, \mathbf{n}, \mathbf{b})$. We claim that

$$\begin{aligned}
\mathbf{t}' &= \kappa \mathbf{n}, \\
\mathbf{n}' &= -\kappa \mathbf{t} - \tau \mathbf{b}, \\
\mathbf{b}' &= \tau \mathbf{n},
\end{aligned}$$

where κ is the curvature, and τ turns out to be the torsion.

We have $\mathbf{t}' = \kappa \mathbf{n}$ by definition of the curvature. Since $\|\mathbf{b}\| = \mathbf{b} \cdot \mathbf{b} = 1$ and $\mathbf{t} \cdot \mathbf{b} = 0$, by taking derivatives we get

$$\mathbf{b} \cdot \mathbf{b}' = 0$$

and

$$\mathbf{t}' \cdot \mathbf{b} = -\mathbf{t} \cdot \mathbf{b}',$$

and thus

$$\mathbf{t} \cdot \mathbf{b}' = -\mathbf{t}' \cdot \mathbf{b} = -\kappa \mathbf{n} \cdot \mathbf{b} = 0.$$

This shows that $\mathbf{b}'$ is collinear to $\mathbf{n}$, and thus that

$$\mathbf{b}' = \tau \mathbf{n},$$

for some real τ. From $\|\mathbf{n}\| = \mathbf{n} \cdot \mathbf{n} = 1$, $\mathbf{n} \cdot \mathbf{t} = 0$, and $\mathbf{n} \cdot \mathbf{b} = 0$, by taking derivatives we get

$$\mathbf{n} \cdot \mathbf{n}' = 0, \quad \mathbf{n}' \cdot \mathbf{t} = -\mathbf{n} \cdot \mathbf{t}', \quad \mathbf{n}' \cdot \mathbf{b} = -\mathbf{n} \cdot \mathbf{b}'.$$

Since $\mathbf{t}' = \kappa \mathbf{n}$ and $\mathbf{b}' = \tau \mathbf{n}$, we get

$$\mathbf{n}' \cdot \mathbf{t} = -\mathbf{n} \cdot \mathbf{t}' = -\mathbf{n} \cdot \kappa \mathbf{n} = -\kappa$$

and

$$\mathbf{n}' \cdot \mathbf{b} = -\mathbf{n} \cdot \mathbf{b}' = -\mathbf{n} \cdot \tau \mathbf{n} = -\tau.$$

But the components of $\mathbf{n}'$ over $(\mathbf{t},\mathbf{n},\mathbf{b})$ are indeed $\mathbf{n}'\cdot\mathbf{t}$, $\mathbf{n}'\cdot\mathbf{n}$, and $\mathbf{n}'\cdot\mathbf{b}$, and thus

$$\mathbf{n}' = -\kappa\mathbf{t} - \tau\mathbf{b}.$$

In matrix form we can write the equations know as the *Frenet (or Frenet–Serret) equations* as

$$\boxed{(\mathbf{t}',\mathbf{n}',\mathbf{b}') = (\mathbf{t},\mathbf{n},\mathbf{b})\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}.}$$

We can now verify that τ agrees with the geometric interpretation given before. The axis of rotation of the rectifying plane is the line given by the intersection of the two planes of equations

$$\mathbf{n}\cdot\overrightarrow{MP} = 0,$$
$$\mathbf{n}'\cdot\overrightarrow{MP} = 0,$$

and since

$$\mathbf{n}' = -\kappa\mathbf{t} - \tau\mathbf{b},$$

the second equation is equivalent to

$$(\kappa\mathbf{t} + \tau\mathbf{b})\cdot\overrightarrow{MP} = 0.$$

This is exactly the equation that we found earlier with $\tau = -1/\alpha$, where

$$\alpha = \frac{\|\mathbf{M}'\times\mathbf{M}''\|^2}{(\mathbf{M}',\mathbf{M}'',\mathbf{M}''')}.$$

Remarks:

(1) Some authors, including Darboux ([6], Livre I, Chapter 1) and Élie Cartan ([5], Chapter VII, Section 2), define the torsion as $-\tau$, in which case

$$\tau = \frac{(\mathbf{M}',\mathbf{M}'',\mathbf{M}''')}{\|\mathbf{M}'\times\mathbf{M}''\|^2},$$

and the Frenet equations take the form

$$(\mathbf{t}',\mathbf{n}',\mathbf{b}') = (\mathbf{t},\mathbf{n},\mathbf{b})\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

A possible advantage of this choice is the elimination of the negative sign in the expression for τ above, and the fact that it may be slightly easier to remember the Frenet matrix, since signs on descending diagonals remain the same. Another possible advantage is that the Frenet matrix has a similar shape in higher dimension (≥ 4). Books on Computer-Aided Gemetric Design seem to prefer this choice. On the other hand, do Carmo [7] and Berger and Gostiaux [2] use the opposite convention (as we do).

(2) It should also be noted that if we let

$$\omega = \tau\mathbf{t} + \kappa\mathbf{b},$$

often called the *Darboux vector*, then (abbreviating three equations in one using a slight abuse of notation)

$$(\mathbf{t}', \mathbf{n}', \mathbf{b}') = \omega \times (\mathbf{t}, \mathbf{n}, \mathbf{b}),$$

which shows that the vectors $\mathbf{t}', \mathbf{n}', \mathbf{b}'$ are the velocities of the tips of the unit frame, and that the unit frame rotates around an instantaneous axis of rotation passing through the origin of the frame, whose direction is the vector $\omega = \tau\mathbf{t} + \kappa\mathbf{b}$.

We now summarize the above considerations in the following definition and lemma.

Definition 19.14. Given a biregular 3D curve $f\colon\ ]a,b[\, \to \mathscr{E}$ (or $f\colon [a,b] \to \mathscr{E}$) of class C^p parametrized by arc length, with $p \geq 3$, given the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ at s, the *curvature* κ *at* s is the nonnegative real number such that $\mathbf{t}' = \kappa\mathbf{n}$, the *torsion* τ *at* s is the real number such that $\mathbf{b}' = \tau\mathbf{n}$, the *radius of curvature at* s is the nonnegative real number $\mathscr{R} = 1/\kappa$, the *radius of torsion at* s is the real number $\mathscr{T} = 1/\tau$, the *center of curvature at* s is the point C on the principal normal such that $C - f(s) = \mathscr{R}\mathbf{n}$, and the *polar axis at* s is the line orthogonal to the osculating plane passing through the center of curvature.

Again, we stress that the curvature κ and the torsion τ are intrinsic *invariants* of the geometric curve defined by f.

Lemma 19.7. *Given a biregular* 3D *curve* $f\colon\ ]a,b[\, \to \mathscr{E}$ *(or* $f\colon [a,b] \to \mathscr{E}$*) of class* C^p *parametrized by arc length, with* $p \geq 3$*, given the Frenet frame* $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ *at s, we have the Frenet (or Frenet–Serret) equations*

$$(\mathbf{t}', \mathbf{n}', \mathbf{b}') = (\mathbf{t}, \mathbf{n}, \mathbf{b}) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}.$$

Given any parametrization for f, the curvature κ *and the torsion* τ *are given by the expressions*

$$\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}$$

and

$$\tau = -\frac{(f'(t), f''(t), f'''(t))}{\|f'(t) \times f''(t)\|^2}.$$

Furthermore, for δ small enough, the normal plane at $t+\delta$ rotates around the polar axis, a line othogonal to the osculating plane and passing through the center of curvature, and the rectifying plane at $t+\delta$ rotates around the line defined by the point of contact at t and the vector $-\tau\mathbf{t}+\kappa\mathbf{b}$ (the Darboux vector).

The torsion measures how the osculating plane rotates around the tangent. Let us show that if f is a biregular curve and if $\tau = 0$ for all t, then f is a plane curve. We can assume that f is parametrized by arc length. Since $\mathbf{b}' = \tau\mathbf{n}$, and we are assuming that $\tau = 0$, we have $\mathbf{b}' = 0$, which means that $\mathbf{b}$ is a constant vector. Since f is biregular, $\mathbf{b} \neq 0$. But now, choosing any origin O and observing that

$$(Of(s)\cdot\mathbf{b})' = f'(s)\cdot\mathbf{b} + Of(s)\cdot\mathbf{b}' = \mathbf{t}\cdot\mathbf{b} + 0 = 0,$$

we conclude that $Of(s)\cdot\mathbf{b} = \lambda$ for some constant λ. Since $\mathbf{b} \neq 0$, we conclude that $f(s)$ is in a plane.

One should be careful to note that the above result is false if f has points that are not biregular, i.e., if $f''(s) = 0$ for some s. We leave as an exercise to find an example of a regular nonplanar curve such that $\tau = 0$.

As an example of the computation of the torsion, consider the circular helix defined by

$$f(t) = (a\cos t,\, a\sin t,\, kt).$$

It is easy to show that the curvature is given by

$$\kappa = \frac{a}{a^2+k^2}$$

and that the torsion is given by

$$\tau = -\frac{k}{a^2+k^2}.$$

Thus, both the curvature and the torsion are constant!

The intrinsic nature of the curvature and the torsion is illustrated by the following result. If $c\colon\ ]a,b[\ \to \mathbb{R}_+$ is a continuous positive C^1 function, $d\colon\ ]a,b[\ \to \mathbb{R}$ is a continuous function, and $s_0 \in\ ]a,b[$, then there is a unique biregular 3D curve $f\colon\ ]a,b[\ \to \mathcal{E}$ such that $f(s_0)$ is any given point, $f'(s_0)$ is any given vector, $f''(s_0)$ is any given vector, and such that $c(s)$ is the curvature of f, and $d(s)$ is the torsion of f. Roughly speaking, the curvature and the torsion determine a biregular curve completely, up to rigid motion.

The hypothesis that $c(s) > 0$ for all s is crucial, and the above result is false if this condition is not satisfied everywhere.

19.9 Osculating Spheres (3D Curves)

We conclude our discussion of curves in 3-space by discussing briefly the notion of osculating sphere. According to Kreyszig [15], osculating spheres were first considered by Fuss in 1806.

Definition 19.15. For any 3D curve $f\colon\ ]a,b[\ \to \mathscr{E}$ (or $f\colon [a,b] \to \mathscr{E}$) of class C^p, with $p \geq 3$, and given any point $M_0 = f(t)$ on the curve, if the polar axis at t exists, f is locally injective at M_0, and the sphere $\Sigma_{t,h}$ centered on the polar axis and passing through the points M_0 and $M_1 = f(t+h)$ has a limit Σ_t as $h \neq 0$ approaches 0, we say that Σ_t is the *osculating sphere to f in $M_0 = f(t)$ at t*. More precisely, if the polar axis at t exists and if there is an open interval $]t-\eta, t+\eta[\ \subseteq\ ]a,b[$ (with $\eta > 0$) such that the point $M_1 = f(t+h)$ is distinct from M_0 for every $h \neq 0$ with $h \in\ ]-\eta, +\eta[$ and the sphere $\Sigma_{t,h}$ centered on the polar axis and passing through the points M_0 and M_1 has a limit Σ_t as $h \neq 0$ approaches 0 (with $h \in\ ]-\eta, +\eta[$), we say that Σ_t is the osculating sphere to f in $M_0 = f(t)$ at t.

Again, the definition is simpler when f is a simple curve. The following lemma gives a simple condition for the existence of the osculating sphere at a point.

Lemma 19.8. *For any* 3D *curve $f\colon\ ]a,b[\ \to \mathscr{E}$ (or $f\colon [a,b] \to \mathscr{E}$) of class C^p parametrized by arc length, with $p \geq 3$, given any point $M_0 = f(s)$ on the curve, if M_0 is a biregular point at s and if $\mathscr{R}'$ is defined, then the osculating sphere to f in M_0 at s exists and has its center Ω on the polar axis Δ, such that $\Omega - C = -\mathscr{T}\mathscr{R}'\mathbf{b}$, where $\mathscr{T}$ is the radius of torsion, $\mathscr{R}$ is the radius of curvature, C is the center of curvature, and* $\mathbf{b}$ *is the binormal, at s.*

According to Kreyszig [15], the formula

$$\Omega - M = \mathscr{R}\mathbf{n} - \mathscr{T}\mathscr{R}'\mathbf{b}$$

is due to de Saint Venant (1845). When s varies, the polar axis generates a surface, and the center Ω of the osculating sphere generates a curve on this surface. In general, this surface consists of the tangents to this curve (called *line of striction* or *edge of regression* of the ruled surface).

Figure 19.12 illustrates the Frenet frame, the polar axis, the center of curvature, and the osculating sphere. It also shows the osculating plane, the normal plane, and the rectifying plane.

The twisted cubic and the locus of the centers of osculating spheres are shown in Figure 19.13. The tangent surface, that is, the surface consisting of the tangent lines to the twisted cubic; the curve of centers of osculating spheres; and two great circles of osculating spheres corresponding to $t = \frac{1}{5}$, are shown in Figure 19.14. The tangent surface is the envelope of the osculating planes. The surface generated by the polar lines is shown in Figure 19.15. This surface is the envelope of the normal planes to the twisted cubic. The curve of the centers of osculating spheres is a *line of striction* (or *edge of regression*) on this surface.

Finally, we discuss the case of curves in Euclidean spaces of dimension $n \geq 4$.

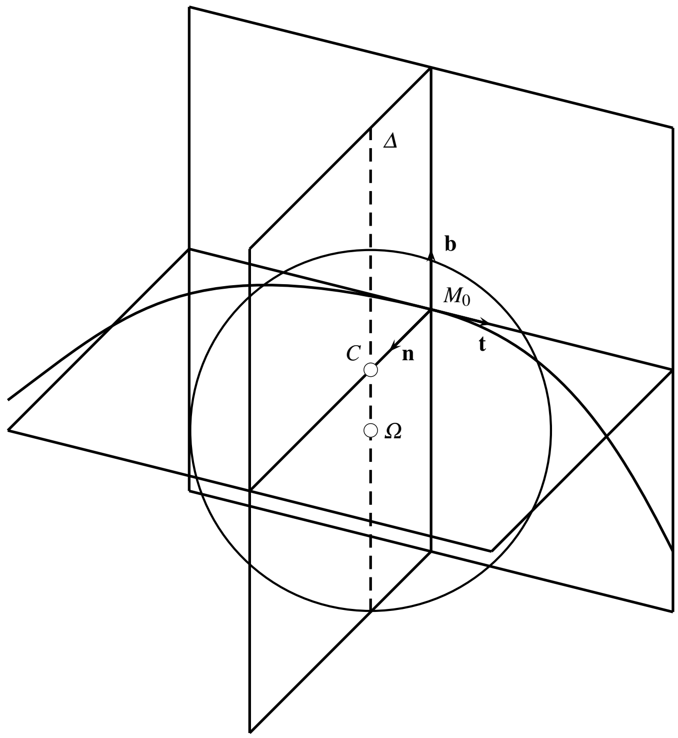

Fig. 19.12 The Frenet frame, polar axis, center of curvature, and osculating sphere.

19.10 The Frenet Frame for nD Curves ($n \geq 4$)

Given a curve $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon\ [a,b]\to \mathbb{E}^n$) of class C^p, with $p \geq n$, it is interesting to consider families $(e_1(t),\ldots,e_n(t))$ of orthonormal frames. Moreover, if for every k, with $1 \leq k \leq n$, the kth derivative $f^{(k)}(t)$ of the curve $f(t)$ is a linear combination of $(e_1(t),\ldots,e_k(t))$ for every $t \in]a,b[$, then such a frame plays the role of a generalized Frenet frame. This leads to the following definition:

Definition 19.16. Let $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon\ [a,b]\to \mathbb{E}^n$) be a curve of class C^p, with $p \geq n$. A family $(e_1(t),\ldots,e_n(t))$ of orthonormal frames, where each $e_i\colon\ ]a,b[\to \mathbb{E}^n$ is C^{n-i}-continuous for $i = 1,\ldots,n-1$ and e_n is C^1-continuous, is called a *moving frame along* f. Furthermore, a moving frame $(e_1(t),\ldots,e_n(t))$ along f such that for every k, with $1 \leq k \leq n$, the kth derivative $f^{(k)}(t)$ of $f(t)$ is a linear combination of $(e_1(t),\ldots,e_k(t))$ for every $t \in]a,b[$, is called a *Frenet n-frame* or *Frenet frame*.

If $(e_1(t),\ldots,e_n(t))$ is a moving frame, then

$$e_i(t)\cdot e_j(t) = \delta_{i\,j} \quad \text{for all } i,j,\ 1 \leq i,j \leq n.$$

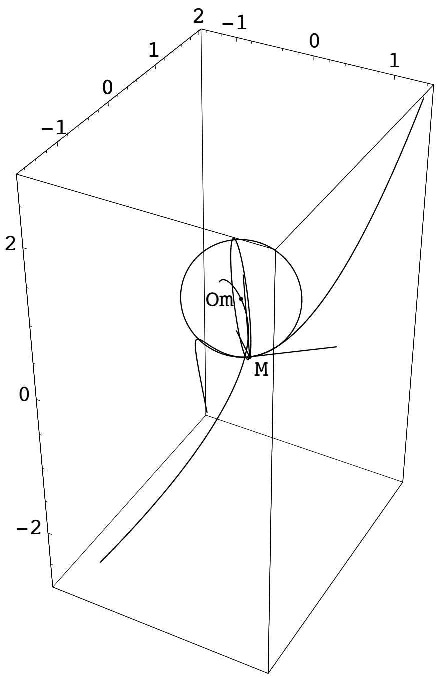

Fig. 19.13 The twisted cubic and the curve of centers of osculating spheres.

Lemma 19.9. *Let $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon [a,b]\to \mathbb{E}^n$) be a curve of class C^p, with $p \geq n$, such that the derivatives $f^{(1)}(t),\ldots,f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t \in]a,b[$. Then there is a unique Frenet n-frame $(e_1(t),\ldots,e_n(t))$ satisfying the following conditions:*

(1) The k-frames $(f^{(1)}(t),\ldots,f^{(k)}(t))$ and $(e_1(t),\ldots,e_k(t))$ have the same orientation for all k, with $1 \leq k \leq n-1$.
(2) The frame $(e_1(t),\ldots,e_n(t))$ has positive orientation.

Proof. Since $(f^{(1)}(t),\ldots,f^{(n-1)}(t))$ is linearly independent, we can use the Gram–Schmidt orthonormalization procedure (see Lemma 6.7) to construct $(e_1(t),\ldots,e_{n-1}(t))$ from $(f^{(1)}(t),\ldots,f^{(n-1)}(t))$. We use the generalized cross product to define e_n, where

$$e_n = e_1 \times \cdots \times e_{n-1}.$$

From the Gram–Schmidt procedure, it is easy to check that $e_k(t)$ is C^{n-k} for $1 \leq k \leq n-1$, and since the components of e_n are certain determinants involving the components of $(e_1,\ldots,e_{n-1})$, it is also clear that e_n is C^1. $\square$

The Frenet n-frame given by Lemma 19.9 is called the *distinguished Frenet n-frame*. We can now prove a generalization of the Frenet–Serret formula that gives an expression of the derivatives of a moving frame in terms of the moving frame itself.

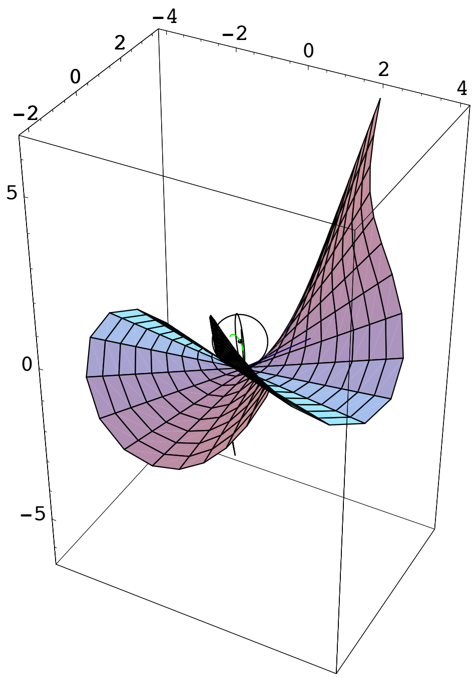

Fig. 19.14 The tangent surface and the centers of osculating spheres.

Lemma 19.10. *Let $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon [a,b]\to \mathbb{E}^n$) be a curve of class C^p, with $p\geq n$, such that the derivatives $f^{(1)}(t),\ldots,f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t\in]a,b[$. Then for any moving frame $(e_1(t),\ldots,e_n(t))$, if we write $\omega_{ij}(t)=e_i'(t)\cdot e_j(t)$, we have*

$$e_i'(t)=\sum_{j=1}^{n}\omega_{ij}(t)e_j(t),$$

with

$$\omega_{ji}(t)=-\omega_{ij}(t),$$

and there are some functions $\alpha_i(t)$ such that

$$f'(t)=\sum_{i=1}^{n}\alpha_i(t)e_i(t).$$

Furthermore, if $(e_1(t),\ldots,e_n(t))$ is the distinguished Frenet n-frame associated with f, then we also have

$$\alpha_1(t)=\|f'(t)\|,\qquad \alpha_i(t)=0\quad \textit{for}\quad i\geq 2,$$

and

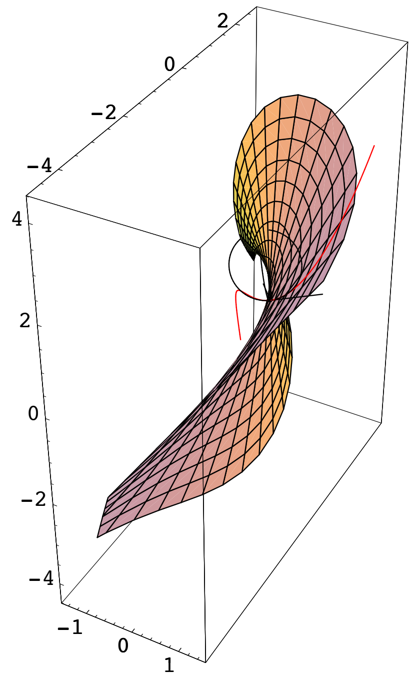

Fig. 19.15 The polar surface and the twisted cubic.

$$\omega_{ij}(t) = 0 \quad for \quad j > i+1.$$

Proof. Since $(e_1(t), \ldots, e_n(t))$ is a moving frame, it is an orthonormal basis, and thus $f'(t)$ and $e_i'(t)$ are linear combinations of $(e_1(t), \ldots, e_n(t))$. Also, we know that

$$e_i'(t) = \sum_{j=1}^{n} (e_i'(t) \cdot e_j(t)) e_j(t),$$

and since $e_i(t) \cdot e_j(t) = \delta_{ij}$, by differentiating, if we write $\omega_{ij}(t) = e_i'(t) \cdot e_j(t)$, we get

$$\omega_{ji}(t) = -\omega_{ij}(t).$$

Now if $(e_1(t), \ldots, e_n(t))$ is the distinguished Frenet frame, by construction, $e_i(t)$ is a linear combination of $f^{(1)}(t), \ldots, f^{(i)}(t)$, and so $e_i'(t)$ is a linear combination of $f^{(2)}(t), \ldots, f^{(i+1)}(t)$, hence of $(e_1(t), \ldots, e_{i+1}(t))$. $\square$

In matrix form, when $(e_1(t), \ldots, e_n(t))$ is the distinguished Frenet frame, the row vector $(e_1'(t), \ldots, e_n'(t))$ can be expressed in terms of the row vector $(e_1(t), \ldots, e_n(t))$ via a skew-symmetric matrix ω, as shown below:

$$(e_1'(t), \ldots, e_n'(t)) = -(e_1(t), \ldots, e_n(t))\omega(t),$$

where

$$\omega = \begin{pmatrix} 0 & \omega_{12} & & & \\ -\omega_{12} & 0 & \omega_{23} & & \\ & -\omega_{23} & 0 & \ddots & \\ & & \ddots & \ddots & \omega_{n-1\,n} \\ & & & -\omega_{n-1\,n} & 0 \end{pmatrix}.$$

The next lemma shows the effect of a reparametrization and of a rigid motion.

Lemma 19.11. *Let $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon\ [a,b]\to\mathbb{E}^n$) be a curve of class C^p, with $p\geq n$, such that the derivatives $f^{(1)}(t),\ldots,f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t\in]a,b[$. Let $h\colon\ \mathbb{E}^n\to\mathbb{E}^n$ be a rigid motion, and assume that the corresponding linear isometry is R. Let $\widetilde{f}=h\circ f$. The following properties hold:*

(1) For any moving frame $(e_1(t),\ldots,e_n(t))$, the n-tuple $(\widetilde{e_1}(t),\ldots,\widetilde{e_n}(t))$, where $\widetilde{e_i}(t)=R(e_i(t))$, is a moving frame along $\widetilde{f}$, and we have

$$\widetilde{\omega_{ij}}(t)=\omega_{ij}(t)\quad \textit{and}\quad \|\widetilde{f}'(t)\|=\|f'(t)\|.$$

(2) For any orientation-preserving diffeormorphism $\rho\colon\ ]c,d[\to]a,b[$ (i.e., $\rho'(t)>0$ for all $t\in]c,d[$), if we write $\widetilde{f}=f\circ\rho$, then for any moving frame $(e_1(t),\ldots,e_n(t))$ on f, the n-tuple $(\widetilde{e_1}(t),\ldots,\widetilde{e_n}(t))$, where $\widetilde{e_i}(t)=e_i(\rho(t))$, is a moving frame on $\widetilde{f}$. Furthermore, if $\|\widetilde{f}'(t)\|\neq 0$, then

$$\frac{\widetilde{\omega_{ij}}(t)}{\|\widetilde{f}'(t)\|}=\frac{\omega_{ij}(\rho(t))}{\|f'(\rho(t))\|}.$$

The proof is straightforward and is omitted.

The above lemma suggests the definition of the curvatures $\kappa_1,\ldots,\kappa_{n-1}$.

Definition 19.17. Let $f\colon\ ]a,b[\to\mathbb{E}^n$ (or $f\colon\ [a,b]\to\mathbb{E}^n$) be a curve of class C^p, with $p\geq n$, such that the derivatives $f^{(1)}(t),\ldots,f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t\in]a,b[$. If $(e_1(t),\ldots,e_n(t))$ is the distinguished Frenet frame associated with f, we define the *ith curvature κ_i of f* by

$$\kappa_i(t)=\frac{\omega_{i\,i+1}(t)}{\|f'(t)\|},$$

with $1\leq i\leq n-1$.

Observe that the matrix $\omega(t)$ can be written as

$$\omega(t)=\|f'(t)\|\kappa(t),$$

where

$$\kappa = \begin{pmatrix} 0 & \kappa_{1\,2} & & & \\ -\kappa_{1\,2} & 0 & \kappa_{2\,3} & & \\ & -\kappa_{2\,3} & 0 & \ddots & \\ & & \ddots & \ddots & \kappa_{n-1\,n} \\ & & & -\kappa_{n-1\,n} & 0 \end{pmatrix}.$$

The matrix κ is sometimes called the *Cartan matrix*.

Lemma 19.12. *Let $f\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon [a,b] \to \mathbb{E}^n$) be a curve of class C^p, with $p \geq n$, such that the derivatives $f^{(1)}(t),\ldots,f^{(n-1)}(t)$ of $f(t)$ are linearly independent for all $t \in]a,b[$. Then for every i, with $1 \leq i \leq n-2$, we have $\kappa_i(t) > 0$.*

Proof. Lemma 19.9 shows that $e_1,\ldots,e_{n-1}$ are expressed in terms of $f^{(1)},\ldots,f^{(n-1)}$ by a triangular matrix (a_{ij}) whose diagonal entries a_{ii} are strictly positive, i.e., we have

$$e_i = \sum_{j=1}^{i} a_{ij} f^{(j)},$$

for $i = 1,\ldots,n-1$, and thus

$$f^{(i)} = \sum_{j=1}^{i} b_{ij} e_j,$$

for $i = 1,\ldots,n-1$, with $b_{ii} = a_{ii}^{-1} > 0$. Then, since $e_{i+1} \cdot f^{(j)} = 0$ for $j \leq i$, we get

$$\|f'\|\kappa_i = \omega_{i\,i+1} = e_i' \cdot e_{i+1} = a_{ii} f^{(i+1)} \cdot e_{i+1} = a_{ii} b_{i+1\,i+1},$$

and since $a_{ii} b_{i+1\,i+1} > 0$, we get $\kappa_i > 0$ ($i = 1,\ldots,n-2$). $\square$

Our previous reasoning in the 3D case is immediately extended to show that the limit of the intersection of the normal hyperplane at $t+\delta$ with the normal hyperplane at t (for δ small) with the osculating plane is a point C such that $C - f(t) = (1/\kappa_1)e_1$. Thus, we obtain a geometric interpretation for the curvature κ_1, and it is also possible to obtain an interpretation for the other κ_i.

We conclude by exploring to what extent the curvatures $\kappa_1,\ldots,\kappa_{n-1}$ determine a curve satisfying the nondegeneracy conditions of Lemma 19.9. Basically, such curves are defined up to a rigid motion.

Lemma 19.13. *Let $f\colon\ ]a,b[\to \mathbb{E}^n$ and $\widetilde{f}\colon\ ]a,b[\to \mathbb{E}^n$ (or $f\colon [a,b] \to \mathbb{E}^n$ and $\widetilde{f}\colon [a,b] \to \mathbb{E}^n$) be two curves of class C^p, with $p \geq n$, and satisfying the nondegeneracy conditions of Lemma 19.9. Denote the distinguished Frenet frames associated with f and $\widetilde{f}$ by $(e_1(t),\ldots,e_n(t))$ and $(\widetilde{e_1}(t),\ldots,\widetilde{e_n}(t))$. If $\kappa_i(t) = \widetilde{\kappa}_i(t)$ for every i, with $1 \leq i \leq n-1$, and $\|f'(t)\| = \|\widetilde{f}'(t)\|$ for all $t \in]a,b[$, then there is a unique rigid motion h such that*

$$\widetilde{f} = h \circ f.$$

Proof. Fix $t_0 \in]a,b[$. First of all, there is a unique rigid motion h such that

$$h(f(t_0)) = \widetilde{f}(t_0) \quad \text{and} \quad R(e_i(t_0)) = \widetilde{e_i}(t_0),$$

for all i, with $1 \leq i \leq n$, where R is the linear isometry associated with h (in fact, a rotation). Consider the curve $\overline{f} = h \circ f$. The hypotheses of the lemma and Lemma 19.11 imply that

$$\overline{\omega_{ij}}(t) = \widetilde{\omega_{ij}}(t) = \omega_{ij}(t), \quad \|\overline{f}'(t)\| = \|\widetilde{f}'(t)\| = \|f'(t)\|,$$

and, by construction, $(\overline{e_1}(t_0), \ldots, \overline{e_n}(t_0)) = (\widetilde{e_1}(t_0), \ldots, \widetilde{e_n}(t_0))$ and $\overline{f}(t_0) = \widetilde{f}(t_0)$. Let

$$\delta(t) = \sum_{i=1}^{n} (\overline{e_i}(t) - \widetilde{e_i}(t)) \cdot (\overline{e_i}(t) - \widetilde{e_i}(t)).$$

Then we have

$$\begin{aligned}
\delta'(t) &= 2\sum_{i=1}^{n} (\overline{e_i}(t) - \widetilde{e_i}(t)) \cdot (\overline{e_i}'(t) - \widetilde{e_i}'(t)) \\
&= -2\sum_{i=1}^{n} (\overline{e_i}(t) \cdot \widetilde{e_i}'(t) + \widetilde{e_i}(t) \cdot \overline{e_i}'(t)).
\end{aligned}$$

Using the Frenet equations, we get

$$\begin{aligned}
\delta'(t) &= -2\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_{ij}\overline{e_i} \cdot \widetilde{e_j} - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_{ij}\overline{e_j} \cdot \widetilde{e_i} \\
&= -2\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_{ij}\overline{e_i} \cdot \widetilde{e_j} - 2\sum_{j=1}^{n}\sum_{i=1}^{n} \omega_{ji}\overline{e_i} \cdot \widetilde{e_j} \\
&= -2\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_{ij}\overline{e_i} \cdot \widetilde{e_j} + 2\sum_{j=1}^{n}\sum_{i=1}^{n} \omega_{ij}\overline{e_i} \cdot \widetilde{e_j} \\
&= 0,
\end{aligned}$$

since ω is skew-symmetric. Thus, $\delta(t)$ is constant, and since the Frenet frames at t_0 agree, we get $\delta(t) = 0$. Then $\overline{e_i}(t) = \widetilde{e_i}(t)$ for all i, and since $\|\overline{f}'(t)\| = \|\widetilde{f}'(t)\|$, we have

$$\overline{f}'(t) = \|\overline{f}'(t)\|\overline{e_1}(t) = \|\widetilde{f}'(t)\|\widetilde{e_1}(t) = \widetilde{f}'(t),$$

so that $\overline{f}(t) - \widetilde{f}(t)$ is constant. However, $\overline{f}(t_0) = \widetilde{f}(t_0)$, and so $\overline{f}(t) = \widetilde{f}(t)$ and $\widetilde{f} = \overline{f} = h \circ f$. $\square$

Finally, the lemma below settles the issue of the existence of a curve with prescribed curvature functions.

Lemma 19.14. *Let $\kappa_1, \ldots, \kappa_{n-1}$ be functions defined on some open $]a,b[$ containing 0 with κ_i C^{n-i-1}-continuous for $i = 1, \ldots, n-1$, and with $\kappa_i(t) > 0$ for $i = 1, \ldots, n-2$ and all $t \in]a,b[$. Then there is curve $f\colon\ ]a,b[\to \mathbb{E}^n$ of class C^p, with $p \geq n$,* *satisfying the nondegeneracy conditions of Lemma 19.9 such that* $\|f'(t)\| = 1$ *and* f *has the* $n-1$ *curvatures* $\kappa_1(t), \ldots, \kappa_{n-1}(t)$.

Proof. Let $X(t)$ be the matrix whose columns consist of the vectors $e_1(t), \ldots, e_n(t)$ of the Frenet frame along f. Consider the system of ODEs,

$$X'(t) = -X(t)\kappa(t),$$

with initial conditions $X(0) = I$, where $\kappa(t)$ is the skew-symmetric matrix of curvatures. By a standard result in ODEs, there is a unique solution $X(t)$.

We claim that $X(t)$ is an orthogonal matrix. For this, note that

$$\begin{aligned}(XX^\top)' &= X'X^\top + X(X^\top)' = -X\kappa X^\top - X\kappa^\top X^\top \\ &= -X\kappa X^\top + X\kappa X^\top = 0.\end{aligned}$$

Since $X(0) = I$, we get $XX^\top = I$. If $F(t)$ is the first column of $X(t)$, we define the curve f by

$$f(s) = \int_0^s F(t)dt,$$

with $s \in]a,b[$. It is easily checked that f is a curve parametrized by arc length, with Frenet frame $X(s)$, and with curvatures κ_is. $\square$

19.11 Applications

Many engineering problems can be reduced to finding curves having some desired properties. This is certainly true of mechanical engineering and robotics, where various trajectories must be computed, and of computer graphics and medical imaging, where contours of shapes, for instance organs, are modeled as curves. In most practical applications it is necessary to consider curves composed of various segments. The problem then arises to join these segments as smoothly as possible, without restricting too much the number of degrees of freedom required for the design. Various kinds of *splines* were invented to solve this problem. If the curve segments are defined parametrically in terms of polynomials, a simple way to achieve continuity is to enforce the agreement of enough derivatives at junction points. This leads to *parametric* C^n*-continuity* and to *B-splines*. The theory of *B*-splines is quite extensive. Among the many references, we recommend Farin [10, 9], Hoschek and Lasser [14], Bartels, Beatty, and Barsky [1], Fiorot and Jeannin [11, 12], Piegl and Tiller [17], or Gallier [13].

Because parametric continuity is easy to formulate, piecewise curves based on parametric continuity are popular. Additionally, there are occasions in which parametric continuity is required. For example, if a spline is used to represent the trajectory of an object, parametric continuity guarantees that the object moves smoothly at the junction between two curve segments. However, there are applications for which parametric continuity is too constraining, since it depends on details of the parametrization that are not relevant to the shape of the curve. For example, if a curve is used to represent the boundary of an object, then only the outline of the curve is important. Thus, more flexible continuity conditions (usually called *geometric continuity*) based only on the geometry of the curve have been investigated. For plane curves, one may consider tangent continuity, or curvature continuity. For space curves, one may consider tangent continuity, curvature continuity, or torsion continuity. One may also want to consider higher-order continuity of the curvature κ and of the torsion τ, which means considering the continuity of higher derivatives of κ and τ. Another notion is *geometric continuity*, or *G^n-continuity*. Roughly speaking, two curves join with G^n-continuity if there is a reparametrization (a diffeomorphism) after which the curves join with parametric C^n-continuity. As a consequence, geometric continuity may be defined using the chain rule, in terms of a certain *connection matrix*. Yet another notion is *Frenet frame continuity*. Again, there is a vast literature on these topics, and we refer the readers to Farin [10, 9], Hoschek and Lasser [14], Bartels, Beatty, and Barsky [1], and Piegl and Tiller [17].

Complex shapes are usually represented in a piecewise fashion, composed of primitive elements smoothly joined. Traditional methods focus on achieving a specific level of interelement continuity, but the resulting shapes often possess bulges and undulations, and thus are of poor quality. They lack *fairness*. Fairness refers to the quality of regularity of the curvature (and torsion, for a space curve) of a curve. For a curve to be fair, it is required that the curvature vary gradually and oscillate as little as possible. Furthermore, the maximum rate of change of curvature should be minimized. This suggests several approaches.

- Minimal energy curve (which bends as little as possible): Minimize

$$\int_C \kappa^2 ds$$

  where κ is the curvature.
- Minimal variation curve (which bends as smoothly as possible): Minimize

$$\int_C \left(\frac{d(\kappa\mathbf{n})}{ds}\right)^2 ds$$

  where κ is the curvature and $\mathbf{n}$ is the principal normal.

Another possibility is to minimize

$$\int_C \left[\left(\frac{d\kappa}{ds}\right)^2 + \left(\frac{d\tau}{ds}\right)^2\right] ds$$

where κ is the curvature and τ is the torsion.

These problems may be cast as constrained optimization problems. Interelement continuity is solved by incorporating a penalty function. Interested readers are referred to the Ph.D. dissertations of Moreton [16] and Welch [18] for more details.

It should also be mentioned that it is possible to define a notion of affine normal and a notion of affine curvature without appealing to the concept of an inner product. For some interesting applications, see Calabi, Olver, and Tannenbaum [4] and Calabi, Olver, Shakiban, Tannenbaum, and Haker [3].

19.12 Problems

19.1. Plot the curve f defined by

$$f(t) = \begin{cases} (-\mathrm{e}^{1/t}, \mathrm{e}^{1/t}\sin(\mathrm{e}^{-1/t})) & \text{if } t < 0; \\ (0,0) & \text{if } t = 0; \\ (\mathrm{e}^{-1/t}, \mathrm{e}^{-1/t}\sin(\mathrm{e}^{1/t})) & \text{if } t > 0. \end{cases}$$

Verify that $f'(0) = 0$ and that the curve oscillates around the origin.

19.2. Plot the curve f defined by

$$f(t) = \begin{cases} (t, t^2\sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0. \end{cases}$$

Show that $f'(0) = (1,0)$ and that $f'(t) = (1, 2t\sin(1/t) - \cos(1/t))$ for $t \neq 0$. Verify that f' is discontinuous at 0.

19.3. Let $f\colon\ ]a,b[\to \mathcal{E}$ be and open curve of class C^∞. For some $t \in]a,b[$, assume that $f'(t) = 0$, but also that there exist some integers p,q with $1 \leq p < q$ such that $f^{(p)}(t)$ is the first derivative not equal to 0 and $f^{(q)}(t)$ is the first derivative not equal to 0 and not collinear to $f^{(p)}(t)$. Show that by Taylor's formula, for $h > 0$ small enough, we have

$$\begin{aligned} f(t+h) - f(t) = \left(\frac{h^p}{p!} + \lambda_{p+1}\frac{h^{p+1}}{(p+1)!} + \cdots + \lambda_{q-1}\frac{h^{q-1}}{(q-1)!}\right) f^{(p)}(t) \\ + \frac{h^q}{q!} f^{(q)}(t) + \frac{h^q}{q!}\varepsilon(h), \end{aligned}$$

where $\lim_{h\to 0, h\neq 0} \varepsilon(h) = 0$.

As a consequence, the curve is tangent to the line of direction $f^{(p)}(t)$ passing through $f(t)$. Show that the curve has the following appearance locally at t:

1. p is odd. The curve traverses every secant through $f(t)$.

1a. q is even. Locally, the curve is entirely on the same side of its tangent at $f(t)$. This looks like an ordinary point.

1b. q is odd. Locally, the curve has an *inflection point* at $f(t)$, i.e., the two arcs of the curve meeting at $f(t)$ are on different sides of the tangent.

2. p is even. The curve does not traverse any secant through $f(t)$. It has a *cusp*.

2a. q is even. In this case, the two arcs of the curve meeting at $f(t)$ are on the same side of the tangent. We say that we have a *cusp of the second kind*.
2b. q is odd. In this case, the two arcs of the curve meeting at $f(t)$ are on different sides of the tangent. We say that we have a *cusp of the first kind*.

Draw examples for $(p=1, q=2)$, $(p=1, q=3)$, $(p=2, q=3)$, and $(p=2, q=4)$.

19.4. Draw the curve defined such that

$$\begin{aligned} x(t) &= \frac{2t^2}{1+t^2}, \\ y(t) &= \frac{2t^3}{1+t^2}. \end{aligned}$$

Show that the point $(0,0)$ is a cusp and that the line of equation $x = 2$ is an asymptote. This curve is called the *cissoid of Diocles*.

19.5. (a) Draw the curve defined such that

$$\begin{aligned} x(t) &= \sin t, \\ y(t) &= \cos t + \log \tan \frac{t}{2}. \end{aligned}$$

Show that the point $(1,0)$ is a cusp and that the line of equation $x = 0$ is an asymptote.

(b) Show that the length of the segment of the tangent of the curve between the point of contact and the y-axis is of constant length 1. For this reason, this curve is called a *tractrix*.

19.6. (a) Given a tractrix specified by

$$\begin{aligned} x(t) &= a \sin t, \\ y(t) &= a \cos t + a \log \tan \frac{t}{2}, \end{aligned}$$

show that the curvature is given by $\kappa = |\tan t|$.

(b) Show that the center of curvature is on the curve

$$\begin{aligned} x(t) &= \frac{a}{\sin t}, \\ y(t) &= a \log \tan \frac{t}{2}. \end{aligned}$$

Show that this curve has the implicit equation

$$x = a \cosh\left(\frac{y}{a}\right).$$

Draw this curve, called a *catenary*.

Note. Recall that the *hyperbolic functions* cosh and sinh are defined by

$$\cosh u = \frac{e^u + e^{-u}}{2} \quad \text{and} \quad \sinh u = \frac{e^u - e^{-u}}{2}.$$

19.7. (a) Draw the curve f defined such that

$$x(t) = a e^{-bt} \cos t,$$
$$y(t) = a e^{-bt} \sin t,$$

where $a, b > 0$.

Show that the curve approaches the origin $(0,0)$ as $t \to +\infty$, spiraling around it. This curve is called a *logarithmic spiral*.

(b) Show that $f'(t) \to (0,0)$ as $t \to +\infty$, and that

$$\lim_{t \to +\infty} \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

is finite. Conclude that f has finite arc length in $[t_0, \infty[$.

19.8. (A square-filling curve due to Hilbert) This version of the Hilbert curve is defined in terms of four maps f_1, f_2, f_3, f_4 defined by

$$\begin{aligned} x' &= \frac{1}{2}x - \frac{1}{2}, & y' &= \frac{1}{2}y + 1, \\ x' &= \frac{1}{2}x + \frac{1}{2}, & y' &= \frac{1}{2}y + 1, \\ x' &= -\frac{1}{2}y + 1, & y' &= \frac{1}{2}x + \frac{1}{2}, \\ x' &= \frac{1}{2}y - 1, & y' &= -\frac{1}{2}x + \frac{1}{2}. \end{aligned}$$

(a) Prove that these maps are affine. Can you describe geometrically what their action is (rotation, translation, scaling?)

(b) Given any polygonal line L, define the following sequence of poygonal lines:

$$S_0 = L,$$
$$S_{n+1} = f_1(S_n) \cup f_2(S_n) \cup f_3(S_n) \cup f_4(S_n).$$

Construct S_1 starting from the polygonal line $L = ((-1,0), (0,1)), ((0,1), (1,0))$.

Can you figure out what S_n looks like in general? (you may want to write a computer program, and iterate at least 6 times).

(c) Prove that S_n has a limit that is a continuous curve not C^1 anywhere and that is space–filling, in the sense that its image is the entire unit square.

19.9. Consider the curve f over $[0,1]$ defined such that

$$f(t) = \begin{cases} (t, t\sin(\pi/t)) & \text{if } t \neq 0, \\ (0,0) & \text{if } t = 0. \end{cases}$$

Show geometrically that the arc length of the portion of curve corresponding to the interval $[1/(n+1),\, 1/n]$ is at least $1/\left(n+\frac{1}{2}\right)$. Use this to show that the length of the curve in the interval $[1/N,\, 1]$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$. Conclude that this curve is not rectifiable.

19.10. Consider a polynomial curve of degree m defined by the control points $(b_0,\ldots,b_m)$ over $[0,1]$. Prove that the curvature at b_0 is

$$\kappa(0) = \frac{m-1}{m}\,\frac{\|\overrightarrow{b_0b_1} \times \overrightarrow{b_1b_2}\|}{\|\overrightarrow{b_0b_1}\|^3},$$

and that the curvature at b_m is given by

$$\kappa(1) = \frac{m-1}{m}\,\frac{\|\overrightarrow{b_{m-1}b_m} \times \overrightarrow{b_{m-2}b_{m-1}}\|}{\|\overrightarrow{b_{m-1}b_m}\|^3}.$$

Show that the torsion at b_0 is given by

$$\tau(0) = -\frac{m-2}{m}\,\frac{(\overrightarrow{b_0b_1},\, \overrightarrow{b_0b_2},\, \overrightarrow{b_0b_3})}{\|\overrightarrow{b_0b_1} \times \overrightarrow{b_1b_2}\|^2}.$$

If $a = \|\overrightarrow{b_0b_1}\|$ and h is the distance from b_2 to the line (b_0, b_1), show that

$$\kappa(0) = \frac{m-1}{m}\,\frac{h}{a^2}.$$

If c is the distance from b_3 to the plane spanned by (b_0, b_1, b_2) (the osculating plane), show that

$$|\tau(0)| = \frac{m-2}{m}\,\frac{c}{ah}.$$

19.11. Consider the curve defined such that

$$f(t) = \begin{cases} (t, t^2 + t^3\sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0. \end{cases}$$

Show that the osculating circle for $t = 0$ is the circle of center $\left(0, \frac{1}{2}\right)$ and that $f''(0)$ is undefined, so that the center of curvature is undefined at $t = 0$.

19.12. Show that the solution of the system

$$\begin{aligned} u'x + v'y &= uu' + vv', \\ u''x + v''y &= uu'' + vv'' + u'^2 + v'^2, \end{aligned}$$

is given by

$$
\begin{aligned}
x &= u - \frac{v'(u'^2 + v'^2)}{u'v'' - v'u''},\\
y &= v + \frac{u'(u'^2 + v'^2)}{u'v'' - v'u''},
\end{aligned}
$$

provided that $u'v'' - v'u'' \neq 0$. Show that the radius of curvature is given by

$$
\mathscr{R} = \frac{(u'^2 + v'^2)^{3/2}}{|u'v'' - v'u''|}.
$$

19.13. (a) Given an ellipse

$$
\begin{aligned}
x &= a\cos\theta,\\
y &= b\sin\theta,
\end{aligned}
$$

show that the radius of curvature is given by

$$
\mathscr{R} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab},
$$

and that the center of curvature is on the curve defined by

$$
\begin{aligned}
x &= \frac{c^2}{a}\cos^3\theta,\\
y &= -\frac{c^2}{b}\sin^3\theta.
\end{aligned}
$$

This curve is called an *astroid*.

(b) Letting $N = \left(\frac{c^2}{a}\cos\theta, 0\right)$ be the intersection of the normal to the point M on the ellipse with Ox, and $d = \|MN\|$ be the distance between M and N, show that the radius of curvature is given by

$$
\mathscr{R} = \frac{a^2}{b^4}d^3.
$$

19.14. Given a parabola of equation $y^2 = 2px$, compute the radius of curvature and show that the center of curvature is on the curve of equation

$$
y^2 = \frac{8}{27p}(x-p)^3.
$$

Show that this is a cuspidal cubic with a cusp at $(p, 0)$.

19.15. Given a hyperbola

$$
\begin{aligned}
x &= a\cosh\theta,\\
y &= b\sinh\theta,
\end{aligned}
$$

compute the radius of curvature and show that the center of curvature is on the curve defined by

$$x = \frac{c^2}{a}\cosh^3\theta,$$
$$y = -\frac{c^2}{b}\sinh^3\theta.$$

Note. The function cosh and sinh are defined in Problem 19.6.

19.16. Given a logarithmic spiral specified by

$$x = a\,\mathrm{e}^{m\theta}\cos\theta,$$
$$y = a\,\mathrm{e}^{m\theta}\sin\theta,$$

where $a > 0$, show that the radius of curvature is

$$\mathscr{R} = a\sqrt{1+m^2}\,\mathrm{e}^{m\theta},$$

and that the center of curvature is on the spiral defined by

$$x = -ma\,\mathrm{e}^{m\theta}\sin\theta,$$
$$y = ma\,\mathrm{e}^{m\theta}\cos\theta.$$

Show that this is the original spiral

19.17. Given a cardioid

$$x = a(1+\cos\theta)\cos\theta,$$
$$y = a(1+\cos\theta)\sin\theta,$$

show that the radius of curvature is

$$\mathscr{R} = \left|\frac{2a}{3}\cos(\theta/2)\right|,$$

and that the center of curvature is on the cardioid defined by

$$x = \frac{2a}{3} + \frac{a}{3}(1-\cos\theta)\cos\theta,$$
$$y = \frac{a}{3}(1-\cos\theta)\sin\theta.$$

19.18. A plane curve is defined in *polar coordinates* if

$$x = \rho(\theta)\cos\theta,$$
$$y = \rho(\theta)\sin\theta,$$

for some function ρ of the polar angle θ.

(a) Prove that the element of arc length is given by

$$ds = \sqrt{\rho^2 + (\rho')^2}\, d\theta.$$

(b) Prove that the curvature is given by

$$\kappa = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{[(\rho')^2 + \rho^2]^{3/2}}.$$

19.19. Give an example of a regular nonplanar curve such that $\tau = 0$.

19.20. A circular helix is defined by

$$f(t) = (a\cos t,\, a\sin t,\, kt).$$

Show that the curvature is given by

$$\kappa = \frac{a}{a^2 + k^2}$$

and that the torsion is given by

$$\tau = -\frac{k}{a^2 + k^2}.$$

19.21. If C is a regular plane curve parametrized by arc length, let $C'(s) = \mathbf{t}$ be the tangent vector at s, and write

$$\mathbf{t} = \cos\varphi\mathbf{i} + \sin\varphi\mathbf{j},$$

where $(\mathbf{i}, \mathbf{j})$ is an orthonormal basis.

(a) Show that the algebraic curvature $k(s)$ is given by

$$k = \frac{d\varphi}{ds}.$$

(b) Letting

$$C(s) = x(s)\mathbf{i} + y(s)\mathbf{j},$$

we have $dx = \cos\varphi\, ds$ and $dy = \sin\varphi\, ds$. If $k(s) = f(s)$ for some C^0-function f, show that

$$\varphi = \int f(s)ds + \varphi_0$$

and thus that

$$x = \int \cos\varphi(s)\, ds + a,$$

$$y = \int \sin\varphi(s)\, ds + b,$$

for some constants φ_0, a, b.

Remark: Integrals of the above form are known as *Fresnel integrals*, and were first encountered by Fresnel (1788–1827) in the context of refraction problems.

(c) Study the curves defined such that $k = cs + d$, for some constants c, d (such curves are called *clothoids*, or *Cornu spirals*).

19.22. Write a computer program that takes as input the parametric equation (not necessarily arc length parametrized) of a curve. Your program will generate a graph of the curve and animate the Frenet frame, osculating circle, and osculating sphere, along the curve. Try your program on a C^2-continuous B-spline to observe discontinuities of the osculating sphere.

19.23. Given a circle C and a point O on C, consider the set of all lines Δ such that if $p \neq O$ is any point on C, the line Δ is the line passing through p and forming an angle with the normal N_p at p equal to the angle of N_p with pO (in other words, Δ is obtained by reflecting pO about the normal N_p at p). When $p = O$, the line Δ is the diameter through O. Prove that the lines Δ are tangent to a cardioid (see Problem 19.17).

Remark: The above problem can be viewed as a problem of optics. If a light source is placed at O, the reflections of the light rays emanating from O will have a cardioid as envelope. Such curves are also called *caustics*.

19.24. Using a recursion scheme in which $[0,1]$ is initially subdivided into four equal intervals and the square $[0,1] \times [0,1]$ is initially subdivided into four equal subsquares, give an analytic definition for the functions $h_n \colon [0,1] \to [0,1] \times [0,1]$ involved in defining the Hilbert curve (see Figure 19.1). Prove that the sequence h_n converges to a continuous function h. Prove that the h_n can be chosen to be injective but that h cannot be injective.

19.25. Two biregular curves f and g in $\mathbb{E}^3$ are called *Bertrand curves* if they have a common principal normal at any of their points.

(a) If f is a plane biregular curve, then prove that any involute of the locus of centers of curvatures of f is a Bertrand curve of f. Any two Bertrand curves are parallel, in the sense that the distance measured along the common principal normal, between corresponding points of the two Bertrand curves, is constant.

(b) If f^* and f are Bertrand curves, then f^* has an equation of the form

$$f^*(t) = f(t) + a(t)\mathbf{n},$$

where $\mathbf{n}$ is the principal normal to f at t. We will prove shortly that $a(t)$ must be a constant.

Assuming that f and f^* are Bertrand curves, using the fact that

$$f^*(t) = f(t) + a(t)\mathbf{n},$$

observe that

$$a^2(t) = (f^* - f) \cdot (f^* - f),$$

and prove that

$$\frac{d}{dt}(a^2) = 2(f^* - f) \cdot \left(\frac{d}{dt}(f^*) - \frac{d}{dt}(f) \right) = 0.$$

Conlude that $a(t)$ is constant.

Let $\mathbf{t}$ and $\mathbf{t}^*$ be the unit tangent vectors to f and f^*, respectively. Using the fact that

$$\frac{d}{dt}(\mathbf{t}^* \cdot \mathbf{t}) = \frac{d\mathbf{t}^*}{dt} \cdot \mathbf{t} + \mathbf{t}^* \cdot \frac{d\mathbf{t}}{dt},$$

prove that

$$\frac{d}{dt}(\mathbf{t}^* \cdot \mathbf{t}) = 0.$$

Let

$$\mathbf{t}^* \cdot \mathbf{t} = \cos\alpha,$$

a constant. Observe that α is the constant angle between the tangents at corresponding points of the Bertrand curves.

Now, assuming that f and f^* are both parametrized by arc lengths, s and s^*, respectively, we have

$$f^*(s) = f(s) + a(s)\mathbf{n}.$$

Prove that

$$\cos\alpha = \frac{ds}{ds^*}(1 - a\kappa).$$

Also prove that

$$\|\mathbf{t}^* \times \mathbf{t}\| = \left\| \frac{ds}{ds^*} a\tau\mathbf{n} \right\|.$$

Conclude that

$$a\tau\frac{ds}{ds^*} = \sin\alpha,$$

where the sign of α is suitably chosen. From

$$\frac{ds}{ds^*}(1 - a\kappa) = \cos\alpha \quad \text{and} \quad a\tau\frac{ds}{ds^*} = \sin\alpha,$$

prove that

$$\frac{1 - a\kappa}{a\tau} = \cot\alpha,$$

and thus, letting $c_1 = a$, $c_2 = a\cot\alpha$, that the linear equation

$$c_1\kappa + c_2\tau = 1$$

holds between κ and τ.

(c) Conversely, assume that that the linear equation

$$c_1\kappa + c_2\tau = 1$$

holds between κ and τ. We shall prove that f has the Bertrand curve

$$f^*(s) = f(s) + c_1\mathbf{n}.$$

Prove that

$$\frac{df^*}{ds} = (1 - c_1\kappa)\mathbf{t} + c_1\tau\mathbf{b}.$$

In view of the equation

$$c_1\kappa + c_2\tau = 1,$$

letting $c = c_2/c_1$, prove that

$$\frac{df^*}{ds} = c_1\tau(c\mathbf{t} + \mathbf{b}).$$

Conclude that the unit tangent vector to C^* is

$$\mathbf{t}^* = \frac{c\mathbf{t} + \mathbf{b}}{\sqrt{1 + c^2}},$$

that

$$\frac{d\mathbf{t}^*}{ds} = \frac{1}{\sqrt{1 + c^2}}\,(c\kappa - \tau)\mathbf{n},$$

and that C and C^* are Bertrand curves.

Thus, we have proved that a curve C has a Bertrand curve iff a linear equation

$$c_1\kappa + c_2\tau = 1$$

holds between κ and τ (Bertrand, 1850).

Extra Credit: Prove that a circular helix is the only nonplanar biregular curve having more than one Bertrand curve.

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