Chapter 4

Basics of Euclidean Geometry

4.1 Inner Products, Euclidean Spaces

In Affine geometry, it is possible to deal with ratios of vectors and barycenters of points, but there is no way to express the notion of length of a line segment, or to talk about orthogonality of vectors.

A Euclidean structure will allow us to deal with *metric notions* such as orthogonality and length (or distance).

We begin by defining inner products and Euclidean Spaces. The Cauchy-Schwarz inequality and the Minkovski inequality are shown.

We define othogonality of vectors and of subspaces, othogonal families, and orthonormal families. We offer a glimpse at Fourier series in terms of the orthogonal families $(\sin px)_{p\geq 1} \cup (\cos qx)_{q\geq 0}$ and $(e^{ikx})_{k\in\mathbb{Z}}$.

We prove that every finite dimensional Euclidean space has orthonormal bases.

The first proof uses duality, and the second one the Gram-Schmidt procedure. The QR-decomposition of matrices is shown as an application.

Linear isometries (also called orthogonal transformations) are defined and studied briefly.

The orthogonal group and orthogonal matrices are studied briefly.

First, we define a Euclidean structure on a vector space, and then, on an affine space.

Definition 4.1.1 A real vector space E is a *Euclidean space* iff it is equipped with a symmetric bilinear form $\varphi: E \times E \to \mathbb{R}$ which is also *positive definite*,which means that

$$\varphi(u, u) > 0, \quad \text{for every } u \neq 0.$$

More explicitly, $\varphi: E \times E \to \mathbb{R}$ satisfies the following axioms:

$$\begin{aligned}
\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v), \\
\varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2), \\
\varphi(\lambda u, v) &= \lambda\varphi(u, v), \\
\varphi(u, \lambda v) &= \lambda\varphi(u, v), \\
\varphi(u, v) &= \varphi(v, u), \\
u \neq 0 \text{ implies that } & \varphi(u, u) > 0.
\end{aligned}$$

The real number $\varphi(u, v)$ is also called the *inner product (or scalar product) of* u *and* v.

We also define the *quadratic form associated with* φ as the function $\Phi\colon E \to \mathbb{R}_+$ such that

$$\Phi(u) = \varphi(u, u),$$

for all $u \in E$.

Since φ is bilinear, we have $\varphi(0, 0) = 0$, and since it is positive definite, we have the stronger fact that

$$\varphi(u, u) = 0 \quad \text{iff} \quad u = 0,$$

that is $\Phi(u) = 0$ iff $u = 0$.

Given an inner product $\varphi\colon E \times E \to \mathbb{R}$ on a vector space E, we also denote $\varphi(u, v)$ by

$$u \cdot v, \quad \text{or} \quad \langle u, v\rangle, \quad \text{or} \quad (u|v),$$

and $\sqrt{\Phi(u)}$ as $\|u\|$.

Example 1. The standard example of a Euclidean space is $\mathbb{R}^n$, under the inner product $\cdot$ defined such that

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Example 2. Let E be a vector space of dimension 2, and let (e_1, e_2) be a basis of E.

If $a > 0$ and $b^2 - ac < 0$, the bilinear form defined such that

$$\varphi(x_1e_1+y_1e_2,\, x_2e_1+y_2e_2) = ax_1x_2+b(x_1y_2+x_2y_1)+cy_1y_2$$

yields a Euclidean structure on E.

In this case,

$$\Phi(xe_1 + ye_2) = ax^2 + 2bxy + cy^2.$$

Example 3. Let $\mathcal{C}[a,b]$ denote the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. It is easily checked that $\mathcal{C}[a,b]$ is a vector space of infinite dimension.

Given any two functions $f, g \in \mathcal{C}[a,b]$, let

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt.$$

We leave as an easy exercise that $\langle -, -\rangle$ is indeed an inner product on $\mathcal{C}[a,b]$.

When $[a,b] = [-\pi, \pi]$ (or $[a,b] = [0, 2\pi]$, this makes basically no difference), one should compute

$$\langle \sin px, \sin qx\rangle, \quad \langle \sin px, \cos qx\rangle,$$
$$\text{and} \quad \langle \cos px, \cos qx\rangle,$$

for all natural numbers $p, q \geq 1$. The outcome of these calculations is what makes Fourier analysis possible!

Let us observe that φ can be recovered from Φ. Indeed, by bilinearity and symmetry, we have

$$\begin{aligned}\Phi(u+v) &= \varphi(u+v,\, u+v)\\ &= \varphi(u,\, u+v) + \varphi(v,\, u+v)\\ &= \varphi(u,\, u) + 2\varphi(u,\, v) + \varphi(v,\, v)\\ &= \Phi(u) + 2\varphi(u,\, v) + \Phi(v).\end{aligned}$$

Thus, we have

$$\varphi(u,\, v) = \frac{1}{2}[\Phi(u+v) - \Phi(u) - \Phi(v)].$$

We also say that φ is the *polar form of* Φ.

One of the very important properties of an inner product φ is that the map $u \mapsto \sqrt{\Phi(u)}$ is a norm.

Lemma 4.1.2 *Let E be a Euclidean space with inner product φ and quadratic form Φ. For all $u, v \in E$, we have the* Cauchy-Schwarz inequality*:*

$$\varphi(u, v)^2 \leq \Phi(u)\Phi(v),$$

the equality holding iff u and v are linearly dependent.

We also have the Minkovski inequality*:*

$$\sqrt{\Phi(u+v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)},$$

the equality holding iff u and v are linearly dependent, where in addition if $u \neq 0$ and $v \neq 0$, then $u = \lambda v$ for some $\lambda > 0$.

Sketch of proof. Define the function $T\colon \mathbb{R} \to \mathbb{R}$, such that

$$T(\lambda) = \Phi(u + \lambda v),$$

for all $\lambda \in \mathbb{R}$. Using bilinearity and symmetry, we can show that

$$\Phi(u + \lambda v) = \Phi(u) + 2\lambda\varphi(u, v) + \lambda^2\Phi(v).$$

Since φ is positive definite, we have $T(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$.

If $\Phi(v) = 0$, then $v = 0$, and we also have $\varphi(u, v) = 0$. In this case, the Cauchy-Schwarz inequality is trivial, If $\Phi(v) > 0$, then

$$\lambda^2\Phi(v) + 2\lambda\varphi(u, v) + \Phi(u) = 0$$

can't have distinct roots, which means that its discriminant

$$\Delta = 4(\varphi(u, v)^2 - \Phi(u)\Phi(v))$$

is zero or negative, which is precisely the Cauchy-Schwarz inequality.

The Minkovski inequality can then be shown.

Let us review the definition of a normed vector space.

Definition 4.1.3 Let E be a vector space over a field K, where K is either the field $\mathbb{R}$ of reals, or the field $\mathbb{C}$ of complex numbers. A *norm on* E is a function $\| \,\| \colon E \to \mathbb{R}_+$, assigning a nonnegative real number $\|u\|$ to any vector $u \in E$, and satisfying the following conditions for all $x, y, z \in E$:

(N1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$. (positivity)

(N2) $\|\lambda x\| = |\lambda| \; \|x\|$. (scaling)

(N3) $\|x + y\| \leq \|x\| + \|y\|$. (triangle inequality)

A vector space E together with a norm $\| \,\|$ is called a *normed vector space*.

From (N3), we easily get

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| \, .$$

The Minkovski inequality

$$\sqrt{\Phi(u+v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}$$

shows that the map $u \mapsto \sqrt{\Phi(u)}$ satisfies the *triangle inequality*, condition (N3) of definition 4.1.3, and since φ is bilinear and positive definite, it also satisfies conditions (N1) and (N2) of definition 4.1.3, and thus, it is a *norm* on E.

The norm induced by φ is called the *Euclidean norm induced by φ*.

Note that the Cauchy-Schwarz inequality can be written as

$$|u \cdot v| \leq \|u\| \, \|v\| \, ,$$

and the Minkovski inequality as

$$\|u + v\| \leq \|u\| + \|v\| \, .$$

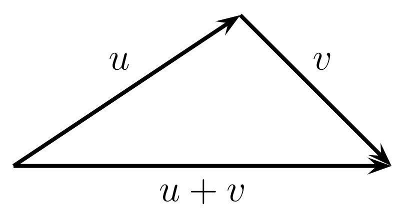

Figure 4.1: The triangle inequality

We now define orthogonality.

4.2 Orthogonality

Definition 4.2.1 Given a Euclidean space E, any two vectors $u, v \in E$ are *orthogonal, or perpendicular* iff $u \cdot v = 0$. Given a family $(u_i)_{i\in I}$ of vectors in E, we say that $(u_i)_{i\in I}$ is *orthogonal* iff $u_i \cdot u_j = 0$ for all $i, j \in I$, where $i \neq j$. We say that the family $(u_i)_{i\in I}$ is *orthonormal* iff $u_i \cdot u_j = 0$ for all $i, j \in I$, where $i \neq j$, and $\|u_i\| = u_i \cdot u_i = 1$, for all $i \in I$. For any subset F of E, the set

$$F^{\perp} = \{v \in E \mid u \cdot v = 0, \text{ for all } u \in F\},$$

of all vectors orthogonal to all vectors in F, is called the *orthogonal complement of F*.

Since inner products are positive definite, observe that for any vector $u \in E$, we have

$$u \cdot v = 0 \quad \text{for all } v \in E \quad \text{iff} \quad u = 0.$$

It is immediately verified that the orthogonal complement $F^{\perp}$ of F is a subspace of E.

Example 4. Going back to example 3, and to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

on the vector space $\mathcal{C}[-\pi, \pi]$, it is easily checked that

$$\langle \sin px, \sin qx\rangle = \begin{cases} \pi & \text{if } p = q,\ p, q \geq 1, \\ 0 & \text{if } p \neq q,\ p, q \geq 1 \end{cases}$$

$$\langle \cos px, \cos qx\rangle = \begin{cases} \pi & \text{if } p = q,\ p, q \geq 1, \\ 0 & \text{if } p \neq q,\ p, q \geq 0 \end{cases}$$

and

$$\langle \sin px, \cos qx\rangle = 0,$$

for all $p \geq 1$ and $q \geq 0$, and of course,
$\langle 1, 1\rangle = \int_{-\pi}^{\pi} dx = 2\pi$.

As a consequence, the family $(\sin px)_{p\geq 1} \cup (\cos qx)_{q\geq 0}$ is orthogonal.

It is not orthonormal, but becomes so if we divide every trigonometric function by $\sqrt{\pi}$, and 1 by $\sqrt{2\pi}$.

Remark: Observe that if we allow complex valued functions, we obtain simpler proofs. For example, it is immediately checked that

$$\int_{-\pi}^{\pi} e^{ikx} dx = \begin{cases} 2\pi & \text{if } k = 0; \\ 0 & \text{if } k \neq 0, \end{cases}$$

because the derivative of e^{ikx} is ike^{ikx}.

However, beware that something strange is going on!

Indeed, unless $k = 0$, we have

$$\langle e^{ikx}, e^{ikx} \rangle = 0,$$

since

$$\langle e^{ikx}, e^{ikx} \rangle = \int_{-\pi}^{\pi} (e^{ikx})^2 dx = \int_{-\pi}^{\pi} e^{i2kx} dx = 0.$$

The inner product $\langle e^{ikx}, e^{ikx} \rangle$ should be strictly positive. What went wrong?

The problem is that we are using the wrong inner product. When we use complex-valued functions, we must use the *Hermitian inner product*

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx,$$

where $\overline{g(x)}$ is the *conjugate* of $g(x)$.

The Hermitian inner product is not symmetric. Instead,

$$\langle g, f\rangle = \overline{\langle f, g\rangle}.$$

(Recall that if $z = a+ib$, where $a, b \in \mathbb{R}$, then $\overline{z} = a-ib$. Also $e^{i\theta} = \cos\theta + i\sin\theta$).

With the Hermitian inner product, everthing works out beautifully! In particular, the family $(e^{ikx})_{k\in\mathbb{Z}}$ is orthogonal.

Lemma 4.2.2 *Given a Euclidean space E, for any family $(u_i)_{i\in I}$ of nonnull vectors in E, if $(u_i)_{i\in I}$ is orthogonal, then it is linearly independent.*

Lemma 4.2.3 *Given a Euclidean space E, any two vectors $u, v \in E$ are orthogonal iff*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 .$$

One of the most useful features of orthonormal bases is that they afford a very simple method for computing the coordinates of a vector over any basis vector.

Indeed, assume that $(e_1, \ldots, e_m)$ is an orthonormal basis. For any vector

$$x = x_1e_1 + \cdots + x_me_m,$$

if we compute the inner product $x \cdot e_i$, we get

$$x \cdot e_i = x_1 e_1 \cdot e_i + \cdots + x_i e_i \cdot e_i + \cdots + x_m e_m \cdot e_i = x_i,$$

since

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

is the property characterizing an orthonormal family.

Thus,

$$x_i = x \cdot e_i,$$

which means that $x_i e_i = (x \cdot e_i) e_i$ is the orthogonal projection of x onto the subspace generated by the basis vector e_i.

If the basis is orthogonal but not necessarily orthonormal, then

$$x_i = \frac{x \cdot e_i}{e_i \cdot e_i} = \frac{x \cdot e_i}{\|e_i\|^2}.$$

All this is true even for an infinite orthonormal (or orthogonal) basis $(e_i)_{i \in I}$.

However, remember that every vector x is expressed as a linear combination

$$x = \sum_{i \in I} x_i e_i$$

where the family of scalars $(x_i)_{i \in I}$ has **finite support**, which means that $x_i = 0$ for all $i \in I - J$, where J is a finite set.

Thus, even though the family $(\sin px)_{p \geq 1} \cup (\cos qx)_{q \geq 0}$ is orthogonal (it is not orthonormal, but becomes one if we divide every trigonometric function by $\sqrt{\pi}$, and 1 by $\sqrt{2\pi}$; we won't because it looks messy!), the fact that a function $f \in \mathcal{C}^0[-\pi, \pi]$ can be written as a Fourier series as

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

does not mean that $(\sin px)_{p \geq 1} \cup (\cos qx)_{q \geq 0}$ is a basis of this vector space of functions, because in general, the families (a_k) and (b_k) **do not** have finite support!

In order for this infinite linear combination to make sense, it is necessary to prove that the partial sums

$$a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of the series converge to a limit when n goes to infinity.

This requires a topology on the space.

Still, a small miracle happens. If $f \in \mathcal{C}[-\pi, \pi]$ can indeed be expressed as a Fourier series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

the coefficients a_0 and $a_k, b_k,\ k \geq 1$, can be computed by projecting f over the basis functions, i.e. by taking inner products with the basis functions in $(\sin px)_{p\geq 1} \cup (\cos qx)_{q\geq 0}$.

Indeed, for all $k \geq 1$, we have

$$a_0 = \frac{\langle f, 1 \rangle}{\|1\|^2},$$

and

$$a_k = \frac{\langle f, \cos kx \rangle}{\|\cos kx\|^2}, \qquad b_k = \frac{\langle f, \sin kx \rangle}{\|\sin kx\|^2},$$

that is

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx,$$

and

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx.$$

If we allow f to be complex-valued and use the family $(e^{ikx})_{k\in\mathbb{Z}}$, which is is indeed orthogonal w.r.t. the Hermitian inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx,$$

we consider functions $f \in \mathcal{C}[-\pi, \pi]$ that can be expressed as the sum of a series

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}.$$

Note that the index k is allowed to be a negative integer. Then, the formula giving the c_k is very nice:

$$c_k = \frac{\langle f, e^{ikx}\rangle}{\left\|e^{ikx}\right\|^2},$$

that is

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$

Note the presence of the negative sign in e^{-ikx}, which is due to the fact that the inner product is Hermitian.

Of course, the real case can be recovered from the complex case. If f is a real-valued function, then we must have

$$a_k = c_k + c_{-k} \quad \text{and} \quad b_k = i(c_k - c_{-k}).$$

Also note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

is not only defined for all discrete values $k \in \mathbb{Z}$, but for all $k \in \mathbb{R}$, and that if f is continuous over $\mathbb{R}$, the integral makes sense.

This suggests defining

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx,$$

called the *Fourier transform* of f. It analyses the function f in the "frequency domain" in terms of its spectrum of harmonics.

Amazingly, there is an inverse Fourier transform (change e^{-ikx} to e^{+ikx} and divide by the scale factor 2π) which reconstructs f (under certain assumptions on f).

A very important property of Euclidean spaces of finite dimension is that the inner product induces a canonical bijection (i.e., independent of the choice of bases) between the vector space E and its dual E^*.

Given a Euclidean space E, for any vector $u \in E$, let $\varphi_u \colon E \to \mathbb{R}$ be the map defined such that

$$\varphi_u(v) = u \cdot v,$$

for all $v \in E$.

Since the inner product is bilinear, the map φ_u is a linear form in E^*.

Thus, we have a map $\flat\colon E \to E^*$, defined such that

$$\flat(u) = \varphi_u.$$

Lemma 4.2.4 *Given a Euclidean space E, the map $\flat\colon E \to E^*$, defined such that*

$$\flat(u) = \varphi_u,$$

is linear and injective. When E is also of finite dimension, the map $\flat\colon E \to E^*$ is a canonical isomorphism.

The inverse of the isomorphism $\flat\colon E \to E^*$ is denoted by $\sharp\colon E^* \to E$.

As a consequence of lemma 4.2.4, if E is a Euclidean space of finite dimension, every linear form $f \in E^*$ corresponds to a unique $u \in E$, such that

$$f(v) = u \cdot v,$$

for every $v \in E$.

In particular, if f is not the null form, the kernel of f, which is a hyperplane H, is precisely the set of vectors that are orthogonal to u.

Lemma 4.2.4 allows us to define the adjoint of a linear map on a Euclidean space.

Let E be a Euclidean space of finite dimension n, and let $f: E \to E$ be a linear map.

For every $u \in E$, the map

$$v \mapsto u \cdot f(v)$$

is clearly a linear form in E^*, and by lemma 4.2.4, there is a unique vector in E denoted as $f^*(u)$, such that

$$f^*(u) \cdot v = u \cdot f(v),$$

for every $v \in E$.

Lemma 4.2.5 *Given a Euclidean space E of finite dimension, for every linear map $f: E \to E$, there is a unique linear map $f^*: E \to E$, such that*

$$f^*(u) \cdot v = u \cdot f(v),$$

for all $u, v \in E$. The map f^* is called the adjoint of f (w.r.t. to the inner product).

Linear maps $f\colon E \to E$ such that $f = f^*$ are called *self-adjoint* maps.

They play a very important role because they have real eigenvalues, and because orthonormal bases arise from their eigenvectors.

Furthermore, many physical problems lead to self-adjoint linear maps (in the form of symmetric matrices).

Linear maps such that $f^{-1} = f^*$, or equivalently

$$f^* \circ f = f \circ f^* = \mathrm{id},$$

also play an important role. They are *isometries*. Rotations are special kinds of isometries.

Another important class of linear maps are the linear maps satisfying the property

$$f^* \circ f = f \circ f^*,$$

called *normal linear maps*.

We will see later on that normal maps can always be diagonalized over orthonormal bases of eigenvectors, but this will require using a Hermitian inner product (over $\mathbb{C}$).

Given two Euclidean spaces E and F, where the inner product on E is denoted as $\langle -, - \rangle_1$ and the inner product on F is denoted as $\langle -, - \rangle_2$, given any linear map $f \colon E \to F$, it is immediately verified that the proof of lemma 4.2.5 can be adapted to show that there is a unique linear map $f^* \colon F \to E$ such that

$$\langle f(u), v \rangle_2 = \langle u, f^*(v) \rangle_1$$

for all $u \in E$ and all $v \in F$. The linear map f^* is also called the adjoint of f.

Remark: Given any basis for E and any basis for F, it is possible to characterize the matrix of the adjoint f^* of f in terms of the matrix of f, and the symmetric matrices defining the inner products. We will do so with respect to orthonormal bases.

We can also use lemma 4.2.4 to show that any Euclidean space of finite dimension has an orthonormal basis.

Lemma 4.2.6 *Given any nontrivial Euclidean space E of finite dimension $n \geq 1$, there is an orthonormal basis $(u_1, \ldots, u_n)$ for E.*

There is a more constructive way of proving lemma 4.2.6, using a procedure known as the *Gram–Schmidt orthonormalization procedure*.

Among other things, the Gram–Schmidt orthonormalization procedure yields the so-called *QR-decomposition for matrices*, an important tool in numerical methods.

Lemma 4.2.7 *Given any nontrivial Euclidean space E of dimension $n \geq 1$, from any basis $(e_1, \ldots, e_n)$ for E, we can construct an orthonormal basis $(u_1, \ldots, u_n)$ for E, with the property that for every k, $1 \leq k \leq n$, the families $(e_1, \ldots, e_k)$ and $(u_1, \ldots, u_k)$ generate the same subspace.*

Proof. We proceed by induction on n. For $n = 1$, let

$$u_1 = \frac{e_1}{\|e_1\|}.$$

For $n \geq 2$, we define the vectors u_k and u'_k as follows.

$$u'_1 = e_1, \qquad u_1 = \frac{u'_1}{\|u'_1\|},$$

and for the inductive step

$$u'_{k+1} = e_{k+1} - \sum_{i=1}^{k} (e_{k+1} \cdot u_i)\, u_i, \qquad u_{k+1} = \frac{u'_{k+1}}{\|u'_{k+1}\|}.$$

We need to show that u'_{k+1} is nonzero, and we conclude by induction.

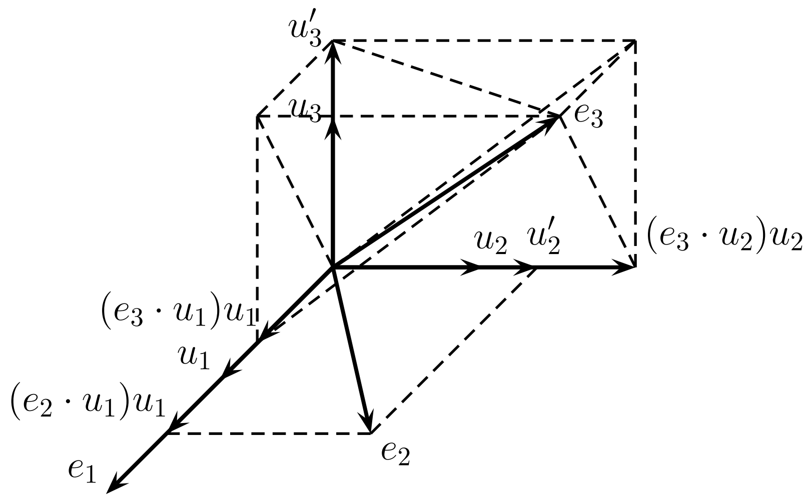

Figure 4.2: The Gram-Schmidt orthonormalization procedure

Remarks:

(1) Note that u'_{k+1} is obtained by subtracting from e_{k+1} the projection of e_{k+1} itself onto the orthonormal vectors $u_1, \ldots, u_k$ that have already been computed. Then, we normalize u'_{k+1}.

The QR-decomposition can now be obtained very easily. We will do this in section 4.4.

(2) We could compute u'_{k+1} using the formula

$$u'_{k+1} = e_{k+1} - \sum_{i=1}^{k} \left(\frac{e_{k+1} \cdot u'_i}{\|u'_i\|^2} \right) u'_i,$$

and normalize the vectors u'_k at the end.

This time, we are subtracting from e_{k+1} the projection of e_{k+1} itself onto the orthogonal vectors $u'_1, \ldots, u'_k$.

This might be preferable when writing a computer program.

(3) The proof of lemma 4.2.7 also works for a countably infinite basis for E, producing a countably infinite orthonormal basis.

Example 5. If we consider polynomials and the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt,$$

applying the Gram–Schmidt orthonormalization procedure to the polynomials

$$1, x, x^2, \ldots, x^n, \ldots,$$

which form a basis of the polynonials in one variable with real coefficients, we get a family of orthonormal polynomials $Q_n(x)$ related to the *Legendre polynomials*.

The Legendre polynomials $P_n(x)$ have many nice properties. They are orthogonal, but their norm is not always 1. The Legendre polynomials $P_n(x)$ can be defined as follows:

If we let f_n be the function

$$f_n(x) = (x^2 - 1)^n,$$

we define $P_n(x)$ as follows:

$$P_0(x) = 1, \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x),$$

where $f_n^{(n)}$ is the nth derivative of f_n.

They can also be defined inductively as follows:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_{n+1}(x) &= \frac{2n+1}{n+1} \, x P_n(x) - \frac{n}{n+1} \, P_{n-1}(x). \end{aligned}$$

It turns out that the polynomials Q_n are related to the Legendre polynomials P_n as follows:

$$Q_n(x) = \frac{2^n (n!)^2}{(2n)!} P_n(x).$$

As a consequence of lemma 4.2.6 (or lemma 4.2.7), given any Euclidean space of finite dimension n, if $(e_1, \ldots, e_n)$ is an orthonormal basis for E, then for any two vectors $u = u_1e_1 + \cdots + u_ne_n$ and $v = v_1e_1 + \cdots + v_ne_n$, the inner product $u \cdot v$ is expressed as

$$u \cdot v = (u_1e_1 + \cdots + u_ne_n) \cdot (v_1e_1 + \cdots + v_ne_n) = \sum_{i=1}^{n} u_iv_i,$$

and the norm $\|u\|$ as

$$\|u\| = \|u_1e_1 + \cdots + u_ne_n\| = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

We can also prove the following lemma regarding orthogonal spaces.

Lemma 4.2.8 *Given any nontrivial Euclidean space E of finite dimension $n \geq 1$, for any subspace F of dimension k, the orthogonal complement $F^{\perp}$ of F has dimension $n - k$, and $E = F \oplus F^{\perp}$. Furthermore, we have $F^{\perp\perp} = F$.*