Spring, 2004 CIS 700-05

Special Topics in Machine Perception

Homework 1

February 27, 2004; Due March 26

You may work in groups of 2 or 3. Please, write up your solutions as clearly and concisely as possible. Be rigorous! You will have to present your solutions of the problems during a special problem session.

Problem B1 (20). (a) If $K = \mathbb{R}$ or $K = \mathbb{C}$, recall that the projective space, $\mathbf{P}(K^{n+1})$, is the set of equivalence classes of the equivalence relation, $\sim$, on $K^{n+1} - \{0\}$, defined so that, for all $u, v \in K^{n+1} - \{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in K - \{0\}.$$

The map, $p\colon (K^{n+1} - \{0\}) \to \mathbf{P}(K^{n+1})$, is the projection mapping any nonzero vector in K^{n+1} to its equivalence class modulo $\sim$. We let $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$ and $\mathbb{CP}^n = \mathbf{P}(\mathbb{C}^{n+1})$.

Prove that for any $n \geq 0$, there is a bijection between $\mathbf{P}(K^{n+1})$ and $K^n \cup \mathbf{P}(K^n)$ (which allows us to identify them).

(b) Prove that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are connected and compact.

Hint. If

$$S^n = \{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

prove that $p(S^n) = \mathbf{P}(K^{n+1})$, and recall that S^n is compact for all $n \geq 0$ and connected for $n \geq 1$. For $n = 0$, $\mathbf{P}(K)$ consists of a single point.

Problem B2 (20). Recall that $\mathbb{R}^2$ and $\mathbb{C}$ can be identified using the bijection $(x, y) \mapsto x + iy$. Also recall that the subset $U(1) \subseteq \mathbb{C}$ consisting of all complex numbers of the form $\cos\theta + i\sin\theta$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If $c\colon U(1) \to U(1)$ is the map defined such that

$$c(z) = z^2,$$

prove that $c(z_1) = c(z_2)$ iff either $z_2 = z_1$ or $z_2 = -z_1$, and thus that c induces a bijective map $\widehat{c}\colon \mathbb{RP}^1 \to S^1$. Prove that $\widehat{c}$ is a homeomorphism (remember that $\mathbb{RP}^1$ is compact).

Problem B3 (40). (i) In $\mathbb{R}^3$, the sphere S^2 is the set of points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$. The point $N = (0, 0, 1)$ is called the *north pole*, and the point $S = (0, 0, -1)$ is called the *south pole*. The *stereographic projection map* $\sigma_N\colon (S^2 - \{N\}) \to \mathbb{R}^2$ is defined as follows: For every point $M \neq N$ on S^2, the point $\sigma_N(M)$ is the intersection of the line through N and M and the plane of equation $z = 0$. Show that if M has coordinates (x, y, z) (with $x^2 + y^2 + z^2 = 1$), then

$$\sigma_N(M) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big).$$

Prove that σ_N is bijective and that its inverse is given by the map $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$, with

$$(x, y) \mapsto \Big(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\Big).$$

Similarly, $\sigma_S \colon (S^2 - \{S\}) \to \mathbb{R}^2$ is defined as follows: For every point $M \neq S$ on S^2, the point $\sigma_S(M)$ is the intersection of the line through S and M and the plane of equation $z = 0$. Show that

$$\sigma_S(M) = \Big(\frac{x}{1+z}, \frac{y}{1+z}\Big).$$

Prove that σ_S is bijective and that its inverse is given by the map $\tau_S \colon \mathbb{R}^2 \to (S^2 - \{S\})$, with

$$(x, y) \mapsto \Big(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1}\Big).$$

Using the complex number $u = x + iy$ to represent the point (x, y), the maps $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$ and $\sigma_N \colon (S^2 - \{N\}) \to \mathbb{R}^2$ can be viewed as maps from $\mathbb{C}$ to $(S^2 - \{N\})$ and from $(S^2 - \{N\})$ to $\mathbb{C}$, defined such that

$$\tau_N(u) = \Big(\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\Big)$$

and

$$\sigma_N(u, z) = \frac{u}{1-z},$$

and similarly for τ_S and σ_S. Prove that if we pick two suitable orientations for the xy-plane, we have

$$\sigma_N(M)\sigma_S(M) = 1,$$

for every $M \in S^2 - \{N, S\}$.

(ii) Identifying $\mathbb{C}^2$ and $\mathbb{R}^4$, for $z = x + iy$ and $z' = x' + iy'$, we define

$$\|(z, z')\| = \sqrt{x^2 + y^2 + x'^2 + y'^2}.$$

The sphere S^3 is the subset of $\mathbb{C}^2$ (or $\mathbb{R}^4$) consisting of those points (z, z') such that $\|(z, z')\|^2 = 1$.

Prove that $\mathbf{P}(\mathbb{C}^2) = p(S^3)$, where $p \colon (\mathbb{C}^2 - \{(0, 0)\}) \to \mathbf{P}(\mathbb{C}^2)$ is the projection map. If we let $u = z/z'$ (where $z, z' \in \mathbb{C}$) in the map

$$u \mapsto \Big(\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\Big)$$

and require that $\|(z,z')\|^2 = 1$, show that we get the map $HF\colon S^3 \to S^2$ defined such that

$$HF((z,z')) = (2z\overline{z'},\ |z|^2 - |z'|^2).$$

Prove that $HF\colon S^3 \to S^2$ induces a bijection $\widehat{HF}\colon \mathbf{P}(\mathbb{C}^2) \to S^2$, and thus that $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$ is homeomorphic to S^2.

(iii) Prove that the inverse image $HF^{-1}(s)$ of every point $s \in S^2$ is a circle. Thus S^3 can be viewed as a union of disjoint circles. The map HF is called the *Hopf fibration*.

Problem B4 (60). (a) Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x,y,z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x,y,z) = \mathcal{H}(x',y',z')$ iff $(x',y',z') = (x,y,z)$ or $(x',y',z') = (-x,-y,-z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\begin{aligned}
\psi_1(u,v) &= \left(\frac{uv}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1}\right),\\
\psi_2(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{u^2-1}{u^2+v^2+1}\right),\\
\psi_3(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1-u^2}{u^2+v^2+1}\right).
\end{aligned}$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1}{u^2+v^2+1}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{u^2+v^2+1}, \frac{1}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}\right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{1}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}\right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian is never zero).

Prove that $\psi_i(\mathbb{R}^2)$ is an open subset, U_i, of $\mathcal{H}(S^2)$ for $i = 1, 2, 3$ and that the union of the U_i's covers $\mathcal{H}(S^2)$.

Prove that each $\psi_i^{-1}: U_i \to \mathbb{R}^2$ is continuous. This is a little tricky. For example, for ψ_1, first prove that if the coordinates in $\mathbb{R}^4$ are (x, y, z, t), then

$$yzt = x(y^2 - z^2).$$

Then, ψ_1^{-1} is defined as follows: If $y \neq 0$ and $z \neq 0$,

$$u = \frac{x}{z} = \frac{yt}{y^2 - z^2}, \quad v = \frac{x}{y} = \frac{zt}{y^2 - z^2}.$$

If $y = 0$ and $z \neq 0$, then

$$u = 0, \quad v = -\frac{t}{z},$$

if $y \neq 0$ and $z = 0$, then

$$u = \frac{t}{y} \quad v = 0,$$

and if $y = z = 0$, then

$$u = 0, \quad v = 0.$$

Finally, you have to show continuity of the above functions, and do a similar thing for ψ_2^{-1} and ψ_3^{-1}.

Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a manifold in $\mathbb{R}^4$.

(c) Investigate the surfaces in $\mathbb{R}^3$ obtained by dropping one of the four coordinates. Show that there are only two of them (the "Steiner Roman surface" and the "crosscap", up to a rigid motion).

Problem B5 (30). (i) Prove that the *Veronese map* $V_2: \mathbb{R}^3 \to \mathbb{R}^6$ defined such that

$$V_2(x, y, z) = (x^2,\, y^2,\, z^2,\, yz,\, zx,\, xy)$$

induces a homeomorphism of $\mathbb{RP}^2$ onto $V_2(S^2)$. Show that $V_2(S^2)$ is a subset of the hyperplane $x_1 + x_2 + x_3 = 1$ in $\mathbb{R}^6$, and thus that $\mathbb{RP}^2$ is homeomorphic to a subset of $\mathbb{R}^5$. Prove that this homeomorphism is smooth.

(ii) Prove that the *Veronese map* $V_3: \mathbb{R}^4 \to \mathbb{R}^{10}$ defined such that

$$V_3(x, y, z, t) = (x^2,\, y^2,\, z^2,\, t^2,\, xy,\, yz,\, xz,\, xt,\, yt,\, zt)$$

induces a homeomorphism of $\mathbb{RP}^3$ onto $V_3(S^3)$. Show that $V_3(S^3)$ is a subset of the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ in $\mathbb{R}^{10}$, and thus that $\mathbb{RP}^3$ is homeomorphic to a subset of $\mathbb{R}^9$. Prove that this homeomorphism is smooth.

Problem B6 (20 pts). Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2 \cosh \omega$.

Prove that the exponential map, $\exp \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathbf{SL}(2, \mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2, \mathbb{C})$.

Problem B7 (10 pts). Prove that the kernel of the homomorphism, $\varphi \colon \mathbf{SL}(2, \mathbb{C}) \to \mathbf{SO}_0(1, 3)$, given on page 26 of the notes on Group actions, Manifolds, etc., is indeed $\{-I, I\}$.

Problem B8 (20 pts). Prove that $\mathbf{GL}(n, \mathbb{C})$, $\mathbf{SL}(n, \mathbb{C})$, $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$ and $\mathbf{SO}(n)$ and are arcwise-connected if $n \geq 1$. What about $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{O}(n)$?
Hint. Use the polar form and/or various exponential maps.

Problem B9 (50 pts). Let A and B be the following 4×4-matrices:

$$A = \begin{pmatrix} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

where $\theta_1, \theta_2 \geq 0$.

(i) Compute A^2, and prove that

$$B = e^A,$$

where

$$e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!} = \sum_{p \geq 0} \frac{A^p}{p!},$$

letting $A^0 = I_n$. Use this to prove that for every orthogonal 4×4-matrix B, there is a skew symmetric matrix A such that

$$B = e^A.$$

(ii) Given a skew-symmetric 4×4-matrix A, prove that there are two skew symmetric matrices A_1 and A_2 and some $\theta_1, \theta_2 \geq 0$, such that

$$\begin{aligned} A &= A_1 + A_2, \\ A_1^3 &= -\theta_1^2 A_1, \\ A_2^3 &= -\theta_2^2 A_2, \\ A_1 A_2 &= A_2 A_1 = 0, \\ tr(A_1^2) &= -2\theta_1^2, \\ tr(A_2^2) &= -2\theta_2^2, \end{aligned}$$

and where $A_i = 0$ if $\theta_i = 0$ and $A_1^2 + A_2^2 = -\theta_1^2 I_4$ if $\theta_2 = \theta_1$.

Using the above, prove that

$$e^A = I_4 + \frac{\sin\theta_1}{\theta_1} A_1 + \frac{\sin\theta_2}{\theta_2} A_2 + \frac{(1-\cos\theta_1)}{\theta_1^2} A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2} A_2^2.$$

(iii) Given an orthogonal 4×4-matrix B, prove that there are two skew symmetric matrices A_1 and A_2 and some $\theta_1, \theta_2 \geq 0$, such that

$$B = I_4 + \frac{\sin\theta_1}{\theta_1} A_1 + \frac{\sin\theta_2}{\theta_2} A_2 + \frac{(1-\cos\theta_1)}{\theta_1^2} A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2} A_2^2.$$

where

$$\begin{aligned} A_1^3 &= -\theta_1^2 A_1, \\ A_2^3 &= -\theta_2^2 A_2, \\ A_1 A_2 &= A_2 A_1 = 0, \\ tr(A_1^2) &= -2\theta_1^2, \\ tr(A_2^2) &= -2\theta_2^2, \end{aligned}$$

and where $A_i = 0$ if $\theta_i = 0$ and $A_1^2 + A_2^2 = -\theta_1^2 I_4$ if $\theta_2 = \theta_1$. Prove that

$$\begin{aligned} 1/2(B - B^\top) &= \frac{\sin\theta_1}{\theta_1} A_1 + \frac{\sin\theta_2}{\theta_2} A_2, \\ 1/2(B + B^\top) &= I_4 + \frac{(1-\cos\theta_1)}{\theta_1^2} A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2} A_2^2, \\ tr(B) &= 2\cos\theta_1 + 2\cos\theta_2. \end{aligned}$$

(iv) Prove that if $\sin\theta_1 = 0$ or $\sin\theta_2 = 0$, then A_1, A_2 and the $\cos\theta_i$ can be computed from B. Prove that if $\theta_2 = \theta_1$, then

$$B = \cos\theta_1 I_4 + \frac{\sin\theta_1}{\theta_1}(A_1 + A_2),$$

and $\cos\theta_1$ and $A_1 + A_2$ can be computed from B.

(v) Prove that

$$\frac{1}{4}tr((B - B^\top)^2) = 2\cos^2\theta_1 + 2\cos^2\theta_2 - 4.$$

Prove that $\cos\theta_1$ and $\cos\theta_2$ are solutions of the equation

$$x^2 - sx + p = 0,$$

where

$$s = \frac{1}{2}tr(B), \quad p = \frac{1}{8}(tr(B))^2 - \frac{1}{16}tr((B - B^\top)^2) - 1.$$

Prove that we also have

$$\cos^2\theta_1 \cos^2\theta_2 = \det\left(1/2(B + B^\top)\right).$$

If $\sin\theta_i \neq 0$ for $i = 1, 2$ and $\cos\theta_2 \neq \cos\theta_1$, prove that the system

$$\begin{aligned} 1/2(B - B^\top) &= \frac{\sin\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2}{\theta_2}A_2, \\ 1/4(B + B^\top)(B - B^\top) &= \frac{\sin\theta_1\cos\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2\cos\theta_2}{\theta_2}A_2, \end{aligned}$$

has a unique solution for A_1 and A_2.

(vi) Prove that $A = A_1 + A_2$ has an orthonormal basis of eigenvectors such that the first two are a basis of the plane w.r.t. which B is a rotation of angle θ_1, and the last two are a basis of the plane w.r.t. which B is a rotation of angle θ_2.

Remark: I don't know a simple way to compute such an orthonormal basis of eigenvectors of $A = A_1 + A_2$, but it should be possible!

TOTAL: 270 points.