is monotone decreasing, for which behavior I can't find any reason. Dini, who recapitulates this proof in [1], seems to assume that for a sequence h_k that goes to 0 the quotients h_m/h_n for $m > n$ are bounded.

And as a last remark: Dini's test for divergence says that Σa_k is divergent if there exists a (weakly) monotone increasing sequence $c_k = m_k a_k$ with $\Sigma 1/m_k$ $(= \Sigma a_k/c_k)$ diverging. The monotonicity condition can be written as $a_k/a_{k+1} \geq (1/m_k)/(1/m_{k+1})$, and by a standard argument that makes Σa_k diverge also. (That is Dini's proof.)

Let us here replace the monotonicity of the c_k by the weaker condition that the c_k are bounded below by a positive number, say by 1. Then we have $a_k \geq a_k/c_k$, with the series of the latter terms diverging by assumption, and thus a slightly improved version of Dini's criterion also turns out to be identical with the comparison test (divergence version)! The interesting aspect is again that writing the c_k as $m_k a_k$ makes it easy to set up specific tests.

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The Derivation of the Exponential Map of Matrices

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The exponential map, which links the Lie algebra to its Lie group, is of course an analytic map, and as such it has a derivative. Although the explicit expression for this derivative is not so complicated, the way to obtain it seems long and difficult. For instance, in **[H]** affine connections and differential equations are used, in **[P]** a Taylor expansion is used and terms of order 2 ($\mathcal{O}(t^2)$) are neglected, in **[V]** a complicated analysis of Taylor series using enveloping algebras is used; in **[MT]** a rather simple argument using differential equations is used, but this argument is only valid for matrices.

We present here a rather elementary way to obtain this derivative. For ease of exposition we will do it for matrices, but only cosmetic changes are needed to make it a valid computation for any Lie group.

Theorem. *Let* $\exp \equiv e\colon M(n,\mathbf{R}) \to M(n,\mathbf{R})$ *denote the exponential map on* $n \times n$ *matrices with real entries. Then*:

$$\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} = e^X \cdot \left(\frac{\mathbf{1} - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)(Y),$$

where $\mathrm{ad}(X)$ *denotes the adjoint representation* $Y \mapsto \mathrm{ad}(X)(Y) \equiv X \cdot Y - Y \cdot X$, *and where the quotient should be interpreted as the formal power series.*

Proof: We introduce the matrix $\Delta(X,Y)$ defined as:

$$\Delta(X,Y) = e^{-X} \cdot \left.\frac{d}{dt}\right|_{t=0} e^{X+tY}.$$

The map Δ is obviously continuous in X and Y, and, moreover, it is linear in Y by definition of the derivative. Applying the Leibnitz rule to the equality $e^{X+tY} = \exp((1/n)X + t(1/n)Y)^n$, valid for any $n \in \mathbf{Z}$, gives us:

$$\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} = \sum_{k=0}^{n-1} \exp\left(\frac{1}{n}X\right)^{n-1-k} \cdot \left(\left.\frac{d}{dt}\right|_{t=0} \exp\left(\frac{1}{n}X + t\frac{1}{n}Y\right)\right) \cdot \exp\left(\frac{1}{n}X\right)^k.$$

Using the definition of Δ we then compute:

$$\begin{aligned}
e^{-X} \cdot \left.\frac{d}{dt}\right|_{t=0} e^{X+tY} &= \sum_{k=0}^{n-1} \exp\left(\frac{1}{n}X\right)^{-k} \cdot \Delta\left(\frac{1}{n}X, \frac{1}{n}Y\right) \cdot \exp\left(\frac{1}{n}X\right)^k \\
&= \frac{1}{n} \cdot \sum_{k=0}^{n-1} \mathrm{Ad}(e^{-X/n})^k \left(\Delta\left(\frac{1}{n}X, Y\right)\right) \\
&= \left(\frac{\mathbf{1} - \mathrm{Ad}(e^{-X})}{n(\mathbf{1} - \mathrm{Ad}(e^{-X/n}))}\right)\left(\Delta\left(\frac{1}{n}X, Y\right)\right) \\
&\xrightarrow{n\to\infty} \left(\frac{\mathbf{1} - e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)(\Delta(\mathbf{0}, Y)).
\end{aligned}$$

To obtain the second equality we used the linearity of Δ in Y and the definition of the Adjoint representation: $\mathrm{Ad}(B)(A) = B \cdot A \cdot B^{-1}$. For the third equality we used the formula for the sum of a geometric progression with factor $\mathrm{Ad}(e^{-X/n})$. For the limit we used the continuity of Δ in X, the fact that ad is the derivative of Ad (for the limit $n \to \infty$ in the denominator!), and that the exponential map intertwines ad and Ad: $\mathrm{Ad}(e^X) = e^{\mathrm{ad}(X)}$. Since an elementary calculation shows that $\Delta(\mathbf{0}, Y) = Y$, the theorem follows when we multiply by e^X.

Remark. Readers who feel uneasy in taking the limit $n \to \infty$ need only check the following convergence of power series of a single complex variable z:

$$\frac{e^z - 1}{n(e^{z/n} - 1)} = \sum_{k=0}^{n-1} \frac{1}{n}(e^{z/n})^k = \sum_{i=0}^{\infty}\left(\sum_{k=0}^{n-1} \frac{k^i}{i!n^{i+1}}\right) \cdot z^i \xrightarrow{n\to\infty} \sum_{i=0}^{\infty} \frac{z^i}{(i+1)!} = \frac{e^z - 1}{z}.$$

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The Kantorovich Inequality

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The inequality appears first in a survey article on functional analysis and applied mathematics by L. V. Kantorovič; it is used in investigations concerning the condition number of operators and has important applications in estimating convergence of methods of steepest descent for solving equations. In a number of subsequent papers the connection of the inequality with an inequality given by Pólya and Szegö was cleared up and a number of proofs, some of considerable complexity, of the inequality and of different variants thereof appeared in the literature.

In view of the importance of the inequality one more note on the subject might be of interest. It is not difficult to see that the result is essentially based on the inequality between the geometric and arithmetic mean; to emphasise this we restate it in a form using the two means which immediately suggests a simple and natural proof.

The Kantorovich inequality. *Suppose $x_1 < x_2 < \cdots < x_n$ are given positive numbers. Let $\lambda_1, \dots, \lambda_n \geq 0$ and $\sum \lambda_j = 1$. Then*

$$\left(\sum \lambda_j x_j\right)\left(\sum \lambda_j x_j^{-1}\right) \leq A^2 G^{-2}$$

where $A = \frac{1}{2}(x_1 + x_n)$ and $G = (x_1 x_n)^{1/2}$.

Proof: Observe that the inequality is homogeneous in the sense that it is invariant with respect to replacing each x_j by a positive multiple αx_j. Accordingly it is possible to assume that $G = 1$ so that $x_n = 1/x_1$. Each x between x_1 and $1/x_1$ satisfies

$$x + \frac{1}{x} \leq x_1 + \frac{1}{x_1}.$$

It follows that $\Sigma \lambda_j x_j + \Sigma \lambda_j x_j^{-1} \leq x_1 + 1/x_1 = 2A$. The conclusion follows by an application of the geometric—arithmetic mean inequality.