The Greatest Mathematical Paper of All Time

A. J. Coleman

You will say that my title is absurd. "Mathematical papers cannot be totally ordered. It's a great pity! Poor old Coleman has obviously gone berserk in his old age." Please read on.

If in 1940 you had asked the starry-eyed Canadian graduate student who was lapping up the λ-calculus from Alonzo Church in Princeton to name the single most important mathematics paper, without doubt I would have chosen Kurt Gödel's bombshell [12] that had rocked the foundations of mathematics a few years before.

In 1970, after my twenty years of refereeing and reviewing, if you had posed the same question, without any hesitation I would have chosen the enormous paper of Walter Feit and John Thompson [11] confirming Burnside's 1911 conjecture [3] that simple finite groups have even order.

Now, in the autumnal serenity of semi-retirement, having finally looked at some of Wilhelm Killing's writings, without any doubt or hesitation I choose his paper dated "Braunsberg, 2 Februar, 1888" as the most significant mathematical paper I have read or heard about in fifty years. Few can contest my choice since apart from Engel, Umlauf, Molien, and Cartan few seem to have read it. Even my friend Hans Zassenhaus, whose *Liesche Ringe* (1940) was a landmark in the subject, admitted over our second beer at the American Mathematical Society meeting in January 1987 that he had not read a word of Killing.

Presupposing that my reader has a rudimentary understanding of linear algebra and group theory, I shall attempt to explain the main new ideas introduced in Killing's paper, describe its remarkable results, and suggest some of its subsequent effects. The paper that,

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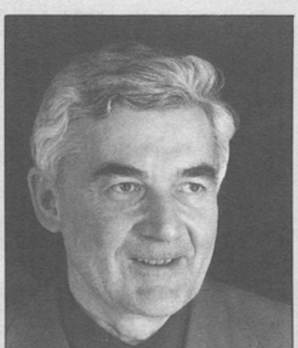


A. John Coleman was born in Toronto, Canada in 1918. He enrolled in the Mathematics and Physics Course at the University of Toronto. Together with I. Kaplansky and N. S. Mendelsohn, in 1938 he was a member of the first team to win the Putnam Contest. He obtained a Master's degree at Princeton studying under Alonzo Church, H. P. Robertson, C. Chevalley, S. Bochner, and E. Wigner.

His 1943 Ph.D. dissertation at Toronto, supervised by Leopold Infeld, was on *Relativistic Quantum Mechanics.* His professors at Toronto included Richard Brauer, J. L. Synge, H. S. M. Coxeter, and Gilbert de B. Robinson.

He has worked mostly on the N-body problem in quantum mechanics. The Proceedings of a Symposium held in his honour in 1986 appeared under the title *Density Matrices and Density Functionals,* edited by R. M. Erdahl and V. Smith, Reidel, 1987. His paper on *The Betti Numbers of Compact Groups* (1958), which employs Killing's determinant for the Coxeter transformation, was recognized in Bourbaki's *Notes Historiques* as significant in the development of the theory of Lie algebras.

In 1980 he ran as the Liberal candidate in the riding of Kingston and the Islands against Flora MacDonald, then Minister of State for External Affairs, reducing her plurality from about 11% to 3%.

following Cartan, I shall refer to as Z.v.G.II, was the second of a series of four [18] about Lie algebras. The series was churned out in Braunsberg, a mathematically isolated spot in East Prussia, during a period when Killing was overburdened with teaching, civic duties, and concerns about his family.

The Ahistoricism of Mathematicians

Most mathematicians seem to have little or no interest in history, so that often the name attached to a key result is that of the follow-up person who exploited an idea or theorem rather than its originator (Jordan form is due to Weierstrass; Wedderburn theory to Cartan and Molien [13]). No one has suffered from this ahistoricism more than Killing. For example, the so-called "Cartan sub-algebra" and "Cartan matrix, $A = (a_{ij})$" were defined and exploited by Killing. The very symbols a_{ij} and ℓ for the rank are in Z.v.G.II. Hawkins [14, p. 290] correctly states:

> Such key notions as the rank of an algebra, semi-simple algebra, Cartan algebra, root systems and Cartan integers originated with Killing, as did the striking theorem enumerating all possible structures for finite-dimensional Lie algebras over the complex numbers. . . . Cartan and Molien also used Killing's results as a paradigm for the development of the structure theory of finite dimensional linear associative algebras over the complex numbers, obtaining thereby the theorem on semisimple algebras later extended by Wedderburn to abstract fields and then applied by Emmy Noether to the matrix representations of finite groups.

In this same paper Killing invented the idea of root systems and of α root-sequences through β. He exhibited the characteristic equation of an arbitrary element of the Weyl group when Weyl was 3 years old and listed the orders of the Coxeter transformation 19 years before Coxeter was born!

I have found no evidence that Hermann Weyl read anything by Killing. Weyl's important papers on the representations of semisimple groups [26], which laid the basis for the subsequent development of abstract harmonic analysis, are based squarely on Killing's results. But Killing's name occurs only in two footnotes in contexts suggesting that Weyl had accepted uncritically the universal myth that Killing's writings were so riddled with egregious errors that Cartan should be regarded as the true creator of the theory of simple Lie algebras. This is nonsense, as must be apparent to anyone who even glances at Z.v.G.II or indeed to anyone who reads Cartan's thesis carefully. Cartan was meticulous in noting his indebtedness to Killing. In Cartan's thesis there are 20 references to Lie and 63 to Killing! For the most part the latter are the theorems or arguments of Killing that Cartan incorporated into his thesis, the first two-thirds of which is essentially a commentary on Z.v.G.II.

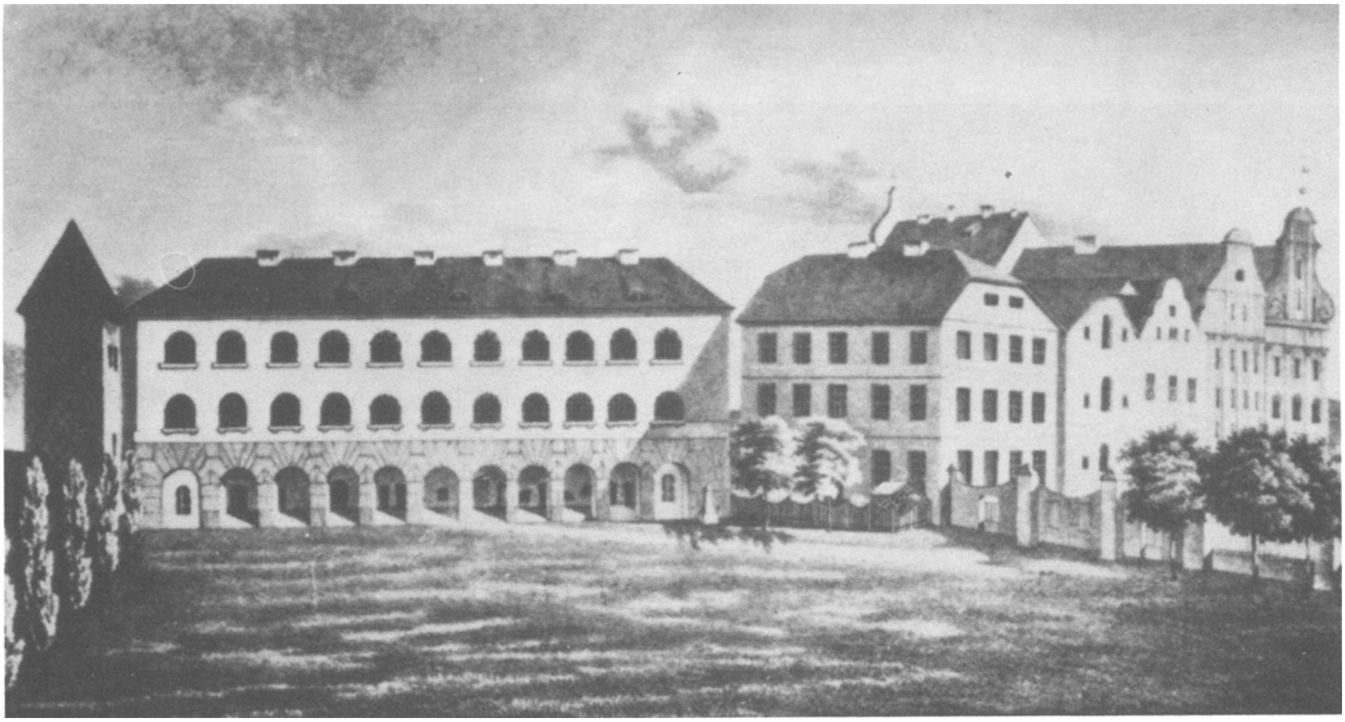
Cartan did give a remarkably elegant and clear exposition of Killing's results. He also made an essential contribution to the logic of the argument by proving that the "Cartan subalgebra" of a simple Lie algebra is abelian. This property was announced by Killing but his proof was invalid. In parts, other than II, of Killing's four papers there are major deficiencies which Cartan corrected, notably in the treatment of nilpotent Lie algebras. In the last third of Cartan's thesis, many new and important results are based upon and go beyond Killing's work. Personally, following the value scheme of my teacher Claude Chevalley, I rank Cartan and Weyl as the two greatest mathematicians of the first half of the twentieth century. Cartan's work on infinite dimensional Lie algebras, exterior differential calculus, differential geometry, and, above all, the representation theory of semisimple Lie algebras was of supreme value. But because one's Ph.D. thesis seems to predetermine one's mathematical life work, perhaps if Cartan had not hit upon the idea of basing his thesis on Killing's epoch-making work he might have ended his days as a teacher in a provincial lycée and the mathematical world would have never heard of him!

The Foothills to Parnassus

Before we enter directly into the content of Z.v.G.II, it may be well to provide some background.

What we now call Lie algebras were invented by the Norwegian mathematician Sophus Lie about 1870 and independently by Killing about 1880 [14]. Lie was seeking to develop an approach to the solution of differential equations analogous to the Galois theory of algebraic equations. Killing's consuming passion was non-Euclidean geometries and their generalizations, so he was led to the problem of classifying infinitesimal motions of a rigid body in any type of space (or *Raumformen*, as he called them). Thus in Euclidean space, the rotations of a rigid body about a fixed point form a group under composition which can be parameterized by three real numbers—the Euler angles, for example. The tangent space at the identity to the parameter space of this group is a three-dimensional linear space of "infinitesimal" rotations. Similarly, for a group that can be parameterized by a smooth manifold of dimension r, there is an r-dimensional tangent space, $\mathcal{L}$, at the identity element. If the product of two elements of the group is continuous and differentiable in the parameters of its factors, it is possible to define a binary operation on $\mathcal{L}$ which we denote by "$\circ$," such that for all $x, y, z, \in \mathcal{L}$, $(x, y) \rightarrow x \circ y$ is linear in each factor,

$$x \circ y + y \circ x = 0 \qquad (1)$$



Lyzeum Hosianum in 1835.

and

$$x \circ (y \circ z) + y \circ (z \circ x) + z \circ (x \circ y) = 0 \qquad (2a)$$

or equivalently,

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z). \qquad (2b)$$

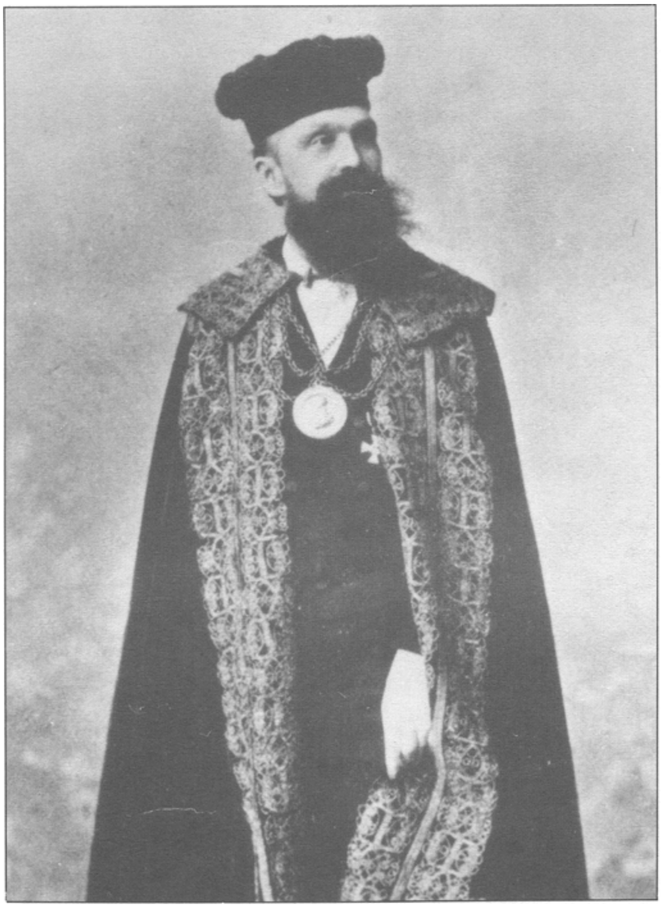
The equation in (2a) is called the *Jacobi identity* and in the form (2b) should remind you of the rule for differentiating a product. $(\mathcal{L}, +, \circ)$ is a Lie algebra with an anti-commutative non-associative product. The Jacobi identity replaces the associativity of familiar rings such as the integers or matrix algebras.

Obviously if $\mathcal{M}$ is a subspace of $\mathcal{L}$ such that $x, y \in \mathcal{M} \Rightarrow x \circ y \in \mathcal{M}$, then $\mathcal{M}$ is a *sub-algebra* of $\mathcal{L}$. Further, if ρ: $(\mathcal{L}_1, +, \circ) \rightarrow (\mathcal{L}_2, +, \circ)$ is a homomorphism of one Lie algebra onto another, the kernel of ρ is not merely a subalgebra but an *ideal*. For if $K = \{x \in \mathcal{L}_1 | \rho(x) = 0\}$, then for any $x \in K$ and any $y \in \mathcal{L}_1$, $\rho(x \circ y) = \rho(x) \circ \rho(y) = 0$. Thus K is not only a sub-algebra but has the property, characteristic of an ideal, that for any $x \in K$, we have $y \circ x \in K$ for every $y \in \mathcal{L}_1$. We can then define a *quotient* algebra $\mathcal{L}_1/K$ isomorphic to $\mathcal{L}_2$, in a manner analogous to groups with normal subgroups. Thus a *Lie algebra* $\mathcal{L}$ whose only ideals are $\{0\}$ and $\mathcal{L}$ is homomorphic only to $\mathcal{L}$ or $\{0\}$. Such an algebra is called *simple.* The simple Lie algebras are the building blocks in terms of which any Lie algebra can be analyzed. Lie recognized rather early that the search for solutions of systems of differential equations would be greatly facilitated if all simple Lie algebras were known. But Lie's attempts to find them ran into the sands very quickly.

In his quest for all uniform spatial forms Killing formulated the problem of classifying *all* Lie algebras over the reals—a task which in the case of nilpotent Lie algebras seems unlikely to have a satisfactory solution. In particular, he was interested in simple *real* Lie algebras; as a step in this direction he was led, with the encouragement of Engel, to the problem of classifying all simple Lie algebras over the complex numbers.

When $\mathcal{A}$ is an associative algebra—for example the set of $n \times n$ matrices over $\mathbf{C}$—then for $X, Y, Z \in \mathcal{A}$, we define $X \circ Y = XY - YX = [X, Y]$—the so-called *commutator* of X and Y. It is then trivial to show that $X \circ Y$ satisfies (1) and (2). Thus any associative algebra $(\mathcal{A}, +, \cdot)$ can be transformed into a Lie algebra $(\mathcal{A}, +, \circ)$ by the simple expedient of defining $X \circ Y = [X, Y]$. This immediately leads us to the notion of a *linear representation* of a Lie algebra $(\mathcal{L}, +, \circ)$ as a mapping ρ of $\mathcal{L}$ into Hom(V), satisfying the following condition: $\rho(x \circ y) = [\rho(x), \rho(y)]$. Although the definition of a representation of a Lie algebra in this simple general form was never given explicitly by Killing or others before 1900, the idea was implicit in what Engel, and Killing following him, called the *adjoint group* [15, p. 143] and what we now call the *adjoint representation.*

In passing, let us note that until about 1930 what we now call Lie groups and Lie algebras were called "continuous groups" and "infinitesimal groups" respectively; see [8], for example. These were the terms Weyl was still using in 1934/5 in his Princeton lectures [27]. However, by 1930 Cartan used the term *groupes de Lie* [4, p. 1166]; the term *Liesche Ringe* appeared in the title of the famous article on enveloping algebras by Witt [28]; and, in his *Classical Groups,* Weyl [1938, p. 260] wrote "In homage to Sophus Lie such an algebra is called a Lie Algebra." Borel [1, p. 71] attributes the term "Lie group" to Cartan and "Lie algebra" to Jacobson.



Killing as rector, 1897–1898.

For the adjoint representation of $\mathcal{L}$ the linear space V, above, is taken to be $\mathcal{L}$ itself and ρ is defined by

$$\rho(x)z = x \circ z \text{ for every } z \in \mathcal{L}. \tag{3}$$

The reader is urged to verify that with this definition of ρ, the Jacobi identity (2b) implies that $\rho(x \circ y) = [\rho(x), \rho(y)]$.

Killing Intervenes

Killing had completed his dissertation under Weierstrass at Berlin in 1872 and knew all about eigenvalues and what we now call the Jordan canonical form of matrices, whereas Lie knew little of the algebra of the contemporary Berlin school. It was therefore Killing rather than Lie who asked the decisive question: "What can one say about the eigenvalues of $X := \rho(x)$ in the adjoint representation for an arbitrary $x \in \mathcal{L}$?" Since $Xx = x \circ x = 0$, X always has zero as an eigenvalue. So Killing looked for the roots of the *characteristic* equation (a term he introduced!):

$$|\omega I - X| = \omega^r - \psi_1(x)\omega^{r-1} + \psi_2(x)\omega^{r-2} - \ldots \pm \psi_{r-1}(x)\omega = 0. \tag{4}$$

He defined k to be the minimum for $x \in \mathcal{L}$ of the multiplicity of zero as a root of (4). This is now called the *rank* of $\mathcal{L}$. But Killing and Cartan used the term *rank* for the number of functionally independent ψ_i regarded as functions of $x \in \mathcal{L}$. Killing noted that $\psi_i(x)$ are polynomial invariants of the Lie group corresponding to the Lie algebra considered. Though expressed in a rather clumsier notation, he realized that

$$\mathcal{H} = \{h \in \mathcal{L} \mid X^p h = 0 \text{ for some } p\}$$

is a subalgebra of $\mathcal{L}$. This follows from a sort of Leibnitz differentiation rule: $X^n(y \circ z) = \Sigma_s[{n \atop s}]X^{n-s}y \circ X^s z$, for $0 \leq s \leq n$. For arbitrary $\mathcal{L}$, if X is such that the $\dim(\mathcal{H})$ is a minimum, the subalgebra is now called a *Cartan subalgebra*. As a Lie algebra itself $\mathcal{H}$ is *nilpotent* or what Killing called an algebra of *zero rank*. For the adjoint representation of $\mathcal{H}$ on $\mathcal{H}$, $|\omega I - H|_{\mathcal{H}} = \omega^k$ for all $h \in \mathcal{H}$, so all ψ_i vanish identically. If $\mathcal{L}$ is simple, $\mathcal{H}$ is in fact abelian. Killing convinced himself of this by an invalid argument. The filling of this lacuna was a significant contribution by Cartan to the classification of simple Lie algebras over **C**. It was a stroke of luck on Killing's part that though his argument was mildly defective, his conclusion on this important matter was correct.

Assuming that $\mathcal{H}$ is abelian, it is trivial to show that in the equation

$$|\omega I - H| = \omega^k \, \Pi_\alpha(\omega - \alpha(h)), \tag{5}$$

the *roots*, $\alpha(h)$, are linear functions of $h \in \mathcal{H}$. Thus $\alpha \in \mathcal{H}^*$, the dual space of $\mathcal{H}$. Following current usage we denote by Δ the set of roots α that occur in (5). Killing proceeded on the assumption that all α had multiplicity one, or that the $r - k$ functions $\alpha(h)$ were distinct. It follows that for each α there is an element $e_\alpha \in \mathcal{L}$ such that $h \circ e_\alpha = \alpha(h)\, e_\alpha$ for all $h \in \mathcal{H}$. Then using (2b) it easily follows that for α, β, $\in \Delta$

$$h \circ (e_\alpha \circ e_\beta) = (\alpha(h) + \beta(h))e_\alpha \circ e_\beta. \tag{6}$$

This equation is the key to the classification of the root systems Δ that can occur for simple Lie algebras. From (6) we can immediately conclude:

$$\text{(i) } e_\alpha \circ e_\beta \neq 0 \Rightarrow \alpha + \beta \in \Delta$$

$$\text{(ii) } \alpha + \beta \notin \Delta \Rightarrow e_\alpha \circ e_\beta = 0$$

$$\text{(iii) } 0 \neq e_\alpha \circ e_\beta \in \mathcal{H} \Rightarrow \alpha + \beta = 0.$$

It turns out that for every $\alpha \in \Delta$, there is a corresponding $-\alpha \in \Delta$ such that $0 \neq h_\alpha := e_\alpha \circ e_{-\alpha} \in \mathcal{H}$. So the number of roots is even, say $2m$, and $r = k + 2m = dim(\mathcal{L})$.

In the adjoint representation let E_α correspond to e_α, and for any $e_\beta \neq 0$ consider the element $E_\alpha^n e_\beta$ for $n \in \mathbf{Z}^+$. Starting from (6) we see by induction that

$$h \circ E_\alpha^n e_\beta = (\beta(h) + n\alpha(h))E_\alpha^n e_\beta.$$

Thus if $E_\alpha^n e_\beta \neq 0$, $\beta + n\alpha \in \Delta$. But vectors with distinct eigenvalues are linearly independent. Thus if $\mathcal{L}$ is finite-dimensional there is a highest value of n for which $E_\alpha^n e_\beta \neq 0$. Call it p. Similarly let q be the largest value of n such that $E_{-\alpha}^n e_\beta \neq 0$. Thus for α, $\beta \in \Delta$ there is an *α-sequence of roots* through β of length $p + q + 1$—what Killing called *Wurzelreihe:*

$$\beta - q\alpha, \beta - (q-1)\alpha, \ldots, \beta, \beta + \alpha, \ldots, \beta + p\alpha. \tag{7}$$

Because $H_\alpha = [E_\alpha, E_{-\alpha}]$, the trace of H_α is zero, which implies

$$2\beta(h_\alpha) + (p-q)\alpha(h_\alpha) = 0. \tag{8}$$

This, in our notation, is equation (7), p. 16, of Z.v.G.II. The dimension of the Cartan subalgebra is now called the *rank* of $\mathcal{L}$. For simple Lie algebras this definition and Killing's definition of rank coincide. That is, for simple Lie algebras $k = \ell$. Hence $\dim(\mathcal{H}^*) = \ell$, so there can be at most ℓ linearly independent roots. Using (8), Killing showed that there exists a basis $B = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ of $\mathcal{H}^*$, where $\alpha_i \in \Delta$ is such that each $\beta \in \Delta$ has rational components in the basis B. Indeed, the $\alpha_i \in B$ can be so chosen that α_i is a top root in any α_j-sequence through it. Thus for each i and j there is a root-sequence

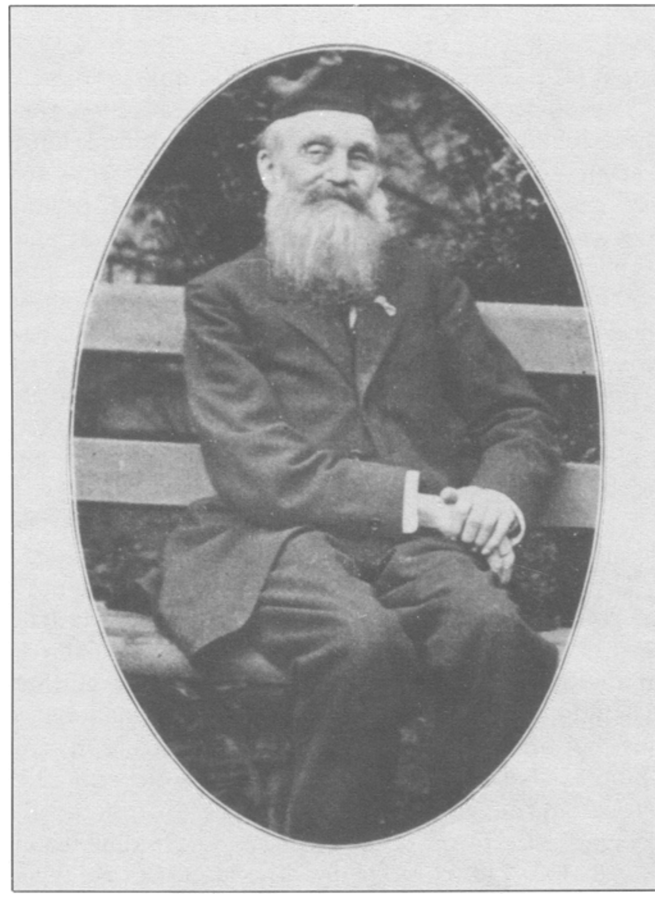
$$\alpha_i, \alpha_i - \alpha_j, \ldots, \alpha_i + a_{ij}\alpha_j \tag{9}$$

where a_{ij} is a non-positive integer. In particular, it turns out that $a_{ii} = -2$.

The Still Point of the Turning World

The definition of the integers a_{ij} was a turning point in mathematical history. It appears at the top of page 16 of II. By page 33 Killing had found the systems Δ for all simple Lie algebras over **C** together with the orders of the associated Coxeter transformations. We continue, using Killing's own words taken from the last paragraph of his introduction, unchanged except for notation:

> If α_i and α_j are two of these ℓ roots, there are two integers a_{ij} and a_{ji} that define a certain relation between the two roots. Here we mention only that together with α_i and α_j both $\alpha_i + a_{ij}\alpha_j$ and $\alpha_j + a_{ji}\alpha_i$ and $\alpha_i + a\alpha_j$ are roots where a is an integer between a_{ij} and 0. The coefficients a_{ii} are all equal to -2; the others are by no means arbitrary; indeed they satisfy many constraining equalities. One series of these constraints derives from the fact that a certain linear transformation, defined in terms of a_{ij}, when iterated gives the identity transformation. Each system of these coefficients is simple or splits into simple systems. These two possibilities are distinguished as follows. Begin with any index i, $1 \leq i \leq \ell$. Adjoin to it all j such that $a_{ij} \neq 0$; then adjoin all k for which an $a_{jk} \neq 0$. Continue as far as possible. Then, if all indices $1, 2, \ldots \ell$ have been included, the *system* of a_{ij} is simple. The roots of a simple system correspond to a simple group. Conversely, the roots of a simple group can be regarded as determined by a simple system. In this way one obtains the simple groups. For each ℓ there are four structures supplemented for $\ell \in \{2, 4, 6, 7, 8\}$ by exceptional simple groups. For these exceptional groups I have various results that are not in fully developed form; I hope later to be able to exhibit these groups in simple form and therefore am not communicating the representations for them that have been found so far.



Wilhelm Killing in his later years.

In reading this, recall that Lie and Killing used the term "group" to include the meaning we now attribute to "Lie algebra." His statement is correct as it stands for $\ell > 3$ but as is apparent from his explicit list of simple algebras he knew that for $\ell = 1$ there is only one isomorphism class and for $\ell = 2$ and 3 there are three. Replacing α_i by $-\alpha_i$ gives rise to integers satisfying $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, which is currently the usual convention. The "certain linear transformation" mentioned by Killing is the Coxeter transformation discussed below. It is worth noting that in Killing's explicit tables the coefficients for all roots in terms of his chosen basis are integers, so he came close to obtaining what we now call a *basis* of simple roots à la

Dynkin. As far as I am aware, such a basis appeared explicitly for the first time in Cartan's beautiful 1927 paper [4, p. 793] on the geometry of simple groups.

The one minor error in Killing's classification was the exhibition of two exceptional groups of rank four. Cartan noticed that Killing's two root systems are easily seen to be equivalent. It is peculiar that Killing overlooked this since his mastery of calculation and algebraic formalism was quite phenomenal. Killing's notation for the various simple Lie algebras, slightly modified by Cartan, is what we still employ: A_n denotes the isomorphism class corresponding to $s\ell(n+1,\mathbf{C})$; B_n corresponds to $so(2n+1)$; C_n to $sp(2n)$; D_n to $so(2n)$. The classes A_n, B_n, D_n were known to Lie and Killing before 1888. Killing was unaware of the existence of type C_n though Lie knew about it, at least for small values of n. On this point see the careful discussion of Hawkins [15, pp. 146–150].

The exceptional algebra of rank two which we now label G_2 was denoted as IIC by Killing. It has dimension 14 and has a linear representation of dimension 7. In a letter to Engel [15, p. 156] Killing remarked that G_2 might occur as a group of point transformations in five, but not fewer, dimensions. That such a representation exists was subsequently verified independently by Cartan and Engel [4, p. 130]. The exceptional algebras F_4, E_6, E_7, E_8 of rank 4, 6, 7, 8 have dimension 52, 78, 133, 248, respectively. The largest of Killing's exceptional groups, E_8 of dimension 248, is now the darling of super-string theorists!

Forward to Coxeter

For an arbitrary simple Lie algebra of rank n, the dimension is $n(h+1)$, where h is the order of a remarkable element of the Weyl group now called the *Coxeter transformation* (because Coxeter expounded its properties as part of his study of finite groups generated by reflections or, as they are now called, *Coxeter groups* [6, 7]). Coxeter employed a graph to classify this type of group. During the 1934/5 lectures by Weyl at Princeton, he noticed that the finite group of permutations of the roots which played a key role in Killing's argument and which is isomorphic to what we now call the Weyl group is in fact generated by involutions. The notes of Weyl's course [27] contain an Appendix by Coxeter in which a set of diagrams equivalent to those of Table 1 appears. Some years later Dynkin independently made use of similar diagrams for characterizing sets of simple roots so that they are now generally described as *Coxeter-Dynkin diagrams*.

The left-hand column of Table 1 encapsulates Killing's classification of simple Lie algebras. By studying the Coxeter transformation for Lie algebras of rank 2, Killing showed [Z.v.G.II, p. 22] that $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$. There is a one-to-one correspondence between the Cartan matrices of finite-dimensional simple Lie algebras and the left-hand column of Table 1. The n nodes of a graph correspond to Killing's indices 1, 2, 3, . . . , n, *or* to the roots of a basis *or* to generators S_i of the Weyl group. A triple bond as in G_2 means that $a_{ij}a_{ji} = 3$. Double and single bonds are interpreted similarly.

On to Kac and Moody!

If we use the current convention that $a_{ii} = 2$ and that a_{ij} is a non-positive integer if $i \neq j$, it is not difficult to see that Killing's conditions imply that $\mathcal{L}$ is a finite-dimensional Lie algebra if and only if the determinant of $A = (a_{ij})$ and those of all its principal minors are strictly positive. Further, Killing's equations (6) [Z.v.G.II. p. 21] imply that A is *symmetrisable*—that is, there exist non-zero numbers d_i such that $d_ia_{ij} = d_ja_{ji}$. In particular, a_{ij} and a_{ji} are zero or non-zero together.

Almost simultaneously in 1967, Victor Kac [16] in the USSR and Robert Moody [22] in Canada noticed that if Killing's conditions on (a_{ij}) were relaxed, it was still possible to associate to the Cartan matrix A a Lie algebra which, necessarily, would be infinite dimensional. The current method of proving the existence of such Lie algebras derives from a short paper of Chevalley [5]. This paper was also basic to the work of my students Bouwer [2] and LeMire [19], who discussed infinite dimensional representations of finite Lie algebras. Chevalley's paper also initiated the current widespread exploitation of the universal associative enveloping algebras of Lie algebras—a concept first rigorously defined by Witt [28].

Among the Kac-Moody algebras the most tractable are the symmetrisable. The most extensively studied and applied are the affine Lie algebras which satisfy all Killing's conditions except that the determinant $|A|$ is 0. The Cartan matrices for the affine Lie algebras are in one-to-one correspondence with the graphs in the right-hand column of Table 1, which first appeared in [27].

Wilhelm Killing the Man

Killing was born in Burbach in Westphalia, Germany, on 10 May 1847 and died in Münster on 11 February 1923. Killing began university study in Münster in 1865 but quickly moved to Berlin and came under the influence of Kummer and Weierstrass. His thesis, completed in March 1872, was supervised by Weierstrass and applied the latter's recently developed theory of elementary divisors of a matrix to "Bundles of Surfaces of the Second Degree." From 1868 to 1882 much of Killing's energy was devoted to teaching at the gymnasium level in Berlin and Brilon (south of Münster). At one stage when Weierstrass was urging him to write up his research on space structures (*Raumformen*) he was spending as much as 36 hours

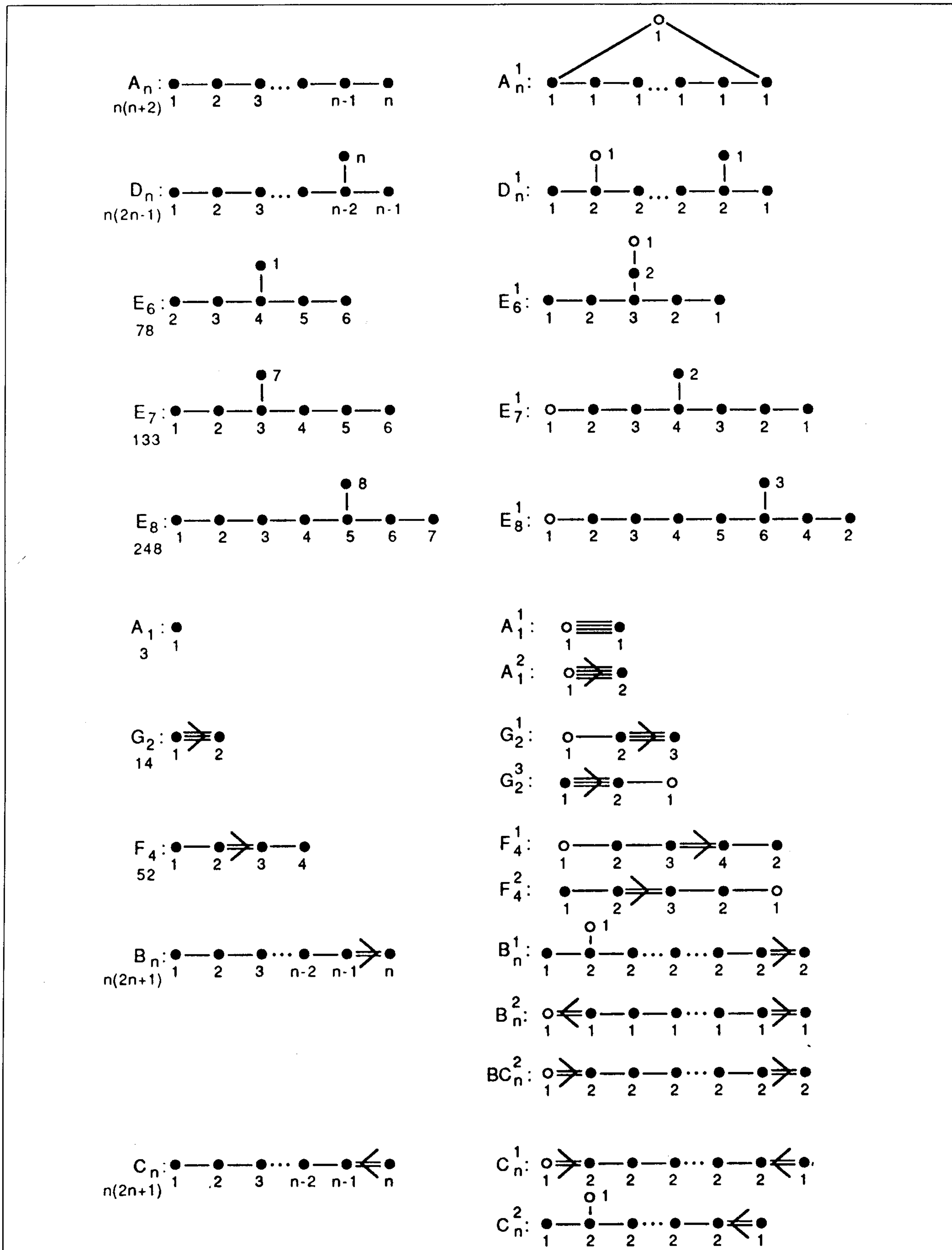


Table 1. Coxeter-Dynkin Diagram of the finite and affine Lie algebras.

Braunsberg, with a view of the thirteenth-century St. Catherine's Church.

per week in the classroom or tutoring. (Now many mathematicians consider 6 hours a week an intolerable burden!) On the recommendation of Weierstrass, Killing was appointed Professor of Mathematics at the Lyzeum Hosianum in Braunsberg in East Prussia (now Braniewo in the region of Olsztyn in Poland). This was a college founded in 1565 by Bishop Stanislaus Hosius, whose treatise on the Christian faith ran into 39 editions!

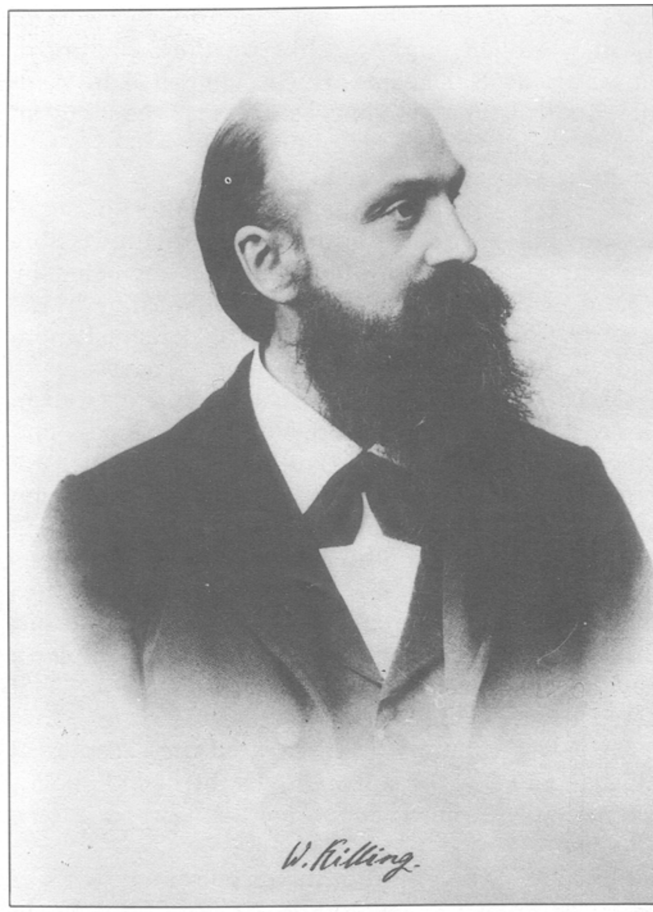
When Killing arrived the building of the Lyzeum must have looked very much as it appears in the accompanying picture. The main object of the college was the training of Roman Catholic clergy, so Killing had to teach a wide range of topics including the reconciliation of faith and science. Although he was isolated mathematically during his ten years in Braunsberg, this was the most creative period in his mathematical life. Killing produced his brilliant work despite worries about the health of his wife and seven children, demanding administrative duties as rector of the college and as a member and chairman of the City Council, and his active role in the church of St. Catherine.

Killing announced his ideas in the form of *Programmschriften* [15] from Braunsberg. These dealt with (i) Non-Euclidean geometries in n-dimensions (1883); (ii) "The Extension of the Concept of Space" (1884); and (iii) his first tentative thoughts about Lie's transformation groups (1886). Killing's original treatment of Lie algebras first appeared in (ii). It was only after this that he learned of Lie's work, most of which was inaccessible to Killing because it never occurred to the college librarian to subscribe to the *Archiv für Mathematik* of the University of Christiana (now, Oslo) where Lie published. Fortunately Engel played a role with respect to Killing similar to that of Halley with Newton, teasing out of him Z.v.G.I–IV, which appeared in the *Math. Annalen.*

In 1892 he was called back to his native Westphalia as professor of mathematics at the University of Münster, where he was quickly submerged in teaching, administration, and charitable activities. He was *Rector Magnificus* for some period and president of the St. Vincent de Paul charitable society for ten years.

Throughout his life Killing evinced a high sense of duty and a deep concern for anyone in physical or spiritual need. He was steeped in what the mathematician Engel characterized as "the rigorous Westphalian Catholicism of the 1850s and 1860s." St. Francis of Assisi was his model, so that at the age of 39 he, together with his wife, entered the Third Order of the Franciscans [24, p. 399]. His students loved and admired Killing because he gave himself unsparingly of time and energy to them, never being satisfied until they understood the matter at hand in depth [23]. Nor was Killing satisfied for them to become narrow specialists, so he spread his lectures over many topics beyond geometry and groups.

Killing was conservative in his political views and vigorously opposed the attempt to reform the examination requirements for graduate students at the University of Münster by deleting the compulsory study of philosophy. Engel comments "Killing could not see that for most candidates the test in philosophy was *vollständig wertlos*" (completely worthless). Nor do my sources suggest that he had much of a sense of humour. He had a profound patriotic love of his country, so that his last years (1918–1923) were deeply pained by the collapse of social cohesion in Germany after the War of 1914–18. Nonetheless, the accompanying photograph of Killing in his old age radiates kindliness and serenity. He was greatly cheered by the award of the Lobachevsky Prize by the Physico-Mathematical Society of Kazan in 1900 for his work in geometry.



Wilhelm Killing, probably about 1889–1891.

Why was Killing's Work Neglected?

Killing was a modest man with high standards; he vastly underrated his own achievement. His interest was geometry and for this he needed *all real* Lie algebras. To obtain merely the simple Lie algebras over the complex numbers did not appear to him to be very significant. Once Z.v.G.IV had appeared, Killing's research energies went back to *Raumformen*. I recall that one day in 1940 during the regular tea-coffee ritual in Fine Hall at Princeton, Marston Morse declaimed "A successful mathematician always believes that his current theorem is the most important piece of mathematics the world has ever seen." Few have lived this philosophy with more élan than Morse! And of course even though I immediately formed a deep-seated dislike for Morse, there *is* something in what he said. If *you* do not think your stuff is important, why will anyone else? But Morse's philosophy is far removed from St. Francis of Assisi!

Also Lie was quite negative about Killing's work. This, I suspect, was partly sour grapes, because Lie admitted that he had merely paged through Z.v.G.II. At the top of page 770 of Lie-Engel III [20] we find the following less than generous comment about Killing's 1886 Programmschrift: "with the exception of the preceding unproved theorem . . . all the theorems that are correct are due to Lie and all the false ones are due to Killing!"

According to Engel [9, p. 221/2] there was no love lost between Lie and Killing. This comes through in the nine references to Killing's work in volume III of [20]. With one exception they are negative and seem to have the purpose of proving that anything of value about transformation groups was first discovered by Lie. Even if this were true, it does not do justice to the fact that there was no possibility of Killing in Braunsberg knowing Lie's results published in Christiana. So if Lie's results are wonderful, Killing's independent discovery of them is equally wonderful!

It seems to me that even Hawkins, who has done more than anyone else to rehabilitate Killing, sometimes allows himself to be too greatly influenced by the widespread negativism surrounding Killing's work. The misunderstanding about the relation of Cartan to Killing would never have occurred if readers of Cartan's thesis had taken the trouble to look up the 63 references to Killing's papers that Cartan supplied.

Conclusion

Why do I think that Z.v.G.II was an epoch-making paper?

(1) It was the paradigm for subsequent efforts to classify the possible structures for any mathematical object. Hawkins [15] documents the fact that Killing's

paper was the immediate inspiration for the work of Cartan, Molien, and Maschke on the structure of linear *associative* algebras which culminated in Wedderburn's theorems. Killing's success was certainly an example which gave Richard Brauer the will to persist in the attempt to classify simple groups.

(2) Weyl's theory of the representation of semisimple Lie groups would have been impossible without ideas, results, and methods originated by Killing in Z.v.G.II. Weyl's fusion of global and local analysis laid the basis for the work of Harish-Chandra and the flowering of abstract harmonic analysis.

(3) The whole industry of root systems evinced in the writings of I. Macdonald, V. Kac, R. Moody, and others started with Killing. For the latest see [21].

(4) The Weyl group and the Coxeter transformation are in Z.v.G.II. There they are realized not as orthogonal motions of Euclidean space but as permutations of the roots. In my view, this is the proper way to think of them for general Kac-Moody algebras. Further, the conditions for symmetrisability which play a key role in Kac's book [17] are given on p. 21 of Z.v.G.II.

(5) It was Killing who discovered the exceptional Lie algebra E_8, which apparently is the main hope for saving Super-String Theory—not that I expect it to be saved!

(6) Roughly one third of the extraordinary work of Elie Cartan was based more or less directly on Z.v.G.II.

Euclid's *Elements* and Newton's *Principia* are more important than Z.v.G.II. But if you can name one paper in the past 200 years of equal significance to the paper which was sent off diffidently to Felix Klein on 2 February 1888 from an isolated outpost of Bismarck's empire, please inform the Editor of the *Mathematical Intelligencer*.

Acknowledgments

My debt to Thomas Hawkins will be obvious to anyone who has explored his fascinating historical writings. I am also most grateful to I. Kiessling of the University library in Münster and to K. Haenel of the library of Göttingen for invaluable information about Killing's life and the pictures which enliven this article.

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