3.2 Separation Theorems

It seems intuitively rather obvious that if A and B are two nonempty disjoint convex sets in $\mathbb{A}^2$, then there is a line, H, separating them, in the sense that A and B belong to the two (disjoint) open half–planes determined by H.

However, this is not always true! For example, this fails if both A and B are closed and unbounded (find an example).

Nevertheless, the result is true if both A and B are open, or if the notion of separation is weakened a little bit.

The key result, from which most separation results follow, is a geometric version of the *Hahn-Banach theorem*.

In the sequel, we restrict our attention to real affine spaces of finite dimension. Then, if X is an affine space of dimension d, there is an affine bijection f between X and $\mathbb{A}^d$.

Now, $\mathbb{A}^d$ is a topological space, under the usual topology on $\mathbb{R}^d$ (in fact, $\mathbb{A}^d$ is a metric space).

Recall that if $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ are any two points in $\mathbb{A}^d$, their **Euclidean distance**, $d(a, b)$, is given by

$$d(a,b) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_d - a_d)^2},$$

which is also the *norm*, $\|\mathbf{ab}\|$, of the vector $\mathbf{ab}$ and that for any $\epsilon > 0$, the *open ball of center a and radius ϵ*, $B(a, \epsilon)$, is given by

$$B(a, \epsilon) = \{b \in \mathbb{A}^d \mid d(a, b) < \epsilon\}.$$

A subset $U \subseteq \mathbb{A}^d$ is *open* (in the *norm topology*) if either U is empty or for every point, $a \in U$, there is some (small) open ball, $B(a, \epsilon)$, contained in U.

A subset $C \subseteq \mathbb{A}^d$ is *closed* iff $\mathbb{A}^d - C$ is open. For example, the *closed balls*, $\overline{B(a, \epsilon)}$, where

$$\overline{B(a, \epsilon)} = \{b \in \mathbb{A}^d \mid d(a, b) \leq \epsilon\},$$

are closed.

A subset $W \subseteq \mathbb{A}^d$ is *bounded* iff there is some ball (open or closed), B, so that $W \subseteq B$.

A subset $W \subseteq \mathbb{A}^d$ is *compact* iff every family, $\{U_i\}_{i \in I}$, that is an open cover of W (which means that $W = \bigcup_{i \in I}(W \cap U_i)$, with each U_i an open set) possesses a finite subcover (which means that there is a finite subset, $F \subseteq I$, so that $W = \bigcup_{i \in F}(W \cap U_i)$).

In $\mathbb{A}^d$, it can be shown that a subset W is compact iff W is closed and bounded.

Given a function, $f\colon \mathbb{A}^m \to \mathbb{A}^n$, we say that f is *continuous* if $f^{-1}(V)$ is open in $\mathbb{A}^m$ whenever V is open in $\mathbb{A}^n$.

If $f\colon \mathbb{A}^m \to \mathbb{A}^n$ is a continuous function, although it is generally **false** that $f(U)$ is open if $U \subseteq \mathbb{A}^m$ is open, it is easily checked that $f(K)$ is compact if $K \subseteq \mathbb{A}^m$ is compact.

An affine space X of dimension d becomes a topological space if we give it the topology for which the open subsets are of the form $f^{-1}(U)$, where U is any open subset of $\mathbb{A}^d$ and $f\colon X \to \mathbb{A}^d$ is an affine bijection.

Given any subset, A, of a topological space X, the smallest closed set containing A is denoted by $\overline{A}$, and is called the *closure* or *adherence of* A.

A subset, A, of X, is *dense in* X if $\overline{A} = X$.

The largest open set contained in A is denoted by $\overset{\circ}{A}$, and is called the *interior of* A.

The set $\mathrm{Fr}\, A = \overline{A} \cap \overline{X - A}$, is called the *boundary* (or *frontier*) of A. We also denote the boundary of A by ∂A.

In order to prove the Hahn-Banach theorem, we will need two lemmas.

Given any two distinct points $x, y \in X$, we let

$$]x, y[= \{(1-\lambda)x + \lambda y \in X \mid 0 < \lambda < 1\}.$$

Lemma 3.2.1 *Let S be a nonempty convex set, and let $x \in \overset{\circ}{S}$ and $y \in \overline{S}$. Then, we have $]x, y[\subseteq \overset{\circ}{S}$.*

Corollary 3.2.2 *If S is convex, then $\overset{\circ}{S}$ is also convex and we have $\overset{\circ}{S} = \overset{\circ}{\overline{S}}$. Further, if $\overset{\circ}{S} \neq \emptyset$, then $\overline{S} = \overline{\overset{\circ}{S}}$.*

There is a simple criterion to test whether a convex set has an empty interior, based on the notion of dimension of a convex set.

Definition 3.2.3 The *dimension* of a nonempty convex subset, S, of X, denoted by $\dim S$, is the dimension of the smallest affine subset $\langle S\rangle$ containing S.

Proposition 3.2.4 *A nonempty convex set S has a nonempty interior iff* $\dim S = \dim X$*.*

Proposition 3.2.4 is false in infinite dimension.

Proposition 3.2.5 *If S is convex, then $\overline{S}$ is also convex.*

One can also easily prove that convexity is preserved under direct image and inverse image by an affine map.

The next lemma, which seems intuitively obvious, is the core of the proof of the Hahn-Banach theorem. This is the case where the affine space has dimension two.

First, we need to define what is a convex cone.

Definition 3.2.6 A convex set, C, is a *convex cone with vertex* x if C is invariant under all central magnifications $H_{x,\lambda}$ of center x and ratio λ, with $\lambda > 0$
(i.e., $H_{x,\lambda}(C) = C$).

Given a convex set, S, and a point $x \notin S$, we can define

$$\mathrm{cone}_x(S) = \bigcup_{\lambda > 0} H_{x,\lambda}(S).$$

It is easy to check that this is a convex cone.

Lemma 3.2.7 *Let B be a nonempty open and convex subset of $\mathbb{A}^2$, and let O be a point of $\mathbb{A}^2$ so that $O \notin B$. Then, there is some line, L, through O, so that $L \cap B = \emptyset$.*

Finally, we come to the Hahn-Banach theorem.

Theorem 3.2.8 *(Hahn-Banach theorem, geometric form) Let X be a (finite-dimensional) affine space, A be a nonempty open and convex subset of X and L be an affine subspace of X so that $A \cap L = \emptyset$. Then, there is some hyperplane, H, containing L, that is disjoint from A.*

Proof. The case where $\dim X = 1$ is trivial. Thus, we may assume that $\dim X \geq 2$. We reduce the proof to the case where $\dim X = 2$. □

Remark: The geometric form of the Hahn-Banach theorem also holds when the dimension of X is infinite, but a more sophisticated proof is required (it uses Zorn's lemma).

Theorem 3.2.8 is false if we omit the assumption that A is open. For a counter-example, let $A \subseteq \mathbb{A}^2$ be the union of the half space $y < 0$ with the close segment $[0, 1]$ on the x-axis and let L be the point $(2, 0)$ on the boundary of A. It is also false if A is closed! (Find a counter-example).

Theorem 3.2.8 has many important corollaries. First, we define the notion of separation. For this, recall the definition of the closed (or open) half–spaces determined by a hyperplane.

Given a hyperplane H, if $f\colon E \to \mathbb{R}$ is any nonconstant affine form defining H (i.e., $H = \mathrm{Ker}\, f$), we define the *closed half-spaces associated with f* by

$$H_+(f) = \{a \in E \mid f(a) \geq 0\},$$
$$H_-(f) = \{a \in E \mid f(a) \leq 0\}.$$

Observe that if $\lambda > 0$, then $H_+(\lambda f) = H_+(f)$, but if $\lambda < 0$, then $H_+(\lambda f) = H_-(f)$, and similarly for $H_-(\lambda f)$.

Thus, the set $\{H_+(f), H_-(f)\}$ only depends on the hyperplane H, and the choice of a specific f defining H amounts to the choice of one of the two half-spaces.

We also define the *open half–spaces associated with f* as the two sets

$$\overset{\circ}{H}_+(f) = \{a \in E \mid f(a) > 0\},$$
$$\overset{\circ}{H}_-(f) = \{a \in E \mid f(a) < 0\}.$$

The set $\{\overset{\circ}{H}_+(f), \overset{\circ}{H}_-(f)\}$ only depends on the hyperplane H.

Clearly, $\overset{\circ}{H}_+(f) = H_+(f) - H$ and $\overset{\circ}{H}_-(f) = H_-(f) - H$.

Definition 3.2.9 Given an affine space, X, and two nonempty subsets, A and B, of X, we say that a hyperplane H *separates (resp. strictly separates) A and B* if A is in one and B is in the other of the two half–spaces (resp. open half–spaces) determined by H.

We will eventually prove that for any two nonempty disjoint convex sets A and B there is a hyperplane separating A and B, but this will take some work.

We begin with the following version of the Hahn-Banach theorem:

Theorem 3.2.10 *(Hahn-Banach, second version) Let X be a (finite-dimensional) affine space, A be a nonempty convex subset of X with nonempty interior and L be an affine subspace of X so that $A \cap L = \emptyset$. Then, there is some hyperplane, H, containing L and separating L and A.*

Corollary 3.2.11 *Given an affine space, X, let A and B be two nonempty disjoint convex subsets and assume that A has nonempty interior ($\overset{\circ}{A} \neq \emptyset$). Then, there is a hyperplane separating A and B.*

Remark: Theorem 3.2.10 and Corollary 3.2.11 also hold in the infinite case.

Corollary 3.2.12 *Given an affine space, X, let A and B be two nonempty disjoint open and convex subsets. Then, there is a hyperplane strictly separating A and B.*

Beware that Corollary 3.2.12 *fails* for *closed* convex sets. However, Corollary 3.2.12 holds if we also assume that A (or B) is compact.

We need to review the notion of distance from a point to a subset.

Let X be a metric space with distance function d. Given any point $a \in X$ and any nonempty subset B of X, we let

$$d(a, B) = \inf_{b \in B} d(a, b)$$

(where inf is the notation for least upper bound).

Now, if X is an affine space of dimension d, it can be given a metric structure by giving the corresponding vector space a metric structure, for instance, the metric induced by a Euclidean structure.

We have the following important property: For any nonempty closed subset, $S \subseteq X$ (not necessarily convex), and any point, $a \in X$, there is some point $s \in S$ "achieving the distance from a to S," i.e., so that

$$d(a, S) = d(a, s).$$

Corollary 3.2.13 *Given an affine space, X, let A and B be two nonempty disjoint closed and convex subsets, with A compact. Then, there is a hyperplane strictly separating A and B.*

Finally, we have the separation theorem announced earlier for arbitrary nonempty convex subsets. (For a different proof, see Berger [**?**], Corollary 11.4.7.)

Corollary 3.2.14 *Given an affine space, X, let A and B be two nonempty disjoint convex subsets. Then, there is a hyperplane separating A and B.*

Remarks:

(1) The reader should compare the proof from Valentine [**?**], Chapter II with Berger's proof using compactness of the projective space $\mathbb{P}^d$ [**?**] (Corollary 11.4.7).

(2) Rather than using the Hahn-Banach theorem to deduce separation results, one may proceed differently and use the following intuitively obvious lemma, as in Valentine [**?**] (Theorem 2.4):

Lemma 3.2.15 *If A and B are two nonempty convex sets such that $A \cup B = X$ and $A \cap B = \emptyset$, then $V = \overline{A} \cap \overline{B}$ is a hyperplane.*

One can then deduce Corollaries 3.2.11 and 3.2.14. Yet another approach is followed in Barvinok [**?**].

(3) How can some of the above results be generalized to infinite dimensional affine spaces, especially Theorem 3.2.8 and Corollary 3.2.11?

One approach is to simultaneously relax the notion of interior and tighten a little the notion of closure, in a more "linear and less topological" fashion, as in Valentine [**?**].

Given any subset $A \subseteq X$ (where X may be infinite dimensional, but is a Hausdorff topological vector space), say that a point $x \in X$ is *linearly accessible from* A iff there is some $a \in A$ with $a \neq x$ and $]a, x[\ \subseteq A$. We let $\mathrm{lina}\, A$ be the set of all points linearly accessible from A and $\mathrm{lin}\, A = A \cup \mathrm{lina}\, A$.

A point $a \in A$ is a *core point of* A iff for every $y \in X$, with $y \neq a$, there is some $z \in]a, y[$, such that $[a, z] \subseteq A$. The set of all core points is denoted $\mathrm{core}\, A$.

It is not difficult to prove that $\mathrm{lin}\, A \subseteq \overline{A}$ and $\overset{\circ}{A} \subseteq \mathrm{core}\, A$. If A has nonempty interior, then $\mathrm{lin}\, A = \overline{A}$ and $\overset{\circ}{A} = \mathrm{core}\, A$.

Also, if A is convex, then core A and lin A are convex. Then, Lemma 3.2.15 still holds (where X is not necessarily finite dimensional) if we redefine V as $V = \text{lin}\, A \cap \text{lin}\, B$ and allow the possibility that V could be X itself.

Corollary 3.2.11 also holds in the general case if we assume that core A is nonempty. For details, see Valentine [**?**], Chapter I and II.

(4) Yet another approach is to define the notion of an algebraically open convex set, as in Barvinok [**?**].

A convex set, A, is *algebraically open* iff the intersection of A with every line, L, is an open interval, possibly empty or infinite at either end (or all of L).

An open convex set is algebraically open. Then, the Hahn-Banach theorem holds provided that A is an algebraically open convex set and similarly, Corollary 3.2.11 also holds provided A is algebraically open.

For details, see Barvinok [**?**], Chapter 2 and 3. We do not know how the notion "algebraically open" relates to the concept of core.

(5) Theorems 3.2.8, 3.2.10 and Corollary 3.2.11 are proved in Lax using the notion of *gauge function* in the more general case where A has some core point (but beware that Lax uses the terminology *interior point* instead of core point!).

An important special case of separation is the case where A is convex and $B = \{a\}$ for some point a in A.

3.3 Supporting Hyperplanes

Definition 3.3.1 Let X be an affine space and let A be any nonempty subset of X. A *supporting hyperplane of* A is any hyperplane, H, containing some point, a, of A, and separating $\{a\}$ and A. We say that H is a *supporting hyperplane of* A *at* a.

Observe that if H is a supporting hyperplane of A at a, then we must have $a \in \partial A$.

Also, if A is convex, then $H \cap \overset{\circ}{A} = \emptyset$.

One should experiment with various pictures and realize that supporting hyperplanes at a point may not exist (for example, if A is not convex), may not be unique, and may have several distinct supporting points!

However, we have the following important proposition first proved by Minkowski (1896):

Proposition 3.3.2 *(Minkowski) Let A be a nonempty closed and convex subset. Then, for every point, $a \in \partial A$, there is a supporting hyperplane to A through a.*

Beware that Proposition 3.3.2 is false when the dimension X of A is infinite and when $\mathring{A} = \emptyset$.

The proposition below gives a sufficient condition for a closed subset to be convex.

Proposition 3.3.3 *Let A be a closed subset with nonempty interior. If there is a supporting hyperplane for every point $a \in \partial A$, then A is convex.*

The condition that A has nonempty interior is crucial!

The proposition below characterizes closed convex sets in terms of (closed) half–spaces. It is another intuitive fact whose rigorous proof is nontrivial.

Proposition 3.3.4 *Let A be a nonempty closed and convex subset. Then, A is the intersection of all the closed half–spaces containing it.*

Next, we consider various types of boundary points of closed convex sets.

3.4 Boundary of a Convex Set: Vertices and Extremal Points

Definition 3.4.1 Let X be an affine space of dimension d. For any nonempty closed and convex subset, A, of dimension d, a point $a \in \partial A$ has *order* $k(a)$ if the intersection of all the supporting hyperplanes of A at a is an affine subspace of dimension $k(a)$. We say that $a \in \partial A$ is a *vertex* if $k(a) = 0$; we say that a is *smooth* if $k(a) = d - 1$, i.e., if the supporting hyperplane at a is unique.

A vertex is a boundary point a such that there are d independent supporting hyperplanes at a.

A d-simplex has boundary points of order $0, 1, \ldots, d-1$. The following proposition is shown in Berger [**?**] (Proposition 11.6.2):

Proposition 3.4.2 *The set of vertices of a closed and convex subset is countable.*

Another important concept is that of an extremal point.

Definition 3.4.3 Let X be an affine space. For any nonempty convex subset A, a point $a \in \partial A$ is *extremal* (or *extreme*) if $A - \{a\}$ is still convex.

It is fairly obvious that a point $a \in \partial A$ is extremal if it does not belong to any closed nontrivial line segment $[x, y] \subseteq A$ $(x \neq y)$.

Observe that a vertex is extremal, but the converse is false.

Also, if $\dim X \geq 3$, the set of extremal points of a compact convex may not be closed.

Actually, it is not at all obvious that a nonempty compact convex possesses extremal points.

In fact, a stronger results holds (Krein and Milman's theorem).

In preparation for the proof of this important theorem, observe that any compact (nontrivial) interval of $\mathbb{A}^1$ has two extremal points, its two endpoints.

Lemma 3.4.4 *Let X be an affine space of dimension n, and let A be a nonempty compact and convex set. Then, $A = \mathcal{C}(\partial A)$, i.e., A is equal to the convex hull of its boundary.*

The following important theorem shows that only extremal points matter as far as determining a compact and convex subset from its boundary.

Theorem 3.4.5 *(Krein and Milman) Let X be an affine space of dimension n. Every compact and convex nonempty subset A is equal to the convex hull of its set of extremal points.*

Observe that Krein and Milman's theorem implies that any nonemty compact and convex set has a nonempty subset of extremal points. This is intuitively obvious, but hard to prove!

Krein and Milman's theorem also holds for infinite dimensional affine spaces, provided that they are locally convex.