2.6 Affine Groups

We now take a quick look at the bijective affine maps.

Given an affine space $E$, the set of affine bijections $f: E \to E$ is clearly a group, called the \textit{affine group of} $E$, and denoted by $GA(E)$.

Recall that the group of bijective linear maps of the vector space $\overrightarrow{E}$ is denoted by $GL(\overrightarrow{E})$. Then, the map $f \mapsto \overrightarrow{f}$ defines a group homomorphism $L: GA(E) \to GL(\overrightarrow{E})$. The kernel of this map is the set of translations on $E$.

The subset of all linear maps of the form $\lambda \text{id}_{\overrightarrow{E}}$, where $\lambda \in \mathbb{R} - \{0\}$, is a subgroup of $GL(\overrightarrow{E})$, and is denoted as $\mathbb{R}^* \text{id}_{\overrightarrow{E}}$. 

The subgroup $\text{DIL}(E) = L^{-1}(\mathbb{R}^*\text{id}_{E})$ of $\text{GA}(E)$ is particularly interesting. It turns out that it is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 1$.

The elements of $\text{DIL}(E)$ are called \textit{affine dilatations (or dilations)}.

Given any point $a \in E$, and any scalar $\lambda \in \mathbb{R}$, a \textit{dilatation (or central dilatation, or magnification, or homothety) of center $a$ and ratio $\lambda$}, is a map $H_{a,\lambda}$ defined such that

$$H_{a,\lambda}(x) = a + \lambda ax,$$

for every $x \in E$.

Observe that $H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by $a$ and $x$, and is obtained by “scaling” $ax$ by $\lambda$. When $\lambda = 1$, $H_{a,1}$ is the identity.
Note that $\overrightarrow{H_{a,\lambda}} = \lambda \id_{\overrightarrow{E}}$. When $\lambda \neq 0$, it is clear that $H_{a,\lambda}$ is an affine bijection.

It is immediately verified that
\[
H_{a,\lambda} \circ H_{a,\mu} = H_{a,\lambda \mu}.
\]

We have the following useful result.

**Lemma 2.6.1** Given any affine space $E$, for any affine bijection $f \in GA(E)$, if $\overrightarrow{f} = \lambda \id_{\overrightarrow{E}}$, for some $\lambda \in \mathbb{R}^*$ with $\lambda \neq 1$, then there is a unique point $c \in E$ such that $f = H_{c,\lambda}$.

Clearly, if $\overrightarrow{f} = \id_{\overrightarrow{E}}$, the affine map $f$ is a translation.

Thus, the group of affine dilatations $\text{DIL}(E)$ is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 0, 1$. Affine dilatations can be given a purely geometric characterization.
2.7 Affine Geometry, a Glimpse

In this section, we state and prove three fundamental results of affine geometry.

Roughly speaking, *affine geometry is the study of properties invariant under affine bijections*. We now prove one of the oldest and most basic results of affine geometry, the **Theorem of Thalés**.

**Lemma 2.7.1** Given any affine space $E$, if $H_1, H_2, H_3$ are any three distinct parallel hyperplanes, and $A$ and $B$ are any two lines not parallel to $H_i$, letting $a_i = H_i \cap A$ and $b_i = H_i \cap B$, then the following ratios are equal:

$$\frac{a_1 a_3}{a_1 a_2} = \frac{b_1 b_3}{b_1 b_2} = \rho.$$

Conversely, for any point $d$ on the line $A$, if $\frac{a_1 d}{a_1 a_2} = \rho$, then $d = a_3$.

The diagram below illustrates the theorem of Thalés.
Figure 2.14: The theorem of Thalés
Lemma 2.7.2 Given any affine space \( E \), given any two distinct points \( a, b \in E \), for any affine dilatation \( f \) different from the identity, if \( a' = f(a) \), \( D = \langle a, b \rangle \) is the line passing through \( a \) and \( b \), and \( D' \) is the line parallel to \( D \) and passing through \( a' \), the following are equivalent:

(i) \( b' = f(b) \);

(ii) If \( f \) is a translation, then \( b' \) is the intersection of \( D' \) with the line parallel to \( \langle a, a' \rangle \) passing through \( b \);

If \( f \) is a dilatation of center \( c \), then \( b' = D' \cap \langle c, b \rangle \).

Figure 2.15: Affine Dilatations
The first case is the parallelogram law, and the second case follows easily from Thalés’ theorem.

We are now ready to prove two classical results of affine geometry, \textit{Pappus’ Theorem and Desargues’ Theorem}. Actually, these results are theorem of projective geometry, and we are stating affine versions of these important results. There are stronger versions which are best proved using projective geometry.

There is a converse to Pappus’ theorem, which yields a fancier version of Pappus’ theorem, but it is easier to prove it using projective geometry.
Lemma 2.7.3 Given any affine plane $E$, given any two distinct lines $D$ and $D'$, for any distinct points $a, b, c$ on $D$, and $a', b', c'$ on $D'$, if $a, b, c, a', b', c'$ are distinct from the intersection of $D$ and $D'$ (if $D$ and $D'$ intersect) and if the lines $\langle a, b' \rangle$ and $\langle a', b \rangle$ are parallel, and the lines $\langle b, c' \rangle$ and $\langle b', c \rangle$ are parallel, then the lines $\langle a, c' \rangle$ and $\langle a', c \rangle$ are parallel.

We now prove an affine version of Desargues’ theorem.
Lemma 2.7.4 Given any affine space \( E \), given any two triangles \((a, b, c)\) and \((a', b', c')\), where \( a, b, c, a', b', c' \) are all distinct, if \( \langle a, b \rangle \) and \( \langle a', b' \rangle \) are parallel and \( \langle b, c \rangle \) and \( \langle b', c' \rangle \) are parallel, then \( \langle a, c \rangle \) and \( \langle a', c' \rangle \) are parallel iff the lines \( \langle a, a' \rangle, \langle b, b' \rangle, \) and \( \langle c, c' \rangle \), are either parallel or concurrent (i.e., intersect in a common point).

![Desargues' theorem (affine version)](image)

There is a fancier version of Desargues’ theorem, but it is easier to prove it using projective geometry.
Desargues’ theorem yields a geometric characterization of the affine dilatations. An affine dilatation \( f \) on an affine space \( E \) is a bijection that maps every line \( D \) to a line \( f(D) \) parallel to \( D \).
2.8 Affine Hyperplanes

In section 2.3, we observed that the set $L$ of solutions of an equation

$$ax + by = c$$

is an affine subspace of $\mathbb{A}^2$ of dimension 1, in fact a line (provided that $a$ and $b$ are not both null).

It would be equally easy to show that the set $P$ of solutions of an equation

$$ax + by + cz = d$$

is an affine subspace of $\mathbb{A}^3$ of dimension 2, in fact a plane (provided that $a, b, c$ are not all null).

More generally, the set $H$ of solutions of an equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = \beta$$

is an affine subspace of $\mathbb{A}^m$, and if $\alpha_1, \ldots, \alpha_m$ are not all null, it turns out that it is a subspace of dimension $m - 1$ called a *hyperplane*. 
We can interpret the equation
\[ \alpha_1 x_1 + \cdots + \alpha_m x_m = \beta \]
in terms of the map \( f : \mathbb{R}^m \to \mathbb{R} \) defined such that
\[ f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m - \beta \]
for all \((x_1, \ldots, x_m) \in \mathbb{R}^m\).

It is immediately verified that this map is affine, and the set \( H \) of solutions of the equation
\[ \alpha_1 x_1 + \cdots + \alpha_m x_m = \beta \]
is the \textit{null set, or kernel}, of the affine map \( f : \mathbb{A}^m \to \mathbb{R} \),
in the sense that
\[ H = f^{-1}(0) = \{ x \in \mathbb{A}^m \mid f(x) = 0 \}, \]
where \( x = (x_1, \ldots, x_m) \).

Thus, it is interesting to consider \textit{affine forms}, which are just affine maps \( f : E \to \mathbb{R} \) from an affine space to \( \mathbb{R} \).
Unlike linear forms $f^*$, for which $\text{Ker } f^*$ is never empty (since it always contains the vector 0), it is possible that $f^{-1}(0) = \emptyset$, for an affine form $f$.

Recall the characterization of hyperplanes in terms of linear forms.

Given a vector space $E$ over a field $K$, a linear map $f: E \to K$ is called a \textit{linear form}. The set of all linear forms $f: E \to K$ is a vector space called the \textit{dual space of $E$}, and denoted as $E^*$. Hyperplanes are precisely the Kernels of nonnull linear forms.
Lemma 2.8.1 Let $E$ be a vector space. The following properties hold:

(a) Given any nonnull linear form $f \in E^*$, its kernel $H = \text{Ker } f$ is a hyperplane.

(b) For any hyperplane $H$ in $E$, there is a (nonnull) linear form $f \in E^*$ such that $H = \text{Ker } f$.

(c) Given any hyperplane $H$ in $E$ and any (nonnull) linear form $f \in E^*$ such that $H = \text{Ker } f$, for every linear form $g \in E^*$, $H = \text{Ker } g$ iff $g = \lambda f$ for some $\lambda \neq 0$ in $K$.

Going back to an affine space $E$, given an affine map $f: E \to \mathbb{R}$, we also denote $f^{-1}(0)$ as $\text{Ker } f$, and we call it the *kernel* of $f$. Recall that an (affine) hyperplane is an affine subspace of codimension 1.

Affine hyperplanes are precisely the Kernels of nonconstant affine forms.
Lemma 2.8.2 Let $E$ be an affine space. The following properties hold:

(a) Given any nonconstant affine form $f: E \to \mathbb{R}$, its kernel $H = \text{Ker} f$ is a hyperplane.

(b) For any hyperplane $H$ in $E$, there is a nonconstant affine form $f: E \to \mathbb{R}$ such that $H = \text{Ker} f$. For any other affine form $g: E \to \mathbb{R}$ such that $H = \text{Ker} g$, there is some $\lambda \in \mathbb{R}$ such that $g = \lambda f$ (with $\lambda \neq 0$).

(c) Given any hyperplane $H$ in $E$ and any (nonconstant) affine form $f: E \to \mathbb{R}$ such that $H = \text{Ker} f$, every hyperplane $H'$ parallel to $H$ is defined by a nonconstant affine form $g$ such that $g(a) = f(a) - \beta$, for all $a \in E$, for some $\beta \in \mathbb{R}$.
2.9 Intersection of Affine Spaces

In this section, we take a closer look at the intersection of affine subspaces.

First, we need a result of linear algebra.

**Lemma 2.9.1** Given a vector space $E$ and any two subspaces $M$ and $N$, we have the Grassmann relation:

\[ \dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N). \]

We now prove a simple lemma about the intersection of affine subspaces.
Lemma 2.9.2 Given any affine space $E$, for any two nonempty affine subspaces $M$ and $N$, the following facts hold:

1. $M \cap N \neq \emptyset$ iff $ab \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$.

2. $M \cap N$ consists of a single point iff $ab \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$, and $\overrightarrow{M} \cap \overrightarrow{N} = \{0\}$.

3. If $S$ is the least affine subspace containing $M$ and $N$, then $\overrightarrow{S} = \overrightarrow{M} + \overrightarrow{N} + K\overrightarrow{ab}$ (the vector space $\overrightarrow{E}$ is defined over the field $K$).

Remarks: (1) The proof of Lemma 2.9.2 shows that if $M \cap N \neq \emptyset$ then $ab \in \overrightarrow{M} + \overrightarrow{N}$ for all $a \in M$ and all $b \in N$.

(2) Lemma 2.9.2 (2) implies that for any two nonempty affine subspaces $M$ and $N$, if $\overrightarrow{E} = \overrightarrow{M} \oplus \overrightarrow{N}$, then $M \cap N$ consists of a single point.
Lemma 2.9.3 Given an affine space $E$ and any two nonempty affine subspaces $M$ and $N$, if $S$ is the least affine subspace containing $M$ and $N$, then the following properties hold:

(1) If $M \cap N = \emptyset$, then
\[
\dim(M) + \dim(N) < \dim(E) + \dim(M + N),
\]
and
\[
\dim(S) = \dim(M) + \dim(N) + 1 - \dim(M \cap N).
\]

(2) If $M \cap N \neq \emptyset$, then
\[
\dim(S) = \dim(M) + \dim(N) - \dim(M \cap N).
\]