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Appendix

17.1 Hyperplanes and Linear Forms

Given a vector space E over a field K, a linear map $f\colon E \to K$ is called a *linear form*. The set of all linear forms $f\colon E \to K$ is a vector space called the *dual space of* E and denoted by E^*. We now prove that hyperplanes are precisely the Kernels of nonnull linear forms.

Lemma 17.1.1 *Let E be a vector space. The following properties hold:*

(a) Given any nonnull linear form $f \in E^*$, its kernel $H = \mathrm{Ker}\, f$ is a hyperplane.

(b) For any hyperplane H in E, there is a (nonnull) linear form $f \in E^*$ such that $H = \mathrm{Ker}\, f$.

(c) Given any hyperplane H in E and any (nonnull) linear form $f \in E^*$ such that $H = \mathrm{Ker}\, f$, for every linear form $g \in E^*$, $H = \mathrm{Ker}\, g$ iff $g = \lambda f$ for some $\lambda \neq 0$ in K.

Proof. (a) If $f \in E^*$ is nonnull, there is some vector $v_0 \in E$ such that $f(v_0) \neq 0$. Let $H = \mathrm{Ker}\, f$. For every $v \in E$, we have

$$f(v - \frac{f(v)}{f(v_0)} v_0) = f(v) - \frac{f(v)}{f(v_0)} f(v_0) = f(v) - f(v) = 0.$$

Thus,

$$v - \frac{f(v)}{f(v_0)} v_0 = h \in H$$

and

$$v = h + \frac{f(v)}{f(v_0)} v_0,$$

that is, $E = H + Kv_0$. Also, since $f(v_0) \neq 0$, we have $v_0 \notin H$, that is, $H \cap Kv_0 = 0$. Thus, $E = H \oplus Kv_0$, and H is a hyperplane.

(b) If H is a hyperplane, $E = H \oplus Kv_0$ for some $v_0 \notin H$. Then every $v \in E$ can be written in a unique way as $v = h + \lambda v_0$. Thus there is a well–defined function $f: E \to K$ such that $f(v) = \lambda$ for every $v = h + \lambda v_0$. We leave as a simple exercise the verification that f is a linear form. Since $f(v_0) = 1$, the linear form f is nonnull. Also, by definition it is clear that $\lambda = 0$ iff $v \in H$, that is, $\operatorname{Ker} f = H$.

(c) Let H be a hyperplane in E, and let $f \in E^*$ be any (nonnull) linear form such that $H = \operatorname{Ker} f$. Clearly, if $g = \lambda f$ for some $\lambda \neq 0$, then $H = \operatorname{Ker} g$. Conversely, assume that $H = \operatorname{Ker} g$ for some nonnull linear form g. From (a) we have $E = H \oplus Kv_0$, for some v_0 such that $f(v_0) \neq 0$ and $g(v_0) \neq 0$. Then observe that

$$g - \frac{g(v_0)}{f(v_0)} f$$

is a linear form that vanishes on H, since both f and g vanish on H, but also vanishes on Kv_0. Thus, $g = \lambda f$, with

$$\lambda = \frac{g(v_0)}{f(v_0)}.$$

□

If E is a vector space of finite dimension n and $(u_1, \ldots, u_n)$ is a basis of E, for any linear form $f \in E^*$ and every $x = x_1 u_1 + \cdots + x_n u_n \in E$, we have

$$f(x) = \lambda_1 x_1 + \cdots + \lambda_n x_n,$$

where $\lambda_i = f(u_i) \in K$, for every i, $1 \leq i \leq n$. Thus, with respect to the basis $(u_1, \ldots, u_n)$, $f(x)$ is a linear combination of the coordinates of x, as expected.

17.2 Metric Spaces and Normed Vector Spaces

Thorough expositions of the material of this section can be found in Lang [109, 110] and Dixmier [50]. We begin with metric spaces. Recall that $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

Definition 17.2.1 A *metric space* is a set E together with a function $d: E \times E \to \mathbb{R}_+$, called a *metric, or distance*, assigning a nonnegative real number $d(x, y)$ to any two points $x, y \in E$ and satisfying the following conditions for all $x, y, z \in E$:

(D1) $d(x, y) = d(y, x)$. (symmetry)

(D2) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$. (positivity)

(D3) $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

Geometrically, condition (D3) expresses the fact that in a triangle with vertices x, y, z, the length of any side is bounded by the sum of the lengths of the other two sides. From (D3), we immediately get

$$|d(x, y) - d(y, z)| \leq d(x, z).$$

Let us give some examples of metric spaces. Recall that the *absolute value* $|x|$ of a real number $x \in \mathbb{R}$ is defined such that $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$, and for a complex number $x = a + ib$, as $|x| = \sqrt{a^2 + b^2}$.

Example 17.1 Let $E = \mathbb{R}$ and $d(x, y) = |x - y|$, the absolute value of $x - y$. This is the so-called natural metric on $\mathbb{R}$.

Example 17.2 Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). We have the Euclidean metric

$$d_2(x, y) = \left(|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2\right)^{\frac{1}{2}},$$

the distance between the points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$.

Example 17.3 For every set E we can define the *discrete metric*, defined such that $d(x, y) = 1$ iff $x \neq y$, and $d(x, x) = 0$.

Example 17.4 For any $a, b \in \mathbb{R}$ such that $a < b$, we define the following sets:

1. $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, (closed interval)
2. $[a, b[= \{x \in \mathbb{R} \mid a \leq x < b\}$, (interval closed on the left, open on the right)
3. $]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$, (interval open on the left, closed on the right)
4. $]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$, (open interval)

Let $E = [a, b]$, and $d(x, y) = |x - y|$. Then $([a, b], d)$ is a metric space.

We now consider a very important special case of metric spaces: Normed vector spaces.

Definition 17.2.2 Let E be a vector space over a field K, where K is either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers. A *norm on* E is a function $\| \, \| \colon E \to \mathbb{R}_+$ assigning a nonnegative real number $\|u\|$ to any vector $u \in E$ and satisfying the following conditions for all $x, y, z \in E$:

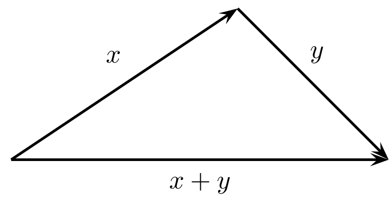

Figure 17.1. The triangle inequality

(N1) $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = 0$. (positivity)

(N2) $\|\lambda x\| = |\lambda| \, \|x\|$. (scaling)

(N3) $\|x + y\| \leq \|x\| + \|y\|$. (convexity inequality)

A vector space E together with a norm $\| \, \|$ is called a *normed vector space*.

Condition (N3) is also called the *triangle inequality*, and it is illustrated in Figure 17.1.

From (N3), we easily get

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Given a normed vector space E, if we define d such that

$$d(x, y) = \|x - y\|,$$

it is easily seen that d is a metric. Thus, every normed vector space is immediately a metric space. Note that the metric associated with a norm is invariant under translation, that is,

$$d(x + u, y + u) = d(x, y).$$

Let us give some examples of normed vector spaces.

Example 17.5 Let $E = \mathbb{R}$ and $\|x\| = |x|$, the absolute value of x. The associated metric is $|x - y|$, as in Example 17.1.

Example 17.6 Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). There are three standard norms. For every $(x_1, \ldots, x_n) \in E$, we have the norm $\|x\|_1$, defined such that

$$\|x\|_1 = |x_1| + \cdots + |x_n|,$$

we have the Euclidean norm $\|x\|_2$, defined such that

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}},$$

and the *sup*-norm $\|x\|_\infty$, defined such that

$$\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}.$$

For geometric applications, we will need to consider affine spaces $\left(E, \overrightarrow{E}\right)$ where the associated space of translations $\overrightarrow{E}$ is a vector space equipped with a norm.

Definition 17.2.3 Given an affine space $\left(E, \overrightarrow{E}\right)$, where the space of translations $\overrightarrow{E}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$, we say that $\left(E, \overrightarrow{E}\right)$ is a *normed affine space* if $\overrightarrow{E}$ is a normed vector space with norm $\| \; \|$.

Given a normed affine space, there is a natural metric on E itself, defined such that

$$d(a, b) = \|\mathbf{ab}\|.$$

Observe that this metric is invariant under translation, that is,

$$d(a + u, b + u) = d(a, b).$$