CIS610, Summer 1, 2009
Advanced Geometric Methods in Computer Science

Jean Gallier
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA 19104, USA
e-mail: jean@cis.upenn.edu

May 27, 2009

Chapter 1

Introduction: Questions, Motivations, Problems

Some problems:

1. Find the intersection of a plane and a line in $\mathbb{R}^3$.

2. Given two triangles, $T_1 = (a_1, b_1, c_1)$ and $T_2 = (a_2, b_2, c_2)$, in the plane $\mathbb{R}^2$, find a *simple (e.g. linear, affine?)* map sending T_1 to T_2.

   When is such a map unique?

3. Same problem as above but for two triangles T_1 and T_2 in $\mathbb{R}^3$ (in 3D space).

4. Given two tetrahedra, $T_1 = (a_1, b_1, c_1, d_1)$ and $T_2 = (a_2, b_2, c_2, d_2)$, in $\mathbb{R}^3$, find a *simple (e.g. linear, affine?)* map sending T_1 to T_2.

   When is such a map unique?

5. More generally, what are “linear gismos” (in $\mathbb{R}^n$)?
6. Can we figure out when two linear gismos intersect?

   How “big” is their intersection?

What’s the difference between *points* and *vectors*?

What’s the difference between *linear subspaces* and *affine subspaces*?

What’s the difference between *linear maps* and *affine maps*?

Linear Gismos

The definition of linear gismos should involve some notion of linear combination. Here are some variants:

1. *linear combinations*: $\lambda_1 v_1 + \cdots + \lambda_k v_k$,
   the v_i's are vectors in $\mathbb{R}^n$ and the $\lambda_i \in \mathbb{R}$ are *unrestricted*.

   $$\mathrm{span}(v_1, \ldots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}\}$$

   is a *linear subspace* of $\mathbb{R}^n$.

2. *positive combinations*: $\lambda_1 v_1 + \cdots + \lambda_k v_k$,
   the v_i's are vectors in $\mathbb{R}^n$ and the $\lambda_i \in \mathbb{R}$ are *nonnegative*, $\lambda_i \geq 0$.

   $$\mathrm{cone}(v_1, \ldots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R},\ \lambda_i \geq 0\}$$

   is a *polyhedral cone* in $\mathbb{R}^n$.

3. *affine combinations*: $\lambda_1 v_1 + \cdots + \lambda_k v_k$,
   the v_i's are vectors in $\mathbb{R}^n$ and the $\lambda_i \in \mathbb{R}$ *add up to 1*: $\lambda_1 + \cdots + \lambda_k = 1$.

   $$\mathrm{aff}(v_1, \ldots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R},\ \sum_i \lambda_i = 1\}$$

   is an *affine subspace* of $\mathbb{R}^n$.

4. *convex combinations*: $\lambda_1 v_1 + \cdots + \lambda_k v_k$,
   the v_i's are vectors in $\mathbb{R}^n$ and the $\lambda_i \in \mathbb{R}$ are *nonnegative and add up to* 1: $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_k = 1$.

   $$\mathrm{conv}(v_1, \ldots, v_k) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1\}$$

   is the *convex hull* of $\{v_1, \ldots, v_k\}$; it is a *convex polytope* in $\mathbb{R}^n$.

What if we mix positive and convex combinations?

Sets of the form

$$\mathrm{conv}(v_1, \ldots, v_p) + \mathrm{cone}(w_1, \ldots, w_q)$$

are called *polyhedra*

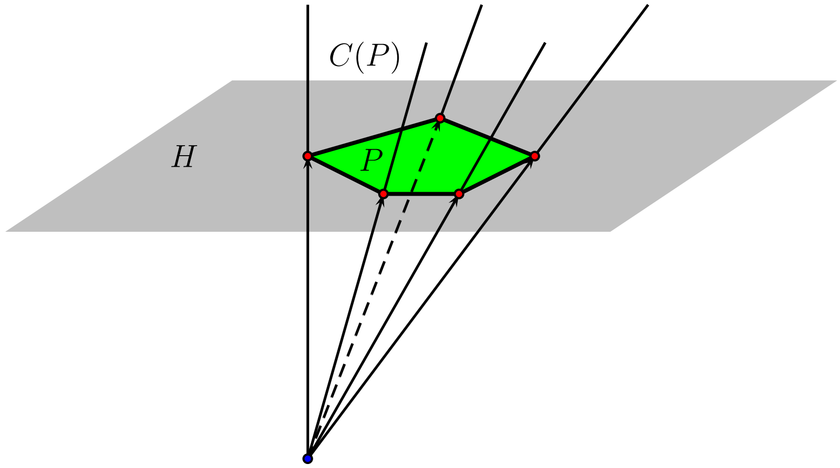

Figure 1.1: Affine subspace H, polyhedral cone $C(P)$ and convex polytope P

In Figure 1.1, the linear subspace spanned by the 5 vectors whose tips are indicated by red dots is $\mathbb{R}^n$;

The polyhedral cone spanned by these vectors is $C(P)$;

The affine subspace spanned by these vectors is the hyperplane H;

The convex hull of these vectors is the polytope P shown in green (including its boundary).

The *Conification Trick*:

Polyhedra are intersections of polyhedral cones with hyperplanes.

Polyhedra are also "cut out" by hyperplanes.

An (affine) *hyperplane*, H, is the set points, $(x_1, \ldots, x_n) \in \mathbb{R}^n$, satisfying an equation

$$a_1x_1 + \cdots + a_nx_n + b = 0$$

with $a_i \neq 0$ for some i.

A hyperplane defines two *(closed) half spaces*

$$H_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n + b \geq 0\}$$

and

$$H_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1x_1 + \cdots + a_nx_n + b \leq 0\}.$$

A major theorem of convex geometry states that a subset of $\mathbb{R}^n$ is a polyhedron (a *$\mathcal{V}$-polyhedron*) iff it is the intersection of a finite number of half-spaces (an *$\mathcal{H}$-polyhedron*).

Such a subset is a polytope (a *$\mathcal{V}$-polytope*) iff it bounded and the intersection of a finite number of half-spaces (an *$\mathcal{H}$-polytope*).

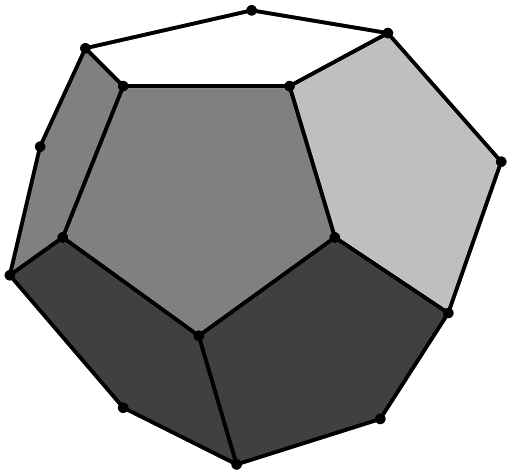

Figure 1.2: Example of a polytope (a dodecahedron)

As we can see in Figure 1.2, a polytope has *faces*, *edges* and *vertices*.

Do these things always exist (in higher dimensions)?

How do we find them?

Vertices (and *extreme points*) are important because a continuous and convex function achieves its maxima (and minima) at extreme points.

What's the difference between *vertices* and *extreme points*?

Polyhedra in Combinatorial Optimization Problems

Consider the *0-1 Knapsack Problem*:

Given a set of n objects, each with a value, p_j, and a list of m "weights", $W_{1j}, \ldots, W_{mj}$, choose a subset of these objects to obtain a collection whose total value is as large as possible and so that the total weights are less than some given limits.

The above can be formalized as

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} p_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} W_{ij} x_j \leq c_i, \quad 1 \leq i \leq m, \\ &&& x_j \in \{0, 1\}, \quad 1 \leq j \leq n, \end{aligned}$$

where the c_i's are maximum capacities (weights, dimension, *etc.*).

This problem is hard to solve (in fact, NP-hard!)

We can try solving an easier problem by *relaxing* the 0-1-constraint to

$$0 \leq x_j \leq 1.$$

We obtain a *fractional packing problem*, which can be stated in matrix form as follows:

$$\begin{aligned} \text{maximize} \quad & \mathbf{p}^\top \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{c} \\ & \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{x}$ denotes $(x_1, \ldots, x_n)$ as a column vector, similarly for $\mathbf{p}$, $A = \begin{pmatrix} W \\ I \end{pmatrix}$ (an $(m+n) \times n$ matrix) and $\mathbf{c}$ is the column vector corresponding to

$$(c_1, \ldots, c_m, \underbrace{1, \ldots, 1}_{n}).$$

The above is a *linear program* (for short, *LP*).

Observe that the constaints define the intersection of some half-spaces, namely, a *polyhedron*.

Thus, it would be useful to understand better the structure of polyhedra and how to maximize (or minimize) functions on them, especially convex functions.

Solving Inconsistent Linear Systems

How does one solve an "inconsistent" linear system

$$Ax = b,$$

e.g., when there are more equations than variables?

Such problems often arise when trying to fit some data. For example, we may have a set of 3D data points,

$$\{p_1, \ldots, p_n\},$$

and we observe that these points are coplanar. We would like to find a plane that "best fits" our data points. If the equation of such a plane is

$$ax + by + cz + d = 0,$$

we would like this equation to be satisfied for all the p_i's, which leads to a system of n equations in 4 unknowns, with $p_i = (x_i, y_i, z_i)$;

$$\begin{aligned} ax_1 + by_1 + cz_1 + d &= 0 \\ &\vdots \qquad\qquad \vdots \\ ax_n + by_n + cz_n + d &= 0. \end{aligned}$$

However, if n is larger than 4, such a system generally has *no solution* besides the trivial solution $a = b = c = d = 0$.

Fortunately, every $n \times m$-matrix A can be written as

$$A = VDU^\top$$

where U and V are orthogonal and D is a rectangular diagonal matrix with non-negative entries (*singular value decomposition, or SVD*).

The SVD can be used solve an "inconsistent" linear system

$$Ax = b.$$

We solve the *least squares problem*:
Minimize $\|Ax - b\|$.

It can be shown that there is a vector x of smallest norm minimizing $\|Ax - b\|$. It is given by the (Penrose) *pseudo-inverse* of A (itself given by the SVD).

For our plane fitting problem, we minimize

$$\sum_{i=1}^{n} (ax_i + by_i + cz_i + d)^2.$$

Actually, take a closer look at this solution of the plane fitting problem. We will need to revisit this issue later.

All this suggests studying some basic of **Affine Geometry**, **Euclidean Geometry** and **Convex Geometry**.