Advanced Geometric Methods In Computer Science CIS610

Jean Gallier
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA 19104, USA
e-mail: jean@saul.cis.upenn.edu

September 4, 2003

Chapter 1

Introduction: Questions, Motivations, Problems

What's the difference between an affine space and a linear space (vector space)?

What are affine maps?

What's Euclidean space?

What's Euclidean geometry?

What's an isometry?

What's a rotation?

What's the difference between a rotation and an orthogonal matrix (with $\det = 1$)?

What's a self-adjoint map?

Why do self-adjoint maps have real eigenvalues? Why can they be diagonalized using orthogonal matrices?

How does one solve an "inconsistent" linear system

$$Ax = b,$$

e.g., when there are more equations than variables?

Some Answers

The set of rotations of Euclidean n-space $\mathbb{E}^n$ forms a group $\mathbf{SO}(n)$.

The group $\mathbf{SO}(n)$ is generated by the hyperplanes reflections. In fact, n reflections suffice.

Rotations in $\mathbf{SO}(3)$ can be "represented" by a quaternion.

Rotations in $\mathbf{SO}(4)$ can be "represented" by two quaternions.

The group $\mathbf{SO}(n)$ is a (path-connected) topological space.

The group $\mathbf{SO}(n)$ has a (smooth) differential structure. The notion of tangent space at a point makes sense.

This makes $\mathbf{SO}(n)$ into a *Lie group*.

The tangent space $\mathfrak{so}(n)$ (at the origin) has some additional structure: it is a *Lie algebra*.

There is a map

$$\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$$

called the *exponential map*. It provides a local representation of the space.

Similarly, the rigid motions of Euclidean affine space form a Lie group $\mathbf{SE}(n)$, etc.

The exponential map can be used to *interpolate* in $\mathbf{SO}(n)$ or $\mathbf{SE}(n)$. There are applications to motion interpolations.

Lie groups pop up naturally in certain problems in robotics and computer vision.

Often, it is necessary to find out the number of "degrees of freedom" and this turns out to be the dimension of the Lie group.

Chris Geyer (Ph.D. under Kostas Daniilidis) used Lie groups in an interesting way in his dissertation on catadioptric sensors (and mirrors).

Every $n \times n$-matrix A can be written as

$$A = QS$$

where Q is orthogonal and S is symmetric with non-negative eigenvalues (the *polar form*).

Every $n \times n$-matrix A can be written as

$$A = VDU^\top$$

where U and V are orthogonal and D is a diagonal matrix with non-negative entries (*singular value decomposition, or SVD*).

The SVD can be used solve an "inconsistent" linear system

$$Ax = b.$$

We solve the *least squares problem*: Minimize $\|Ax - b\|$.

It can be shown that there is a vector x of smallest norm minimizing $\|Ax - b\|$. It is given by the (Penrose) *pseudo-inverse* (itself given by the SVD).

All this suggests studying some basic of
Euclidean Geometry and
Lie groups and Lie algebras.

I. Some problems in 3D-Mesh Generation

There is a class of problems, including 3D-mesh generation in medical imaging, notably brain imaging, where it is necessary to construct a 3D-tetrahedral mesh (a "tetrahedrization") from an input polyhedron (image data) possibly with boundaries.

One of the major difficulties is that such a polyhedron is usually **not** convex. If the input is convex, a 3D-Delaunay triangulation (i.e., a decomposition of the convex body into tetrahedra, a "tetrahedrization") is a very satisfactory answer.

Furthermore, a nonconvex polyhedron **cannot always** be triangulated by tetrahedra without adding extra vertices!

How do we cope with these problems?

There are at least two possible attacks:

(1) Conformal Delaunay triangulations. This means, Delaunay tetrahedral triangulations whose vertices and faces include the vertices and faces of the original input polyhedron.

Their construction requires the addition of new vertices and faces. No upper bound known. Quality of the tetrahedra not always good.

(2) Use the medial axis concept (see below).

These problems are pretty much open.

II. Dirichlet-Voronoi Diagrams and Delaunay Triangulations in Euclidean Space. Applications to Tetrahedral Mesh Generation

In a Euclidean space, given a finite set $P = \{p_1, \ldots, p_m\}$ of points, the Voronoi region $V(p_i)$ of p_i consists of all points that are closer to p_i than to any $p_j \neq p_i$. The Voronoi region $V(p_i)$ is the intersection of the half planes containing p_i defined by the bisector hyperplanes of the pairs of points (p_i, p_j).

Voronoi diagrams and their duals, Delaunay triangulations, have many applications, as we will see below.

The figure below shows the Voronoi diagram of a set of twelve points.

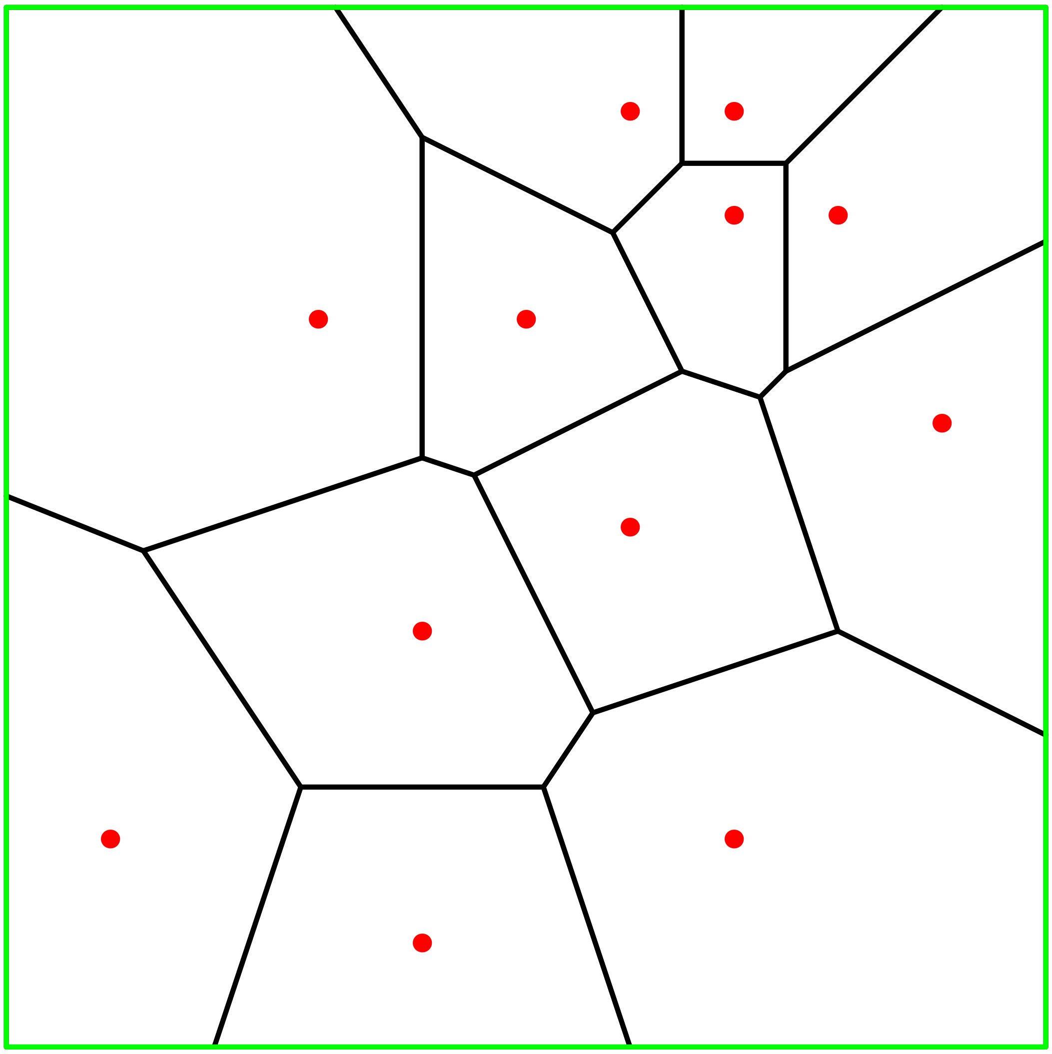

Figure 1.1: A Voronoi diagram

The figure below shows the Delaunay triangulation associated with the earlier Voronoi diagram.

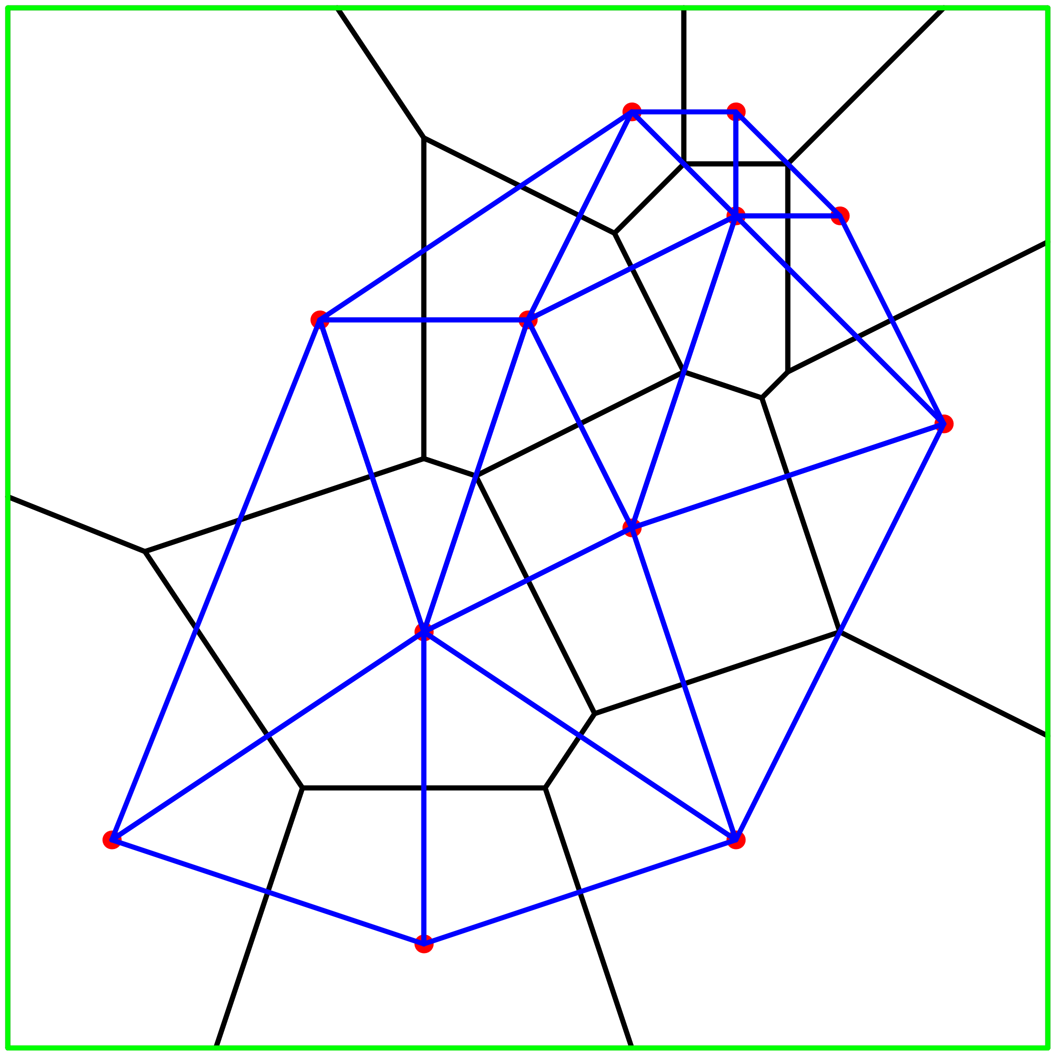

Figure 1.2: Delaunay triangulation associated with a Voronoi diagram

Delaunay trianguations are used to provide "nice" triangulations for defining complicated shapes, or in the method of finite elements.

III. Medial Axis Representation

We can generalize the notion of a medial line. This leads to the notion of medial axis. The basic idea is to represent a complex shape in terms of a kind of *skeleton* and a boundary representation.

The Blum *medial axis* of a planar shape S is defined (roughly) as the locus of centers of all circles bitangent to the boundary of S in two distinct points.

A similar notion can be defined for 3D-shapes, using spheres instead of circles.

Medial axis representations are useful in medical applications, for instance: Geometric representation of the kidneys, liver, prostate, etc.

How are medial axes computed? For instance, as pruned trees of Voronoi diagrams.