2.6 Affine Groups

We now take a quick look at the bijective affine maps.

Given an affine space E, the set of affine bijections $f\colon E \to E$ is clearly a group, called the *affine group of E*, and denoted by $\mathrm{GA}(E)$.

Recall that the group of bijective linear maps of the vector space $\overrightarrow{E}$ is denoted by $\mathrm{GL}(\overrightarrow{E})$. Then, the map $f \mapsto \overrightarrow{f}$ defines a group homomorphism $L\colon \mathrm{GA}(E) \to \mathrm{GL}(\overrightarrow{E})$. The kernel of this map is the set of translations on E.

The subset of all linear maps of the form $\lambda\,\mathrm{id}_{\overrightarrow{E}}$, where $\lambda \in \mathbb{R} - \{0\}$, is a subgroup of $\mathrm{GL}(\overrightarrow{E})$, and is denoted as $\mathbb{R}^*\mathrm{id}_{\overrightarrow{E}}$.

The subgroup $\mathrm{DIL}(E) = L^{-1}(\mathbb{R}^* \mathrm{id}_{\overrightarrow{E}})$ of $\mathrm{GA}(E)$ is particularly interesting. It turns out that it is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 1$.

The elements of $\mathrm{DIL}(E)$ are called *affine dilatations (or dilations)*.

Given any point $a \in E$, and any scalar $\lambda \in \mathbb{R}$, a *dilatation (or central dilatation, or magnification, or homothety) of center a and ratio λ*, is a map $H_{a,\lambda}$ defined such that

$$H_{a,\lambda}(x) = a + \lambda \mathbf{ax},$$

for every $x \in E$.

Observe that $H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by a and x, and is obtained by "scaling" $\mathbf{ax}$ by λ. When $\lambda = 1$, $H_{a,1}$ is the identity.

Note that $\overrightarrow{H_{a,\lambda}} = \lambda\,\mathrm{id}_{\overrightarrow{E}}$. When $\lambda \neq 0$, it is clear that $H_{a,\lambda}$ is an affine bijection.

It is immediately verified that

$$H_{a,\lambda} \circ H_{a,\mu} = H_{a,\lambda\mu}.$$

We have the following useful result.

Lemma 2.6.1 *Given any affine space E, for any affine bijection $f \in GA(E)$, if $\overrightarrow{f} = \lambda\,\mathrm{id}_{\overrightarrow{E}}$, for some $\lambda \in \mathbb{R}^*$ with $\lambda \neq 1$, then there is a unique point $c \in E$ such that $f = H_{c,\lambda}$.*

Clearly, if $\overrightarrow{f} = \mathrm{id}_{\overrightarrow{E}}$, the affine map f is a translation.

Thus, the group of affine dilatations $\mathrm{DIL}(E)$ is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 0, 1$. Affine dilatations can be given a purely geometric characterization.

2.7 Affine Geometry, a Glimpse

In this section, we state and prove three fundamental results of affine geometry.

Roughly speaking, *affine geometry is the study of properties invariant under affine bijections*. We now prove one of the oldest and most basic results of affine geometry, the *Theorem of Thalés*.

Lemma 2.7.1 *Given any affine space E, if H_1, H_2, H_3 are any three distinct parallel hyperplanes, and A and B are any two lines not parallel to H_i, letting $a_i = H_i \cap A$ and $b_i = H_i \cap B$, then the following ratios are equal:*

$$\frac{\mathbf{a_1 a_3}}{\mathbf{a_1 a_2}} = \frac{\mathbf{b_1 b_3}}{\mathbf{b_1 b_2}} = \rho.$$

Conversely, for any point d on the line A, if $\frac{\mathbf{a_1 d}}{\mathbf{a_1 a_2}} = \rho$, then $d = a_3$.

The diagram below illustrates the theorem of Thalés.

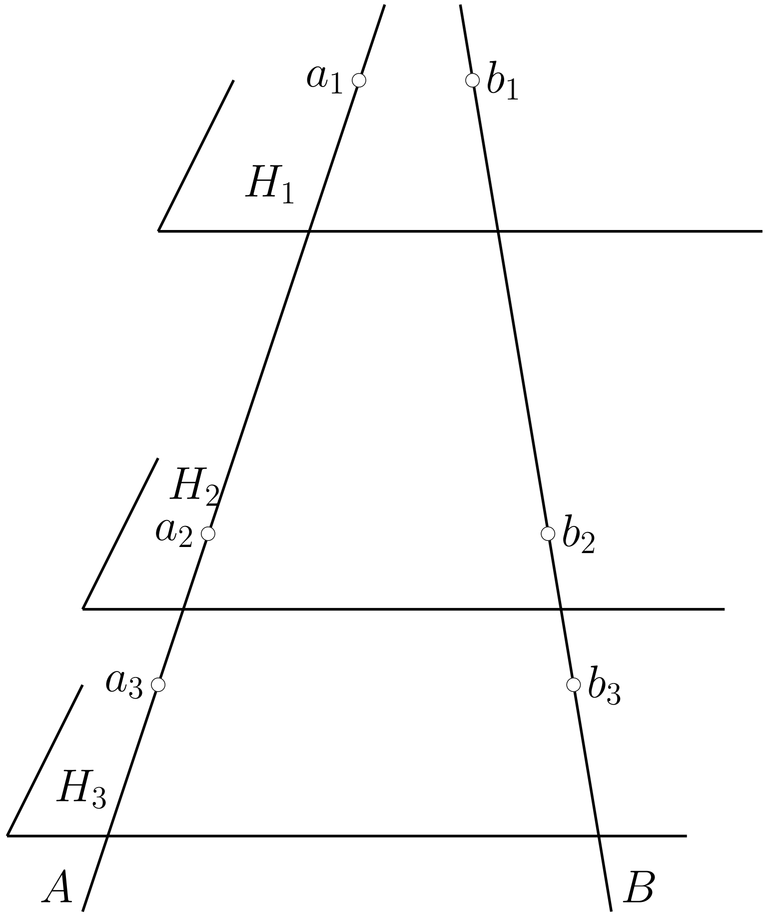

Figure 2.14: The theorem of Thalés

Lemma 2.7.2 *Given any affine space E, given any two distinct points $a, b \in E$, for any affine dilatation f different from the identity, if $a' = f(a)$, $D = \langle a, b \rangle$ is the line passing through a and b, and D' is the line parallel to D and passing through a', the following are equivalent:*

(i) $b' = f(b)$;

(ii) *If f is a translation, then b' is the intersection of D' with the line parallel to $\langle a, a' \rangle$ passing through b;*

If f is a dilatation of center c, then $b' = D' \cap \langle c, b \rangle$.

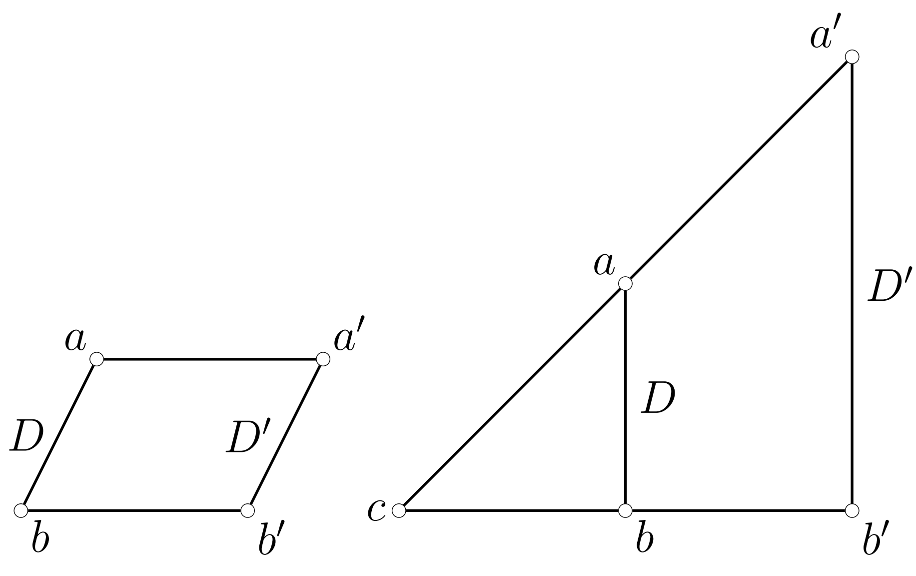

Figure 2.15: Affine Dilatations

The first case is the parallelogram law, and the second case follows easily from Thalés' theorem.

We are now ready to prove two classical results of affine geometry, *Pappus' Theorem and Desargues' Theorem*. Actually, these results are theorem of projective geometry, and we are stating affine versions of these important results. There are stronger versions which are best proved using projective geometry.

There is a converse to Pappus' theorem, which yields a fancier version of Pappus' theorem, but it is easier to prove it using projective geometry.

Lemma 2.7.3 *Given any affine plane E, given any two distinct lines D and D', for any distinct points a, b, c on D, and a', b', c' on D', if a, b, c, a', b', c' are distinct from the intersection of D and D' (if D and D' intersect) and if the lines $\langle a, b' \rangle$ and $\langle a', b \rangle$ are parallel, and the lines $\langle b, c' \rangle$ and $\langle b', c \rangle$ are parallel, then the lines $\langle a, c' \rangle$ and $\langle a', c \rangle$ are parallel.*

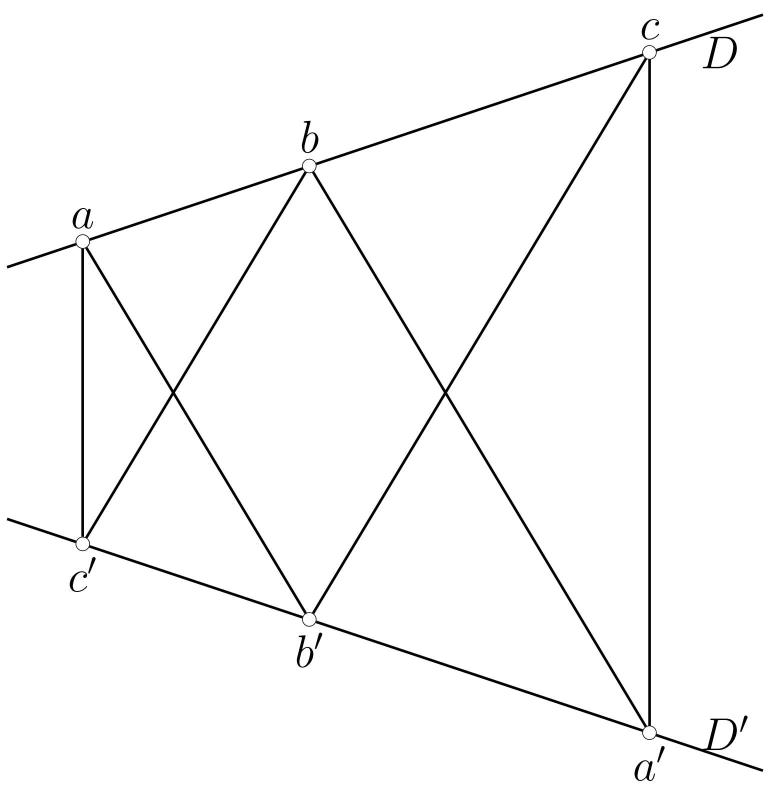

Figure 2.16: Pappus' theorem (affine version)

We now prove an affine version of Desargues' theorem.

Lemma 2.7.4 *Given any affine space E, given any two triangles (a, b, c) and (a', b', c'), where a, b, c, a', b', c' are all distinct, if $\langle a, b\rangle$ and $\langle a', b'\rangle$ are parallel and $\langle b, c\rangle$ and $\langle b', c'\rangle$ are parallel, then $\langle a, c\rangle$ and $\langle a', c'\rangle$ are parallel iff the lines $\langle a, a'\rangle$, $\langle b, b'\rangle$, and $\langle c, c'\rangle$, are either parallel or concurrent (i.e., intersect in a common point).*

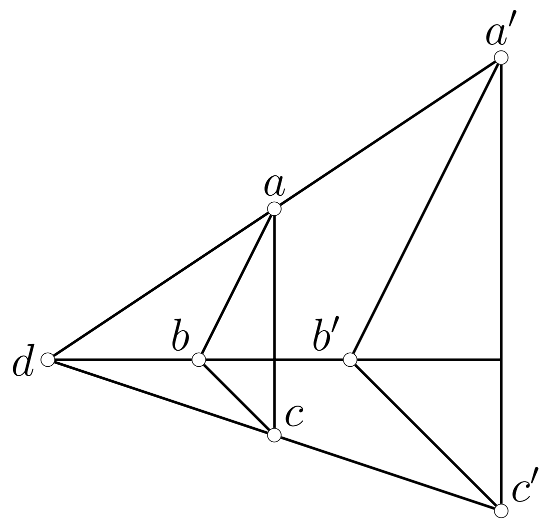

Figure 2.17: Desargues' theorem (affine version)

There is a fancier version of Desargues' theorem, but it is easier to prove it using projective geometry.

Desargues' theorem yields a geometric characterization of the affine dilatations. An affine dilatation f on an affine space E is a bijection that maps every line D to a line $f(D)$ parallel to D.

2.8 Affine Hyperplanes

In section 2.3, we observed that the set L of solutions of an equation

$$ax + by = c$$

is an affine subspace of $\mathbb{A}^2$ of dimension 1, in fact a line (provided that a and b are not both null).

It would be equally easy to show that the set P of solutions of an equation

$$ax + by + cz = d$$

is an affine subspace of $\mathbb{A}^3$ of dimension 2, in fact a plane (provided that a, b, c are not all null).

More generally, the set H of solutions of an equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = \beta$$

is an affine subspace of $\mathbb{A}^m$, and if $\alpha_1, \ldots, \alpha_m$ are not all null, it turns out that it is a subspace of dimension $m - 1$ called a *hyperplane*.

We can interpret the equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = \beta$$

in terms of the map $f\colon \mathbb{R}^m \to \mathbb{R}$ defined such that

$$f(x_1, \ldots, x_m) = \alpha_1 x_1 + \cdots + \alpha_m x_m - \beta$$

for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$.

It is immediately verified that this map is affine, and the set H of solutions of the equation

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = \beta$$

is the *null set, or kernel,* of the affine map $f\colon \mathbb{A}^m \to \mathbb{R}$, in the sense that

$$H = f^{-1}(0) = \{x \in \mathbb{A}^m \mid f(x) = 0\},$$

where $x = (x_1, \ldots, x_m)$.

Thus, it is interesting to consider *affine forms*, which are just affine maps $f\colon E \to \mathbb{R}$ from an affine space to $\mathbb{R}$.

Unlike linear forms f^*, for which $\mathrm{Ker}\, f^*$ is never empty (since it always contains the vector 0), it is possible that $f^{-1}(0) = \emptyset$, for an affine form f.

Recall the characterization of hyperplanes in terms of linear forms.

Given a vector space E over a field K, a linear map $f\colon E \to K$ is called a *linear form*. The set of all linear forms $f\colon E \to K$ is a vector space called the *dual space of E*, and denoted as E^*.

Hyperplanes are precisely the Kernels of nonnull linear forms.

Lemma 2.8.1 *Let E be a vector space. The following properties hold:*

(a) Given any nonnull liner form $f \in E^*$, its kernel $H = \mathrm{Ker}\, f$ is a hyperplane.

(b) For any hyperplane H in E, there is a (nonnull) linear form $f \in E^*$ such that $H = \mathrm{Ker}\, f$.

(c) Given any hyperplane H in E and any (nonnull) linear form $f \in E^*$ such that $H = \mathrm{Ker}\, f$, for every linear form $g \in E^*$, $H = \mathrm{Ker}\, g$ iff $g = \lambda f$ for some $\lambda \neq 0$ in K.

Going back to an affine space E, given an affine map $f\colon E \to \mathbb{R}$, we also denote $f^{-1}(0)$ as $\mathrm{Ker}\, f$, and we call it the *kernel* of f. Recall that an (affine) hyperplane is an affine subspace of codimension 1.

Affine hyperplanes are precisely the Kernels of nonconstant affine forms.

Lemma 2.8.2 *Let E be an affine space. The following properties hold:*

(a) Given any nonconstant affine form $f\colon E \to \mathbb{R}$, its kernel $H = \mathrm{Ker}\, f$ is a hyperplane.

(b) For any hyperplane H in E, there is a nonconstant affine form $f\colon E \to \mathbb{R}$ such that $H = \mathrm{Ker}\, f$. For any other affine form $g\colon E \to \mathbb{R}$ such that $H = \mathrm{Ker}\, g$, there is some $\lambda \in \mathbb{R}$ such that $g = \lambda f$ (with $\lambda \neq 0$).

(c) Given any hyperplane H in E and any (nonconstant) affine form $f\colon E \to \mathbb{R}$ such that $H = \mathrm{Ker}\, f$, every hyperplane H' parallel to H is defined by a nonconstant affine form g such that $g(a) = f(a) - \beta$, for all $a \in E$, for some $\beta \in \mathbb{R}$.

2.9 Intersection of Affine Spaces

In this section, we take a closer look at the intersection of affine subspaces.

First, we need a result of linear algebra.

Lemma 2.9.1 *Given a vector space E and any two subspaces M and N, we have the Grassmann relation:*

$$\dim(M) + \dim(N) = \dim(M+N) + \dim\,(M \cap N).$$

We now prove a simple lemma about the intersection of affine subspaces.

Lemma 2.9.2 *Given any affine space E, for any two nonempty affine subspaces M and N, the following facts hold:*

(1) $M \cap N \neq \emptyset$ iff $\mathbf{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$.

(2) $M \cap N$ consists of a single point iff $\mathbf{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$, and $\overrightarrow{M} \cap \overrightarrow{N} = \{0\}$.

(3) If S is the least affine subspace containing M and N, then $\overrightarrow{S} = \overrightarrow{M} + \overrightarrow{N} + K\mathbf{ab}$ (the vector space $\overrightarrow{E}$ is defined over the field K).

Remarks: (1) The proof of Lemma 2.9.2 shows that if $M \cap N \neq \emptyset$ then $\mathbf{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for all $a \in M$ and all $b \in N$.

(2) Lemma 2.9.2 (2) implies that for any two nonempty affine subspaces M and N, if $\overrightarrow{E} = \overrightarrow{M} \oplus \overrightarrow{N}$, then $M \cap N$ consists of a single point.

Lemma 2.9.3 *Given an affine space E and any two nonempty affine subspaces M and N, if S is the least affine subspace containing M and N, then the following properties hold:*

(1) If $M \cap N = \emptyset$, then

$$\dim(M) + \dim(N) < \dim(E) + \dim(\overrightarrow{M} + \overrightarrow{N}),$$

and

$$\dim(S) = \dim(M) + \dim(N) + 1 - \dim(\overrightarrow{M} \cap \overrightarrow{N}).$$

(2) If $M \cap N \neq \emptyset$, then

$$\dim(S) = \dim(M) + \dim(N) - \dim(M \cap N).$$