Preface

Many problems arising in engineering, and notably in computer science and mechanical engineering, require geometric tools and concepts. This is especially true of problems arising in computer graphics, geometric modeling, computer vision, and motion planning, just to mention some key areas. This book is an introduction to fundamental geometric concepts and tools needed for solving problems of a geometric nature with a computer. In a previous text, Gallier [70], we focused mostly on affine geometry and on its applications to the design and representation of polynomial curves and surfaces (and B-splines). The main goal of this book is to provide an introduction to more sophisticated geometric concepts needed in tackling engineering problems of a geometric nature. Many problems in the above areas require some nontrivial geometric knowledge, but in our opinion, books dealing with the relevant geometric material are either too theoretical, or else rather specialized. For example, there are beautiful texts entirely devoted to projective geometry, Euclidean geometry, and differential geometry, but reading each one represents a considerable effort (certainly from a nonmathematician!). Furthermore, these topics are usually treated for their own sake (and glory), with little attention paid to applications.

This book is an attempt to fill this gap. We present a coherent view of geometric methods applicable to many engineering problems at a level that can be understood by a senior undergraduate with a good math background. Thus, this book should be of interest to a wide audience including computer scientists (both students and professionals), mathematicians, and engineers interested in geometric methods (for example, mechanical engineers). In particular, we provide an introduction to affine geometry, projective geometry, Euclidean geometry, basics of differential geometry and Lie groups, and a glimpse of computational geometry (convex sets, Voronoi diagrams, and Delaunay triangulations). This material provides the foundations for the algorithmic treatment of curves and surfaces, some basic tools of geometric modeling. The right dose of projective geometry also leads to a rigorous and yet smooth presentation of rational curves and surfaces. However, to keep the size of this book reasonable, a number of topics could not be included. Nevertheless, they can be found in the additional material on the web site: see http://www.cis.upenn.edu/~jean/gbooks/geom2.html, abbreviated as web page. This is the case of the material on rational curves and surfaces.

This book consists of sixteen chapters and an appendix. The additional material on the web site consists of eight chapters and an appendix: see web page.

- The book starts with a brief introduction (Chapter 1).

- Chapter 2 provides an introduction to affine geometry. This ensures that readers are on firm ground to proceed with the rest of the book, in particular, projective geometry. This is also useful to establish the notation and terminology. Readers proficient in geometry may omit this section, or use it *as needed*. On the other hand, readers totally unfamiliar with this material will probably have a hard time with the rest of the book. These readers are advised do some extra reading in order to assimilate some basic knowledge of geometry. For example, we highly recommend Pedoe [136], Coxeter [35], Snapper and Troyer [160], Berger [12, 13], Fresnel [66], Samuel [146], Hilbert and Cohn–Vossen [84], Boehm and Prautzsch [17], and Tisseron [169].

- Basic properties of convex sets and convex hulls are discussed in Chapter 3. Three major theorems are proved: Carthéodory's theorem, Radon's theorem, and Helly's theorem.

- Chapter 4 presents a construction (the "hat construction") for embedding an affine space into a vector space. An important application of this construction is the projective completion of an affine space, presented in the next chapter. Other applications are treated in Chapter 20, which is on the web site, see web page.

- Chapter 5 provides an introduction to projective geometry. Since we are not writing a treatise on projective geometry, we cover only the most fundamental concepts, including projective spaces and subspaces, frames, projective maps, multiprojectve maps, the projective completion of an affine space, cross-ratios, duality, and the complexification of a real projective space. This material also provides the foundations for our algorithmic treatment of rational curves and surfaces, to be found on the web site (Chapters 18, 19,21, 22, 23, 24); see web page.

- Chapters 6, 7, and 8, provide an introduction to Euclidean geometry, to the groups of isometries $\mathbf{O}(n)$ and $\mathbf{SO}(n)$, the groups of affine rigid motions $\mathbf{Is}(n)$ and $\mathbf{SE}(n)$, and to the quaternions. Several versions of the Cartan–Dieudonné theorem are proved in Chapter 7. The QR-decomposition of matrices is explained geometrically, both in terms of the Gram–Schmidt procedure and in terms of Householder transformations. These chapters are crucial to a firm understanding of the differential geometry of curves and surfaces, and computational geometry.

- Chapter 9 gives a short introduction to some fundamental topics in computational geometry: Voronoi diagrams and Delaunay triangulations.

- Chapter 10 provides an introduction to Hermitian geometry, to the groups of isometries $\mathbf{U}(n)$ and $\mathbf{SU}(n)$, and the groups of affine rigid motions $\mathbf{Is}(n, \mathbb{C})$ and $\mathbf{SE}(n, \mathbb{C})$. The generalization of the Cartan–Dieudonné theorem to Hermitian spaces can be found on the web site: see web page (Chapter 25). An introduction to Hilbert spaces, including the projection theorem, and the isomorphism of every Hilbert space with some space $l^2(K)$, can also be found on the web site: see web page.

- Chapter 11 provides a presentation of the spectral theorems in Euclidean and Hermitian spaces, including normal, self-adjoint, skew self-adjoint, and orthogonal linear maps. Normal form (in terms of block diagonal matrices) for various types of linear maps are presented.

- The singular value decomposition (SVD) and the polar form of linear maps are discussed quite extensively in Chapter 12. The pseudo-inverse of a matrix and its characterization using the Penrose properties are presented.

- Chapter 13 presents some applications of Euclidean geometry to various optimization problems. The method of least squares is presented, as well as the applications of the SVD and QR-decomposition to solve least squares problems. We also describe a method for minimizing positive definite quadratic forms, using Lagrange multipliers.

- Chapter 14 provides an introduction to the linear Lie groups, via a presentation of some of the classical groups and their Lie algebras, using the exponential map. The surjectivity of the exponential map is proved for $\mathbf{SO}(n)$ and $\mathbf{SE}(n)$.

- An introduction to the local differential geometry of curves is given in Chapter 15 (curvature, torsion, the Frenet frame, etc).
- An introduction to the local differential geometry of surfaces based on some lectures by Eugenio Calabi is given in Chapter 16. This chapter is rather unique, as it reflects decades of experience from a very distinguished geometer.
- Chapter 17 is an appendix consisting of short sections consisting of basics of linear algebra and analysis. This chapter has been included to make the material self-contained. Our advice is to use it *as needed*!

A very elegant presentation of rational curves and surfaces can be given using some notions of affine and projective geometry. We push this approach quite far in the material on the web site: see web page. However, we provide only a cursory coverage of CAGD methods. Luckily, there are excellent texts on CAGD, including Bartels, Beatty, and Barsky [10], Farin [58, 57], Fiorot and Jeannin [60, 61], Riesler [142], Hoschek and Lasser [90], and Piegl and Tiller [139]. Although we cover affine, projective, and Euclidean geometry in some detail, we are far from giving a comprehensive treatment of these topics. For such a treatment, we highly recommend Berger [12, 13], Samuel [146], Pedoe [136], Coxeter [37, 36, 34, 35], Snapper and Troyer [160], Fresnel [66], Tisseron [169], Sidler [159], Dieudonné [46], and Veblen and Young [172, 173], a great classic.

Similarly, although we present some basics of differential geometry and Lie groups, we only scratch the surface. For instance, we refrain from discussing manifolds in full generality. We hope that our presentation is a good preparation for more advanced texts, such as Gray [78], do Carmo [51], Berger and Gostiaux [14], and Lafontaine [106]. The above are still fairly elementary. More advanced texts on differential geometry include do Carmo [52, 53], Guillemin and Pollack [80], Warner [176], Lang [108], Boothby [19], Lehmann and Sacré [113], Stoker [163], Gallot, Hulin, and Lafontaine [71], Milnor [127], Sharpe [156], Malliavin [117], and Godbillon [74].

It is often possible to reduce interpolation problems involving polynomial curves or surfaces to solving systems of linear equations. Thus, it is very helpful to be aware of efficient methods for numerical matrix analysis. For instance, we present the QR-decomposition of matrices, both in terms of the (modified) Gram–Schmidt method and in terms of Householder transformations, in a novel geometric fashion. For further information on these topics, readers are referred to the excellent texts by Strang [166], Golub and Van Loan [75], Trefethen and Bau [170], Ciarlet [33], and Kincaid and Cheney [100]. Strang's beautiful book on applied mathematics is also highly recommended as a general reference [165]. There are other interesting applications of geometry to computer vision, computer graphics, and solid modeling. Some good references are Trucco and Verri [171], Koenderink [103], and Faugeras [59] for computer vision; Hoffman [87] for solid modeling; and Metaxas [125] for physics-based deformable models.

Novelties

As far as we know, there is no fully developed modern exposition integrating the basic concepts of affine geometry, projective geometry, Euclidean geometry, Hermitian geometry, basics of Hilbert spaces with a touch of Fourier series, basics of Lie groups and Lie algebras, as well as a presentation of curves and surfaces both from the standard differential point of view and from the algorithmic point of view in terms of control points (in the polynomial and rational case).

New Treatment, New Results

This books provides an introduction to affine geometry, projective geometry, Euclidean geometry, Hermitian geometry, Hilbert spaces, a glimpse at Lie groups and Lie algebras, and the basics of local differential geometry of curves and surfaces. We also cover some classics of convex geometry, such as Carathéodory's theorem, Radon's theorem, and Helly's theorem. However, in order to help the reader assimilate all these concepts with the least amount of pain, we begin with some basic notions of affine geometry in Chapter 2. Basic notions of Euclidean geometry come later only in Chapters 6, 7, 8. Generally, noncore material is relegated to appendices or to the web site: see web page.

We cover the standard local differential properties of curves and surfaces at an elementary level, but also provide an in-depth presentation of polynomial and rational curves and surfaces from an algorithmic point of view. The approach (sometimes called *blossoming*) consists in multilinearizing everything in sight (getting *polar forms*), which leads very naturally to a presentation of polynomial and rational curves and surfaces in terms of control points (Bézier curves and surfaces). We present many algorithms for subdividing and drawing curves and surfaces, all implemented in *Mathematica*. A clean and elegant presentation of control points with weights (and control vectors) is obtained by using a construction for embedding an affine space into a vector space (the so-called "hat construction," originating in Berger [12]). We also give several new methods for drawing efficiently closed rational curves and surfaces, and a method for resolving base points of triangular rational surfaces. We give a quick introduction to the concepts of Voronoi diagrams and Delaunay triangulations, two of the most fundamental concepts in computational geometry. As a general rule, we try to be rigorous, but we always keep the algorithmic nature of the mathematical objects under consideration in the forefront.

Many problems and programming projects are proposed (over 230). Some are routine, some are (very) difficult.

Applications

Although it is core mathematics, geometry has many practical applications. Whenever possible, we point out some of these applications, For example, we mention some (perhaps unexpected) applications of projective geometry to computer vision (camera calibration), efficient communication, error correcting codes, and cryptography (see Section 5.13). As applications of Euclidean geometry, we mention motion interpolation, various normal forms of matrices including QR-decomposition in terms of Householder transformations and SVD, least squares problems (see Section 13.1), and the minimization of quadratic functions using Lagrange multipliers (see Section 13.2). Lie groups and Lie algebras have applications in robot kinematics, motion interpolation, and optimal control. They also have applications in physics. As applications of the differential geometry of curves and surfaces, we mention geometric continuity for splines, and variational curve and surface design (see Section 15.11 and Section 16.12). Finally, as applications of Voronoi diagrams and Delaunay triangulations, we mention the nearest neighbors problem, the largest empty circle problem, the minimum spanning tree problem, and motion planning (see Section 9.5). Of course, rational curves and surfaces have many applications to computer-aided geometric design (CAGD), manufacturing, computer graphics, and robotics.

Many Algorithms and Their Implementation

Although one of our main concerns is to be mathematically rigorous, which implies that we give precise definitions and prove almost all of the results in this book, we are primarily interested in the representation and the implementation of concepts and tools used to solve geometric problems. Thus, we devote a great deal of efforts to the development and implemention of algorithms to manipulate curves, surfaces, triangulations, etc. As a matter of fact, we provide *Mathematica* code for most of the geometric algorithms presented in this book. We also urge the reader to write his own algorithms, and we propose many challenging programming projects.

Open Problems

Not only do we present standard material (although sometimes from a fresh point of view), but whenever possible, we state some open problems, thus taking the reader to the cutting edge of the field. For example, we describe very clearly the problem of resolving base points of rectangular rational surfaces (this material is on the web site, see web page).

What's Not Covered in This Book

Since this book is already quite long, we have omitted solid modeling techniques, methods for rendering implicit curves and surfaces, the finite elements method, and wavelets. The first two topics are nicely covered in

Hoffman [87], and the finite element method is the subject of so many books that we will not attempt to mention any references besides Strang and Fix [167]. As to wavelets, we highly recommend the classics by Daubechies [44], and Strang and Truong [168], among the many texts on this subject. It would also have been nice to include chapters on the algebraic geometry of curves and surfaces. However, this is a very difficult subject that requires a lot of algebraic machinery. Interested readers may consult Fulton [67] or Harris [83].

How to Use This Book for a Course

This books covers three complementary but fairly disjoint topics:

(1) Projective geometry and its applications to rational curves and surfaces (Chapters 5, 18, 19, 21, 22, 23, 24);

(2) Euclidean geometry, Voronoi diagrams, and Delaunay triangulations, Hermitian geometry, basics of Hilbert spaces, spectral theorems for special kinds of linear maps, SVD, polar form, and basics of Lie groups and Lie algebras (Chapters 6, 7, 8, 9, 10, 11, 12, 13, 14);

(3) Basics of the differential geometry of curves and surfaces (Chapters 15 and 16).

Chapter 17 is an appendix consisting of background material and should be used only *as needed*.

Our experience is that there is too much material to cover in a one–semester course. The ideal situation is to teach the material in the entire book in two semesters. Otherwise, a more algebraically inclined teacher should teach the first or second topic, whereas a more differential-geometrically inclined teacher should teach the third topic. In either case, Chapter 2 on affine geometry should be covered. Chapter 4 is required for the first topic, but not for the second. A graph showing the dependencies of chapters is shown in Figure 1.

Problems are found at the end of each chapter. They range from routine to very difficult. Some programming assignments have been included. They are often quite open-ended, and may require a considerable amount of work. The end of a proof is indicated by a square box ($\square$). The word *iff* is an abbreviation for *if and only if*.

References to the web page:
http://www.cis.upenn.edu/~jean/gbooks/geom2.html will be abbreviated as web page.

Hermann Weyl made the following comment in the preface (1938) of his beautiful book [180]:

> The gods have imposed upon my writing the yoke of a foreign tongue that was not sung at my cradle .... Nobody is more aware than myself of the attendant loss in vigor, ease and lucidity of expression.

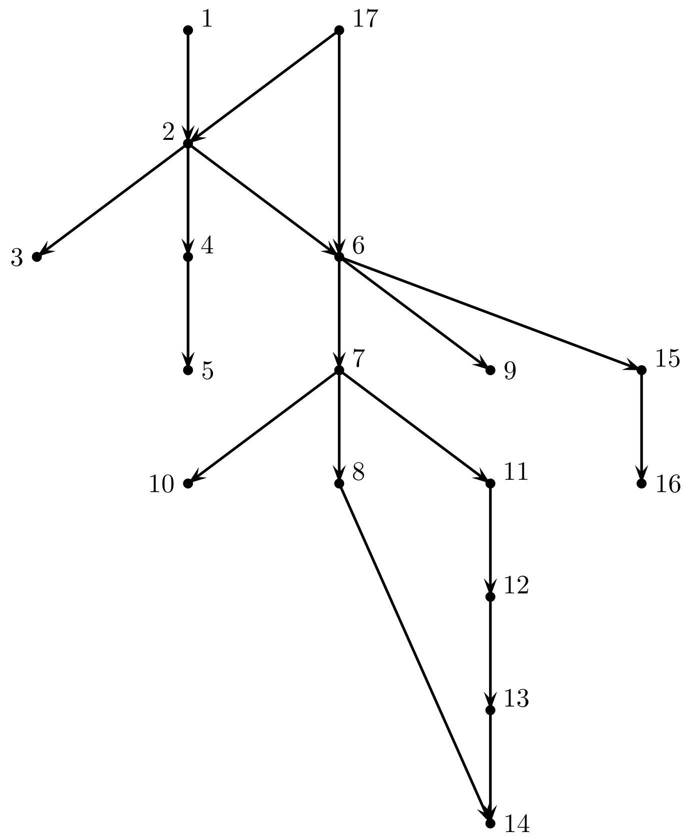

Figure 1. Dependency of chapters

Being in a similar position, I hope that I was at least successful in conveying my enthusiasm and passion for geometry, and that I have inspired my readers to study some of the books that I respect and admire.

Acknowledgments

This book grew out of lectures notes that I have written as I have been teaching CIS610, *Advanced Geometric Methods in Computer Science*, for the past two years. Many thanks to the copyeditor, David Kramer, who did a superb job. I also wish to thank some students and colleagues for their comments, including Koji Ashida, Doug DeCarlo, Jaydev Desai, Will Dickinson, Charles Erignac, Steve Frye, Edith Haber, Andy Hicks, Paul Hughett, David Jelinek, Marcus Khuri, Hartmut Liefke, Shih-Schon Lin, Ying Liu, Nilesh Mankame, Dimitris Metaxas, Viorel Mihalef, Albert Montillo, Youg-jin Park, Harold Sun, Deepak Tolani, Dianna Xu, and Hui Zhang. Also thanks to Norm Badler for triggering my interest in geometric modeling, and to Marcel Berger, Chris Croke, Ron Donagi, Herman Gluck, David Harbater, Alexandre Kirillov, and Steve Shatz for sharing

some of their geometric secrets with me. Finally, many thanks to Eugenio Calabi for teaching me what I know about differential geometry (and much more!). I am very grateful to Professor Calabi for allowing me to write up his lectures on the differential geometry of curves and surfaces given in an undergraduate course in Fall 1994 (as Chapter 16).

Philadelphia Jean Gallier

Contents

1
Introduction

> Je ne crois donc pas avoir fait une œuvre inutile en écrivant le présent Mémoire; je regrette seulement qu'il soit trop long; mais quand j'ai voulu me restreindre, je suis tombé dans l'obscurité; j'ai préféré passer pour un peu bavard.
> —**Henri Poincaré,** *Analysis Situ*, 1895

1.1 Geometries: Their Origin, Their Uses

What is geometry? According to Veblen and Young [172], geometry deals with the properties of figures in space. Etymologically, geometry means the practical science of measurement. No wonder geometry plays a fundamental role in mathematics, physics, astronomy, and engineering. Historically, as explained in more detail by Coxeter [34], geometry was studied in Egypt about 2000 B.C. Then, it was brought to Greece by Thales (640–456 B.C.). Thales also began the process of abstracting positions and straight edges as points and lines, and studying incidence properties. This line of work was greatly developed by Pythagoras and his disciples, among which we should distinguish Hippocrates. Indeed, Hippocrates attempted a presentation of geometry in terms of logical deductions from a few definitions and assumptions. But it was Euclid (about 300 B.C.) who made fundamental contributions to geometry, recorded in his immortal *Elements*, one of the most widely read books in the world.

Euclid's basic assumptions consist of basic notions concerning magnitudes, and five postulates. Euclid's fifth postulate, sometimes called the "parallel postulate," is historically very significant. It prompted mathematicians to question the traditional foundations of geometry, and led them to realize that there are different kinds of geometries. The fifth postulate can be stated in the following way:

V. *If a straight line meets two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.*

Euclid's fifth postulate is definitely not self-evident. It is also not simple or natural, and after Euclid, many people tried to deduce it from the other postulates. However, they succeeded only in replacing it by various equivalent assumptions, of which we only mention two:

V$'$. *Two parallel lines are equidistant.* (Posidonius, first century B.C.).

V$''$. *The sum of the angles of a triangle is equal to two right angles.* (Legendre, 1752–1833).

According to Euclid, two lines are parallel if they are coplanar without intersecting.

It is remarkable that until the eighteenth century, no serious attempts at proving or disproving Euclid's fifth postulate were made. Saccheri (1667–1733) and Lambert (1728–1777) attempted to prove Euclid's fifth postulate, but of course, this was impossible. This was shown by Lobachevski (1793–1856) and Bolyai (1802–1860), who proposed some models of non-Euclidean geometries. Actually, Gauss (1777–1855) was the first to consider seriously the possibility that a geometry denying Euclid's fifth postulate was of some interest. However, this was such a preposterous idea in those days that he kept these ideas to himself until others had published them independently.

Thus, circa the 1830s, it was finally realized that there is not just one geometry, but *different kinds of geometries* (spherical, hyperbolic, elliptic). The next big step was taken by Riemann, (1826–1866) who introduced the "infinitesimal approach" to geometry, wherein the differential of distance is expressed as the square root of the sum of the squares of the differentials of the coordinates. Riemann studied spherical spaces of higher dimension, and showed that their geometry is non-Euclidean. Finally, Cayley (1821–1895) and especially Klein (1849–1925) reached a clear understanding of the various geometries and their relationships. Basically, all geometries can be viewed as embedded in a universal geometry, *projective geometry*. Projective geometry itself is non-Euclidean, since two coplanar lines always intersect in a single point.

Projective geometry was developed in the nineteenth century, mostly by Monge, Poncelet, Chasles, Steiner, and Von Staudt (but anticipated by

Kepler (1571–1630) and Desargues (1593–1662)). Klein also realized that "a geometry" can be defined by the set of properties invariant under a certain group of transformations. For example, projective properties are invariant under the group of projectivities, affine properties are invariant under the group of affine bijections, and Euclidean properties are invariant under rigid motions. Although it is possible to define these various groups of transformations as certain subgroups of the group of projectivities, such an approach is quite bewildering to a novice. In order to appreciate such acrobatics, one has to already know about projective geometry, affine geometry, and Euclidean geometry.

Since the fifties, geometry has been built on top of linear algebra, as opposed to axiomatically (as in Veblen and Young [172, 173] or Samuel [146]). Even though geometry loses some of its charm presented that way, it has the advantage of receiving a more unified and simpler treatment.

Affine geometry is basically the geometry of linear algebra. Well, not quite, since affine maps are not linear maps. The additional ingredient is that affine geometry is invariant under translations, which are not linear maps! Instead of linear combinations of vectors, we need to consider affine combinations of points, or barycenters (where the scalars add up to 1). Affine maps preserve barycenters. In some sense, affine geometry is the geometry of systems of particles and forces acting on them. Angles and distances are undefined, but parallelism is well defined. The crucial notion is the notion of ratio. Given any two points a, b and any scalar λ, the point $c = (1 - \lambda)a + \lambda b$ is the point on the line (a, b) (assuming $a \neq b$) such that $\mathbf{ac} = \lambda\mathbf{ab}$, i.e., the point c is "λ of the way between a and b." Even though such a geometry may seem quite restrictive, it allows the handling of polynomial curves and surfaces.

Euclidean geometry is obtained by adding an inner product to affine geometry. This way, angles and distances can be defined. The maps that preserve the inner product are the rigid motions. In Euclidean geometry, orthogonality can be defined. This is a very rich geometry. The structure of rigid motions (rotations and rotations followed by a flip) is well understood, and plays an important role in rigid body mechanics.

Projective geometry is, roughly speaking, linear algebra "up to a scalar." There is no notion of angle or distance, and projective maps are more general than affine maps. What is remarkable is that every affine space can be embedded into a projective space, its projective completion. In such a projective completion, there is a special hyperplane of "points at infinity." Affine maps are the projectivities that preserve (globally) this hyperplane at infinity. Thus, affine geometry can be viewed as a specialization of projective geometry. What is remarkable is that if we consider projective spaces over the complex field, it is possible to introduce the notion of angle in a projective manner (via the cross-ratio). This discovery, due to Poncelet, Laguerre, and Cayley, can be exploited to show that Euclidean geometry is a specialization of projective geometry.

Besides projective geometry and its specializations, there are other important and beautiful facets of geometry, notably differential geometry and algebraic geometry. Nowdays, each one is a major area of mathematics, and it is out of the question to discuss both in any depth. We will present some basics of the differential geometry of curves and surfaces. This topic was studied by many, including Euler and Gauss, who made fundamental contributions. However, we will limit ourselves to the study of local properties and not even attempt to touch manifolds.

These days, projective geometry is rarely tought at any depth in mathematics departments, and similarly for basic differential geometry. Typically, projective spaces are defined at the begining of an algebraic geometry course, but modern algebraic geometry courses deal with much more advanced topics, such as varieties and schemes. Similarly, differential geometry courses proceed quickly to manifolds and Riemannian metrics, but the more elementary "geometry in the small" is cursorily covered, if at all.

Paradoxically, with the advent of faster computers, it was realized by manufacturers (for instance of cars and planes) that it was possible and desirable to use computer-aided methods for their design. Computer vision problems (and some computer graphics problems) can often be formulated in the framework of projective geometry. Thus, there seems to be an interesting turn of events. After being neglected for decades, stimulated by computer science, old-fashioned geometry seems to be making a comeback as a fundamental tool used in manufacturing, computer graphics, computer vision, and motion planning, just to mention some key areas.

We are convinced that geometry will play an important role in computer science and engineering in the years to come. The demand for technology using 3D graphics, virtual reality, animation techniques, etc., is increasing fast, and it is clear that storing and processing complex images and complex geometric models of shapes (face, limbs, organs, etc.) will be required. This book represents an attempt at presenting a coherent view of geometric methods used to tackle problems of a geometric nature with a computer. We believe that this can be a great way of learning some old-fashioned (and some new!) geometry while having fun. Furthermore, there are plenty of opportunities for applying these methods to real-world problems.

While we are interested in the standard (local) differential properties of curves and surfaces (torsion, curvature), we concentrate on methods for discretizing curves and surfaces in order to store them and display them efficiently. However, in order to gain a deeper understanding of this theory of curves and surfaces, we present the underlying geometric concepts in some detail, in particular, affine, projective, and Euclidean geometry.

1.2 Prerequisites and Notation

It is assumed that the reader is familiar with the basics of linear algebra, at the level of Strang [166]. The reader may also consult appropriate chapters on linear algebra in Lang [107]. For the material on the differential geometry of curves and surfaces and Lie groups, familiarity with some basics of analysis are assumed. Lang's text [110] is more than sufficient as background. A general background in classical geometry is helpful, but not mandatory. Two excellent sources are Coxeter [35] and Pedoe [136].

We denote the set $\{0, 1, 2, \ldots\}$ of natural numbers by $\mathbb{N}$, the ring $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers by $\mathbb{Z}$, the field of rationals by $\mathbb{Q}$, the field of real numbers by $\mathbb{R}$, and the field of complex numbers by $\mathbb{C}$. The multiplicative group $\mathbb{R} - \{0\}$ of reals is denoted by $\mathbb{R}^*$, and similarly, the multiplicative field of complex numbers is denoted by $\mathbb{C}^*$. We let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ denote the set of nonnegative reals.

The n-dimensional vector space of real n-tuples is denoted by $\mathbb{R}^n$, and the complex n-dimensional vector space of complex n-tuples is denoted by $\mathbb{C}^n$.

Given a vector space E, vectors are usually denoted by lowercase letters from the end of the alphabet, in italic or boldface; for example, u, v, w, $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

The null vector $(0, \ldots, 0)$ is abbreviated as 0 or $\mathbf{0}$. A vector space consisting only of the null vector is called a *trivial vector space*. A trivial vector space $\{0\}$ is sometimes denoted by 0. A vector space $E \neq \{0\}$ is called a *nontrivial vector space*.

When dealing with affine spaces, we will use an arrow in order to distinguish between spaces of points (E, U, etc.) and the corresponding spaces of vectors ($\overrightarrow{E}, \overrightarrow{U}$, etc.).

The dimension of the vector space E is denoted by $\dim(E)$. The direct sum of two vector spaces U, V is denoted by $U \oplus V$. The dual of a vector space E is denoted by E^*. The kernel of a linear map $f\colon E \to F$ is denoted by $\mathrm{Ker}\, f$, and the image by $\mathrm{Im}\, f$. The transpose of a matrix A is denoted by $A^\top$. The identity function is denoted by id, and the $n \times n$-identity matrix is denoted by I_n, or I. The determinant of a matrix A is denoted by $\det(A)$ or $D(A)$.

The cardinality of a set S is denoted by $|S|$. Set difference is denoted by

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

A list of symbols in their order of appearance in this book is given after the bibliography.