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Appendix

17.1 Hyperplanes and Linear Forms

Given a vector space $E$ over a field $K$, a linear map $f: E \to K$ is called a linear form. The set of all linear forms $f: E \to K$ is a vector space called the dual space of $E$ and denoted by $E^*$. We now prove that hyperplanes are precisely the Kernels of nonnull linear forms.

Lemma 17.1.1 Let $E$ be a vector space. The following properties hold:

(a) Given any nonnull linear form $f \in E^*$, its kernel $H = \text{Ker } f$ is a hyperplane.

(b) For any hyperplane $H$ in $E$, there is a (nonnull) linear form $f \in E^*$ such that $H = \text{Ker } f$.

(c) Given any hyperplane $H$ in $E$ and any (nonnull) linear form $f \in E^*$ such that $H = \text{Ker } f$, for every linear form $g \in E^*$, $H = \text{Ker } g$ iff $g = \lambda f$ for some $\lambda \neq 0$ in $K$.

Proof. (a) If $f \in E^*$ is nonnull, there is some vector $v_0 \in E$ such that $f(v_0) \neq 0$. Let $H = \text{Ker } f$. For every $v \in E$, we have

$$f(v - \frac{f(v)}{f(v_0)}v_0) = f(v) - \frac{f(v)}{f(v_0)}f(v_0) = f(v) - f(v) = 0.$$ 

Thus,

$$v - \frac{f(v)}{f(v_0)}v_0 = h \in H$$
and
\[ v = h + \frac{f(v)}{f(v_0)}v_0, \]
that is, \( E = H + K v_0 \). Also, since \( f(v_0) \neq 0 \), we have \( v_0 \notin H \), that is, \( H \cap K v_0 = \emptyset \). Thus, \( E = H \oplus K v_0 \), and \( H \) is a hyperplane.

(b) If \( H \) is a hyperplane, \( E = H \oplus K v_0 \) for some \( v_0 \notin H \). Then every \( v \in E \) can be written in a unique way as \( v = h + \lambda v_0 \). Thus there is a well-defined function \( f: E \to K \) such that \( f(v) = \lambda \) for every \( v = h + \lambda v_0 \). We leave as a simple exercise the verification that \( f \) is a linear form. Since \( f(v_0) = 1 \), the linear form \( f \) is nonnull. Also, by definition it is clear that \( \lambda = 0 \) iff \( v \in H \), that is, \( \text{Ker } f = H \).

(c) Let \( H \) be a hyperplane in \( E \), and let \( f \in E^* \) be any (nonnull) linear form such that \( H = \text{Ker } f \). Clearly, if \( g = \lambda f \) for some \( \lambda \neq 0 \), then \( H = \text{Ker } g \). Conversely, assume that \( H = \text{Ker } g \) for some nonnull linear form \( g \). From (a) we have \( E = H \oplus K v_0 \), for some \( v_0 \) such that \( f(v_0) \neq 0 \) and \( g(v_0) \neq 0 \). Then observe that
\[
g - \frac{g(v_0)}{f(v_0)}f
\]
is a linear form that vanishes on \( H \), since both \( f \) and \( g \) vanish on \( H \), but also vanishes on \( K v_0 \). Thus, \( g = \lambda f \), with
\[
\lambda = \frac{g(v_0)}{f(v_0)}.
\]

If \( E \) is a vector space of finite dimension \( n \) and \( (u_1, \ldots, u_n) \) is a basis of \( E \), for any linear form \( f \in E^* \) and every \( x = x_1 u_1 + \cdots + x_n u_n \in E \), we have
\[
f(x) = \lambda_1 x_1 + \cdots + \lambda_n x_n,
\]
where \( \lambda_i = f(u_i) \in K \), for every \( i, 1 \leq i \leq n \). Thus, with respect to the basis \( (u_1, \ldots, u_n) \), \( f(x) \) is a linear combination of the coordinates of \( x \), as expected.

### 17.2 Metric Spaces and Normed Vector Spaces

Thorough expositions of the material of this section can be found in Lang [109, 110] and Dixmier [50]. We begin with metric spaces. Recall that \( \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \} \).

**Definition 17.2.1** A metric space is a set \( E \) together with a function \( d: E \times E \to \mathbb{R}_+ \), called a metric, or distance, assigning a nonnegative real
number $d(x, y)$ to any two points $x, y \in E$ and satisfying the following conditions for all $x, y, z \in E$:

(D1) $d(x, y) = d(y, x)$. \hspace{1cm} (symmetry)

(D2) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$. \hspace{1cm} (positivity)

(D3) $d(x, z) \leq d(x, y) + d(y, z)$. \hspace{1cm} (triangle inequality)

Geometrically, condition (D3) expresses the fact that in a triangle with vertices $x, y, z$, the length of any side is bounded by the sum of the lengths of the other two sides. From (D3), we immediately get

$$|d(x, y) - d(y, z)| \leq d(x, z).$$

Let us give some examples of metric spaces. Recall that the absolute value $|x|$ of a real number $x \in \mathbb{R}$ is defined such that $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$, and for a complex number $x = a + ib$, as $|x| = \sqrt{a^2 + b^2}$.

**Example 17.1** Let $E = \mathbb{R}$ and $d(x, y) = |x - y|$, the absolute value of $x - y$. This is the so-called natural metric on $\mathbb{R}$.

**Example 17.2** Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). We have the Euclidean metric

$$d_2(x, y) = (|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2)^{\frac{1}{2}},$$

the distance between the points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$.

**Example 17.3** For every set $E$ we can define the discrete metric, defined such that $d(x, y) = 1$ iff $x \neq y$, and $d(x, x) = 0$.

**Example 17.4** For any $a, b \in \mathbb{R}$ such that $a < b$, we define the following sets:

1. $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, \hspace{1cm} (closed interval)
2. $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$, \hspace{1cm} (interval closed on the left, open on the right)
3. $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$, \hspace{1cm} (interval open on the left, closed on the right)
4. $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$, \hspace{1cm} (open interval)

Let $E = [a, b]$, and $d(x, y) = |x - y|$. Then $([a, b], d)$ is a metric space.

We now consider a very important special case of metric spaces: Normed vector spaces.

**Definition 17.2.2** Let $E$ be a vector space over a field $K$, where $K$ is either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers. A norm on $E$ is a function $\|\| : E \to \mathbb{R}_+$ assigning a nonnegative real number $\|u\|$ to any vector $u \in E$ and satisfying the following conditions for all $x, y, z \in E$:
A vector space \( E \) together with a norm \( \| \cdot \| \) is called a \textit{normed vector space}.

Condition (N3) is also called the \textit{triangle inequality}, and it is illustrated in Figure 17.1.

From (N3), we easily get
\[
\|\|x\|-\|y\|\| \leq \|x-y\|. 
\]

Given a normed vector space \( E \), if we define \( d \) such that
\[
d(x, y) = \|x-y\|, 
\]
it is easily seen that \( d \) is a metric. Thus, every normed vector space is immediately a metric space. Note that the metric associated with a norm is invariant under translation, that is,
\[
d(x+u, y+u) = d(x, y). 
\]

Let us give some examples of normed vector spaces.

**Example 17.5** Let \( E = \mathbb{R} \) and \( \|x\| = |x| \), the absolute value of \( x \). The associated metric is \( |x-y| \), as in Example 17.1.

**Example 17.6** Let \( E = \mathbb{R}^n \) (or \( E = \mathbb{C}^n \)). There are three standard norms. For every \((x_1, \ldots, x_n) \in E\), we have the norm \( \|x\|_1 \), defined such that
\[
\|x\|_1 = |x_1| + \cdots + |x_n|, 
\]
we have the Euclidean norm \( \|x\|_2 \), defined such that
\[
\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}, 
\]
and the \textit{sup}-norm \( \|x\|_\infty \), defined such that
\[
\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}. 
\]

For geometric applications, we will need to consider affine spaces \((E, \overline{E})\) where the associated space of translations \( \overline{E} \) is a vector space equipped with a norm.
Definition 17.2.3 Given an affine space \((E, \overrightarrow{E})\), where the space of translations \(\overrightarrow{E}\) is a vector space over \(\mathbb{R}\) or \(\mathbb{C}\), we say that \((E, \overrightarrow{E})\) is a \textit{normed affine space} if \(\overrightarrow{E}\) is a normed vector space with norm \(|\cdot|\).

Given a normed affine space, there is a natural metric on \(E\) itself, defined such that

\[
d(a, b) = |ab|.
\]

Observe that this metric is invariant under translation, that is,

\[
d(a + u, b + u) = d(a, b).
\]