Chapter 5

Lie Groups, Lie Algebras and the Exponential Map

5.1 Lie Groups and Lie Algebras

In Chapter 2, we defined the notion of a Lie group as a certain type of manifold embedded in $\mathbb{R}^N$, for some $N \geq 1$. Now that we have the general concept of a manifold, we can define Lie groups in more generality.

Definition 5.1.1 A *Lie group* is a nonempty subset, G, satisfying the following conditions:

(a) G is a group (with identity element denoted e or 1).

(b) G is a smooth manifold.

(c) G is a topological group. In particular, the group operation, $\cdot : G \times G \to G$, and the inverse map, ${}^{-1}\colon G \to G$, are smooth.

We have already met a number of Lie groups: $\mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$, $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{SL}(n, \mathbb{C})$, $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{E}(n, \mathbb{R})$. Also, every linear Lie group (i.e., a closed subgroup of $\mathbf{GL}(n, \mathbb{R})$) is a Lie group.

We saw in the case of linear Lie groups that the tangent space to G at the identity, $\mathfrak{g} = T_1G$, plays a very important role. In particular, this vector space is equipped with a (non-associative) multiplication operation, the Lie bracket, that makes $\mathfrak{g}$ into a Lie algebra. This is again true in this more general setting.

Definition 5.1.2 A *(real) Lie algebra*, $\mathcal{A}$, is a real vector space together with a bilinear map, $[\cdot, \cdot]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the *Lie bracket* on $\mathcal{A}$ such that the following two identities hold for all $a, b, c \in \mathcal{A}$:

$$[a,\, a] = 0,$$

and the so-called *Jacobi identity*

$$[a,\, [b,\, c]] + [c,\, [a,\, b]] + [b,\, [c,\, a]] = 0.$$

It is immediately verified that $[b, a] = -[a, b]$.

Let us also recall the definition of homomorphisms of Lie groups and Lie algebras.

Definition 5.1.3 Given two Lie groups G_1 and G_2, a *homomorphism (or map) of Lie groups* is a function, $f\colon G_1 \to G_2$, that is a homomorphism of groups and a smooth map (between the manifolds G_1 and G_2). Given two Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *homomorphism (or map) of Lie algebras* is a function, $f\colon \mathcal{A}_1 \to \mathcal{A}_2$, that is a linear map between the vector spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ and that preserves Lie brackets, i.e.,

$$f([A, B]) = [f(A), f(B)]$$

for all $A, B \in \mathcal{A}_1$.

An *isomorphism of Lie groups* is a bijective function f such that both f and f^{-1} are maps of Lie groups, and an *isomorphism of Lie algebras* is a bijective function f such that both f and f^{-1} are maps of Lie algebras.

The Lie bracket operation on $\mathfrak{g}$ can be defined in terms of the so-called adjoint representation.

Given a Lie group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Because multiplication and the inverse maps are smooth, the maps L_a and R_a are diffeomorphisms, and their derivatives play an important role.

The inner automorphisms $R_{a^{-1}} \circ L_a$ (also written $R_{a^{-1}}L_a$ or $\mathbf{Ad}_a$) also play an important role. Note that

$$R_{a^{-1}}L_a(b) = aba^{-1}.$$

The derivative

$$d(R_{a^{-1}}L_a)_1 \colon \mathfrak{g} \to \mathfrak{g}$$

of $R_{a^{-1}}L_a$ at 1 is an isomorphism of Lie algebras, denoted by $\mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g}$.

The map $a \mapsto \mathrm{Ad}_a$ is a map of Lie groups

$$\mathrm{Ad} \colon G \to \mathbf{GL}(\mathfrak{g}),$$

called the *adjoint representation of G* (where $\mathbf{GL}(\mathfrak{g})$ denotes the Lie group of all bijective linear maps on $\mathfrak{g}$).

In the case of a linear group, one can verify that

$$\mathrm{Ad}(a)(X) = \mathrm{Ad}_a(X) = aXa^{-1}$$

for all $a \in G$ and all $X \in \mathfrak{g}$.

The derivative

$$d\mathrm{Ad}_1\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

of Ad at 1 is map of Lie algebras, denoted by

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

called the *adjoint representation of* $\mathfrak{g}$ (where $\mathfrak{gl}(\mathfrak{g})$ denotes the Lie algebra, $\mathrm{End}(\mathfrak{g}, \mathfrak{g})$, of all linear maps on $\mathfrak{g}$).

In the case of a linear group, it can be verified that

$$\mathrm{ad}(A)(B) = [A,\ B]$$

for all $A, B \in \mathfrak{g}$.

One can also check (in general) that the Jacobi identity on $\mathfrak{g}$ is equivalent to the fact that ad preserves Lie brackets, i.e., ad is a map of Lie algebras:

$$\mathrm{ad}([u,\ v]) = [\mathrm{ad}(u),\ \mathrm{ad}(v)],$$

for all $u, v \in \mathfrak{g}$ (where on the right, the Lie bracket is the commutator of linear maps on $\mathfrak{g}$).

This is the key to the definition of the Lie bracket in the case of a general Lie group (not just a linear Lie group).

Definition 5.1.4 Given a Lie group, G, the tangent space, $\mathfrak{g} = T_1 G$, at the identity with the Lie bracket defined by

$$[u, v] = \mathrm{ad}(u)(v), \quad \text{for all } u, v \in \mathfrak{g},$$

is the *Lie algebra of the Lie group G*.

Actually, we have to justify why $\mathfrak{g}$ really is a Lie algebra. For this, we have

Proposition 5.1.5 *Given a Lie group, G, the Lie bracket, $[u, v] = \mathrm{ad}(u)(v)$, of Definition 5.1.4 satisfies the axioms of a Lie algebra (given in Definition 5.1.2). Therefore, $\mathfrak{g}$ with this bracket is a Lie algebra.*

Remark: After proving that $\mathfrak{g}$ is isomorphic to the vector space of left-invariant vector fields on G, we get another proof of Proposition 5.1.5.

5.2 Left and Right Invariant Vector Fields, the Exponential Map

A fairly convenient way to define the exponential map is to use left-invariant vector fields.

Definition 5.2.1 If G is a Lie group, a vector field, X, on G is *left-invariant* (resp. *right-invariant*) iff

$$d(L_a)_b(X(b)) = X(L_a(b)) = X(ab), \quad \text{for all } a, b \in G.$$

(resp.

$$d(R_a)_b(X(b)) = X(R_a(b)) = X(ba), \quad \text{for all } a, b \in G.)$$

Equivalently, a vector field, X, is left-invariant iff the following diagram commutes (and similarly for a right-invariant vector field):

$$\begin{tikzcd} T_bG \arrow[r, "d(L_a)_b"] & T_{ab}G \\ G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u, "X"'] \end{tikzcd}$$

If X is a left-invariant vector field, setting $b = 1$, we see that

$$X(a) = d(L_a)_1(X(1)),$$

which shows that X is determined by its value, $X(1) \in \mathfrak{g}$, at the identity (and similarly for right-invariant vector fields).

Conversely, given any $v \in \mathfrak{g}$, we can define the vector field, v^L, by

$$v^L(a) = d(L_a)_1(v), \quad \text{for all } a \in G.$$

We claim that v^L is left-invariant. This follows by an easy application of the chain rule:

$$\begin{aligned} v^L(ab) &= d(L_{ab})_1(v) \\ &= d(L_a \circ L_b)_1(v) \\ &= d(L_a)_b(d(L_b)_1(v)) \\ &= d(L_a)_b(v^L(b)). \end{aligned}$$

Furthermore, $v^L(1) = v$.

Therefore, we showed that the map, $X \mapsto X(1)$, establishes an isomorphism between the space of left-invariant vector fields on G and $\mathfrak{g}$.

In fact, the map $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^L(a)$ is an isomorphism between $G \times \mathfrak{g}$ and the tangent bundle, TG.

Remark: Given any $v \in \mathfrak{g}$, we can also define the vector field, v^R, by

$$v^R(a) = d(R_a)_1(v), \quad \text{for all } a \in G.$$

It is easily shown that v^R is right-invariant and we also have an isomorphism $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^R(a)$.

Another reason left-invariant (resp. right-invariant) vector fields on a Lie group are important is that they are complete, i.e., they define a flow whose domain is $\mathbb{R} \times G$. To prove this, we begin with the following easy proposition:

Proposition 5.2.2 *Given a Lie group, G, if X is a left-invariant (resp. right-invariant) vector field and Φ is its flow, then*

$$\Phi(t, g) = g\Phi(t, 1) \quad (\textit{resp.} \quad \Phi(t, g) = \Phi(t, 1)g),$$

for all $(t, g) \in \mathcal{D}(X)$.

Proposition 5.2.3 *Given a Lie group, G, for every $v \in \mathfrak{g}$, there is a unique smooth homomorphism, $h_v \colon (\mathbb{R}, +) \to G$, such that $\dot{h}_v(0) = v$. Furthermore, $h_v(t)$ is the maximal integral curve of both v^L and v^R with initial condition 1 and the flows of v^L and v^R are defined for all $t \in \mathbb{R}$.*

Since $h_v \colon (\mathbb{R}, +) \to G$ is a homomorphism, the integral curve, h_v, is often referred to as a *one-parameter group*.

Proposition 5.2.3 yields the definition of the exponential map.

Definition 5.2.4 Given a Lie group, G, the *exponential map*, $\exp \colon \mathfrak{g} \to G$, is given by

$$\exp(v) = h_v(1) = \Phi_1^v(1), \quad \text{for all } v \in \mathfrak{g},$$

where Φ_t^v denotes the flow of v^L.

It is not difficult to prove that exp is smooth.

Observe that for any fixed $t \in \mathbb{R}$, the map

$$s \mapsto h_v(st)$$

is a smooth homomorphism, h, such that $\dot{h}(0) = tv$. By uniqueness, we have

$$h_v(st) = h_{tv}(s).$$

Setting $s = 1$, we find that

$$h_v(t) = \exp(tv), \quad \text{for all } v \in \mathfrak{g} \text{ and all } t \in \mathbb{R}.$$

Then, differentiating with respect to t at $t = 0$, we get

$$v = d\exp_0(v),$$

i.e., $d\exp_0 = \mathrm{id}_{\mathfrak{g}}$.

By the inverse function theorem, exp is a local diffeomorphism at 0. This means that there is some open subset, $U \subseteq \mathfrak{g}$, containing 0, such that the restriction of exp to U is a diffeomorphism onto $\exp(U) \subseteq G$, with $1 \in \exp(U)$.

In fact, by left-translation, the map $v \mapsto g\exp(v)$ is a local diffeomorphism between some open subset, $U \subseteq \mathfrak{g}$, containing 0 and the open subset, $\exp(U)$, containing g.

The exponential map is also natural in the following sense:

Proposition 5.2.5 *Given any two Lie groups, G and H, for every Lie group homomorphism, $f\colon G \to H$, the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "df_1"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

As useful corollary of Proposition 5.2.5 is:

Proposition 5.2.6 *Let G be a connected Lie group and H be any Lie group. For any two homomorphisms, $\varphi_1\colon G \to H$ and $\varphi_2\colon G \to H$, if $d(\varphi_1)_1 = d(\varphi_2)_1$, then $\varphi_1 = \varphi_2$.*

The above proposition shows that if G is connected, then a homomorphism of Lie groups, $\varphi\colon G \to H$, is uniquely determined by the Lie algebra homomorphism,
$d\varphi_1\colon \mathfrak{g} \to \mathfrak{h}$.

We obtain another useful corollary of Proposition 5.2.5 when we apply it to the adjoint representation of G,

$$\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$$

and to the conjugation map,

$$\mathbf{Ad}_a\colon G \to G,$$

where $\mathbf{Ad}_a(b) = aba^{-1}$. In the first case, $d\mathrm{Ad}_1 = \mathrm{ad}$, with $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and in the second case,
$d(\mathbf{Ad}_a)_1 = \mathrm{Ad}_a$.

Proposition 5.2.7 *Given any Lie group, G, the following properties hold:*

(1)

$$\mathrm{Ad}(\exp(u)) = e^{\mathrm{ad}(u)}, \qquad \text{for all } u \in \mathfrak{g},$$

where $\exp\colon \mathfrak{g} \to G$ *is the exponential of the Lie group, G, and $f \mapsto e^f$ is the exponential map given by*

$$e^f = \sum_{k=0}^{\infty} \frac{f^k}{k!},$$

for any linear map (matrix), $f \in \mathfrak{gl}(\mathfrak{g})$. Equivalently, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathbf{GL}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathfrak{gl}(\mathfrak{g}). \arrow[u, "f \mapsto e^f"'] \end{tikzcd}$$

(2)

$$\exp(t\mathrm{Ad}_g(u)) = g \exp(tu) g^{-1},$$

for all $u \in \mathfrak{g}$, all $g \in G$ and all $t \in \mathbb{R}$. Equivalently, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathbf{Ad}_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}_g"'] & \mathfrak{g}. \arrow[u, "\exp"'] \end{tikzcd}$$

Since the Lie algebra $\mathfrak{g} = T_1G$ is isomorphic to the vector space of left-invariant vector fields on G and since the Lie bracket of vector fields makes sense (see Definition 4.3.6), it is natural to ask if there is any relationship between, $[u, v]$, where $[u, v] = \mathrm{ad}(u)(v)$, and the Lie bracket, $[u^L, v^L]$, of the left-invariant vector fields associated with $u, v \in \mathfrak{g}$.

The answer is: Yes, they coincide (*via* the correspondence $u \mapsto u^L$).

This fact is recorded in the proposition below whose proof involves some rather acrobatic uses of the chain rule found in Warner [**?**] (Chapter 3), Bröcker and tom Dieck [**?**] (Chapter 1, Section 2), or Marsden and Ratiu [**?**] (Chapter 9).

Proposition 5.2.8 *Given a Lie group, G, we have*

$$[u^L, v^L](1) = \mathrm{ad}(u)(v), \quad \textit{for all } u, v \in \mathfrak{g}.$$

We can apply Proposition 3.4.9 and use the exponential map to prove a useful result about Lie groups.

If G is a Lie group, let G_0 be the connected component of the identity. We know G_0 is a topological normal subgroup of G and it is a submanifold in an obvious way, so it is a Lie group.

Proposition 5.2.9 *If G is a Lie group and G_0 is the connected component of 1, then G_0 is generated by $\exp(\mathfrak{g})$. Moreover, G_0 is countable at infinity.*

5.3 Homomorphisms of Lie Groups and Lie Algebras, Lie Subgroups

If G and H are two Lie groups and $\varphi \colon G \to H$ is a homomorphism of Lie groups, then $d\varphi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is a linear map between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H.

In fact, it is a Lie algebra homomorphism, as shown below.

Proposition 5.3.1 *If G and H are two Lie groups and $\varphi \colon G \to H$ is a homomorphism of Lie groups, then*

$$d\varphi_1 \circ \mathrm{Ad}_g = \mathrm{Ad}_{\varphi(g)} \circ d\varphi_1, \quad \textit{for all } g \in G,$$

that is, the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\varphi_1"] \arrow[d, "\mathrm{Ad}_g"'] & \mathfrak{h} \arrow[d, "\mathrm{Ad}_{\varphi(g)}"] \\ \mathfrak{g} \arrow[r, "d\varphi_1"'] & \mathfrak{h} \end{tikzcd}$$

and $d\varphi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Remark: If we identify the Lie algebra, $\mathfrak{g}$, of G with the space of left-invariant vector fields on G, the map $d\varphi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is viewed as the map such that, for every left-invariant vector field, X, on G, the vector field $d\varphi_1(X)$ is the unique left-invariant vector field on H such that

$$d\varphi_1(X)(1) = d\varphi_1(X(1)),$$

i.e., $d\varphi_1(X) = d\varphi_1(X(1))^L$. Then, we can give another proof of the fact that $d\varphi_1$ is a Lie algebra homomorphism using the notion of φ-related vector fields.

Proposition 5.3.2 *If G and H are two Lie groups and $\varphi \colon G \to H$ is a homomorphism of Lie groups, if we identify $\mathfrak{g}$ (resp. $\mathfrak{h}$) with the space of left-invariant vector fields on G (resp. left-invariant vector fields on H), then,*

(a) X and $d\varphi_1(X)$ are φ-related, for every left-invariant vector field, X, on G;

(b) $d\varphi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

We now consider Lie subgroups. As a preliminary result, note that if $\varphi: G \to H$ is an injective Lie group homomorphism, then $d\varphi_g: T_g G \to T_{\varphi(g)} H$ is injective for all $g \in G$.

As $\mathfrak{g} = T_1 G$ and $T_g G$ are isomorphic for all $g \in G$ (and similarly for $\mathfrak{h} = T_1 H$ and $T_h H$ for all $h \in H$), it is sufficient to check that $d\varphi_1: \mathfrak{g} \to \mathfrak{h}$ is injective.

However, by Proposition 5.2.5, the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\varphi_1"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes, and since the exponential map is a local diffeomorphism at 0, as φ is injective, then $d\varphi_1$ is injective, too. Therefore, if $\varphi: G \to H$ is injective, it is automatically an immersion.

Definition 5.3.3 Let G be a Lie group. A set, H, is an *immersed (Lie) subgroup* of G iff

(a) H is a Lie group;

(b) There is an injective Lie group homomorphism, $\varphi\colon H \to G$ (and thus, φ is an immersion, as noted above).

We say that H is a *Lie subgroup* (or *closed Lie subgroup*) of G iff H is a Lie group that is a subgroup of G and also a submanifold of G.

Observe that an immersed Lie subgroup, H, is an immersed submanifold, since φ is an injective immersion.

However, $\varphi(H)$ may *not* have the subspace topology inherited from G and $\varphi(H)$ may not be closed.

As we will see below, a Lie subgroup, is always closed.

We borrowed the terminology "immersed subgroup" from Fulton and Harris [**?**] (Chapter 7), but we warn the reader that most books call such subgroups "Lie subgroups" and refer to the second kind of subgroups (that are submanifolds) as "closed subgroups".

Theorem 5.3.4 *Let G be a Lie group and let (H, φ) be an immersed Lie subgroup of G. Then, φ is an embedding iff $\varphi(H)$ is closed in G. As as consequence, any Lie subgroup of G is closed.*

Proof. The proof can be found in Warner [**?**] (Chapter 1, Theorem 3.21) and uses a little more machinery than we have introduced.

However, we can prove easily that a Lie subgroup, H, of G is closed.

We also have the following important and useful theorem: If G is a Lie group, say that a subset, $H \subseteq G$, is an *abstract subgroup* iff it is just a subgroup of the underlying group of G (i.e., we forget the topology and the manifold structure).

Theorem 5.3.5 *Let G be a Lie group. An abstract subgroup, H, of G is a submanifold (i.e., a Lie subgroup) of G iff H is closed (i.e, H with the induced topology is closed in G).*

5.4 The Correspondence Lie Groups–Lie Algebras

Historically, Lie was the first to understand that a lot of the structure of a Lie group is captured by its Lie algebra, a simpler object (since it is a vector space). In this short section, we state without proof some of the "Lie theorems", although not in their original form.

Definition 5.4.1 If $\mathfrak{g}$ is a Lie algebra, a *subalgebra*, $\mathfrak{h}$, of $\mathfrak{g}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$, for all $u, v \in \mathfrak{h}$. If $\mathfrak{h}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$ for all $u \in \mathfrak{h}$ and all $v \in \mathfrak{g}$, we say that $\mathfrak{h}$ is an *ideal* in $\mathfrak{g}$.

For a proof of the theorem below, see Warner [**?**] (Chapter 3) or Duistermaat and Kolk [**?**] (Chapter 1, Section 10).

Theorem 5.4.2 *Let G be a Lie group with Lie algebra, $\mathfrak{g}$, and let (H, φ) be an immersed Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $d\varphi_1\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, for each subalgebra, $\widetilde{\mathfrak{h}}$, of $\mathfrak{g}$, there is a unique connected immersed subgroup, (H, φ), of G so that $d\varphi_1\mathfrak{h} = \widetilde{\mathfrak{h}}$. In fact, as a group, $\varphi(H)$ is the subgroup of G generated by $\exp(\widetilde{\mathfrak{h}})$. Furthermore, normal subgroups correspond to ideals.*

Theorem 5.4.2 shows that there is a one-to-one correspondence between connected immersed subgroups of a Lie group and subalgebras of its Lie algebra.

Theorem 5.4.3 *Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For every homomorphism,*
$\psi\colon \mathfrak{g} \to \mathfrak{h}$, there is a unique Lie group homomorphism,
$\varphi\colon G \to H$, so that $d\varphi_1 = \psi$.

Again a proof of the theorem above is given in Warner [**?**] (Chapter 3) or Duistermaat and Kolk [**?**] (Chapter 1, Section 10).

Corollary 5.4.4 *If G and H are connected and simply connected Lie groups, then G and H are isomorphic iff $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

It can also be shown that for every finite-dimensional Lie algebra, $\mathfrak{g}$, there is a connected and simply connected Lie group, G, such that $\mathfrak{g}$ is the Lie algebra of G.

This is a consequence of deep theorem (whose proof is quite hard) known as *Ado's theorem*. For more on this, see Knapp [**?**], Fulton and Harris [**?**], or Bourbaki [**?**].

In summary, following Fulton and Harris, we have the following two principles of the Lie group/Lie algebra correspondence:

First Principle: If G and H are Lie groups, with G connected, then a homomorphism of Lie groups,
$\varphi\colon G \to H$, is uniquely determined by the Lie algebra homomorphism, $d\varphi_1\colon \mathfrak{g} \to \mathfrak{h}$.

Second Principle: Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras.

A linear map, $\psi\colon \mathfrak{g} \to \mathfrak{h}$, is a Lie algebra map iff there is a unique Lie group homomorphism, $\varphi\colon G \to H$, so that $d\varphi_1 = \psi$.

5.5 More on the Lorentz Group $\mathbf{SO}_0(n,1)$

In this section, we take a closer look at the Lorentz group $\mathbf{SO}_0(n,1)$ and, in particular, at the relationship between $\mathbf{SO}_0(n,1)$ and its Lie algebra, $\mathfrak{so}(n,1)$.

The Lie algebra of $\mathbf{SO}_0(n,1)$ is easily determined by computing the tangent vectors to curves, $t \mapsto A(t)$, on $\mathbf{SO}_0(n,1)$ through the identity, I. Since $A(t)$ satisfies

$$A^\top JA = J,$$

differentiating and using the fact that $A(0) = I$, we get

$$A'^\top J + JA' = 0.$$

Therefore,

$$\mathfrak{so}(n,1) = \{A \in \mathrm{Mat}_{n+1,n+1}(\mathbb{R}) \mid A^\top J + JA = 0\}.$$

This means that JA is skew-symmetric and so,

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} B & u \\ u^\top & 0 \end{pmatrix} \mid u \in \mathbb{R}^n, \quad B^\top = -B \right\}.$$

Observe that every matrix $A \in \mathfrak{so}(n, 1)$ can be written uniquely as

$$\begin{pmatrix} B & u \\ u^\top & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0^\top & 0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix},$$

where the first matrix is skew-symmetric, the second one is symmetric and both belong to $\mathfrak{so}(n, 1)$.

Thus, it is natural to define

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0^\top & 0 \end{pmatrix} \mid B^\top = -B \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix} \mid u \in \mathbb{R}^n \right\}.$$

It is immediately verified that both $\mathfrak{k}$ and $\mathfrak{p}$ are subspaces of $\mathfrak{so}(n, 1)$ (as vector spaces) and that $\mathfrak{k}$ is a Lie subalgebra isomorphic to $\mathfrak{so}(n)$, but $\mathfrak{p}$ is *not* a Lie subalgebra of $\mathfrak{so}(n, 1)$ because it is not closed under the Lie bracket.

Still, we have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Clearly, we have the direct sum decomposition

$$\mathfrak{so}(n, 1) = \mathfrak{k} \oplus \mathfrak{p},$$

known as *Cartan decomposition*.

There is also an automorphism of $\mathfrak{so}(n, 1)$ known as the *Cartan involution*, namely,

$$\theta(A) = -A^{\top},$$

and we see that

$$\mathfrak{k} = \{A \in \mathfrak{so}(n, 1) \mid \theta(A) = A\}$$

abd

$$\mathfrak{p} = \{A \in \mathfrak{so}(n, 1) \mid \theta(A) = -A\}.$$

Unfortunately, there does not appear to be any simple way of obtaining a formula for $\exp(A)$, where $A \in \mathfrak{so}(n, 1)$ (except for small n—there is such a formula for $n = 3$ due to Chris Geyer).

However, it is possible to obtain an explicit formula for the matrices in $\mathfrak{p}$. This is because for such matrices, A, if we let $\omega = \|u\| = \sqrt{u^\top u}$, we have

$$A^3 = \omega^2 A.$$

Thus, we get

Proposition 5.5.1 *For every matrix, $A \in \mathfrak{p}$, of the form*

$$A = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix},$$

we have

$$\begin{aligned} e^A &= \begin{pmatrix} I + \dfrac{(\cosh\omega - 1)}{\omega^2} uu^\top & \dfrac{\sinh\omega}{\omega} u \\ \dfrac{\sinh\omega}{\omega} u^\top & \cosh\omega \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{I + \dfrac{\sinh^2\omega}{\omega^2} uu^\top} & \dfrac{\sinh\omega}{\omega} u \\ \dfrac{\sinh\omega}{\omega} u^\top & \cosh\omega \end{pmatrix}. \end{aligned}$$

Now, it clear from the above formula that each e^B, with $B \in \mathfrak{p}$ is a Lorentz boost. Conversely, every Lorentz boost is the exponential of some $B \in \mathfrak{p}$, as shown below.

Proposition 5.5.2 *Every Lorentz boost,*

$$A = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

with $c = \sqrt{\|v\|^2 + 1}$, *is of the form* $A = e^B$, *for* $B \in \mathfrak{p}$, *i.e., for some* $B \in \mathfrak{so}(n, 1)$ *of the form*

$$B = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}.$$

Remarks:

(1) It is easy to show that the eigenvalues of matrices

$$B = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}$$

are 0, with multiplicity $n-1$, $\|u\|$ and $-\|u\|$. Eigenvectors are also easily determined.

(2) The matrices $B \in \mathfrak{so}(n,1)$ of the form

$$B = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha \\ 0 & \cdots & \alpha & 0 \end{pmatrix}$$

are easily seen to form an abelian Lie subalgebra, $\mathfrak{a}$, of $\mathfrak{so}(n,1)$ (which means that for all $B, C \in \mathfrak{a}$, $[B,C] = 0$, i.e., $BC = CB$).

One will easily check that for any $B \in \mathfrak{a}$, as above, we get

$$e^B = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix}$$

The matrices of the form e^B, with $B \in \mathfrak{a}$, form an abelian subgroup, A, of $\mathbf{SO}_0(n,1)$ isomorphic to $\mathbf{SO}_0(1,1)$.

As we already know, the matrices $B \in \mathfrak{so}(n,1)$ of the form

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where B is skew-symmetric, form a Lie subalgebra, $\mathfrak{k}$, of $\mathfrak{so}(n,1)$.

Clearly, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(n)$ and using the exponential, we get a subgroup, K, of $\mathbf{SO}_0(n,1)$ isomorphic to $\mathbf{SO}(n)$.

It is also clear that $\mathfrak{k} \cap \mathfrak{a} = (0)$, but $\mathfrak{k} \oplus \mathfrak{a}$ is *not* equal to $\mathfrak{so}(n,1)$. What is the missing piece?

Consider the matrices $N \in \mathfrak{so}(n,1)$ of the form

$$N = \begin{pmatrix} 0 & -u & u \\ u^\top & 0 & 0 \\ u^\top & 0 & 0 \end{pmatrix},$$

where $u \in \mathbb{R}^{n-1}$.

The reader should check that these matrices form an abelian Lie subalgebra, $\mathfrak{n}$, of $\mathfrak{so}(n,1)$ and that

$$\mathfrak{so}(n,1) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

This is the *Iwasawa decomposition* of the Lie algebra $\mathfrak{so}(n,1)$.

Furthermore, the reader should check that every $N \in \mathfrak{n}$ is nilpotent; in fact, $N^3 = 0$. (It turns out that $\mathfrak{n}$ is a nilpotent Lie algebra, see Knapp [**?**]).

The connected Lie subgroup of $\mathbf{SO}_0(n,1)$ associated with $\mathfrak{n}$ is denoted N and it can be shown that we have the *Iwasawa decomposition* of the Lie group $\mathbf{SO}_0(n,1)$:

$$\mathbf{SO}_0(n,1) = KAN.$$

It is easy to check that $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$, so $\mathfrak{a} \oplus \mathfrak{n}$ is a Lie subalgebra of $\mathfrak{so}(n,1)$ and $\mathfrak{n}$ is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$.

This implies that N is normal in the group corresponding to $\mathfrak{a} \oplus \mathfrak{n}$, so AN is a subgroup (in fact, solvable) of $\mathbf{SO}_0(n, 1)$.

For more on the Iwasawa decomposition, see Knapp [**?**]. Observe that the image, $\overline{\mathfrak{n}}$, of $\mathfrak{n}$ under the Cartan involution, θ, is the Lie subalgebra

$$\overline{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & u & u \\ -u^\top & 0 & 0 \\ u^\top & 0 & 0 \end{pmatrix} \mid u \in \mathbb{R}^{n-1} \right\}.$$

It is easy to see that the centralizer of $\mathfrak{a}$ is the Lie subalgebra

$$\mathfrak{m} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{n+1,n+1}(\mathbb{R}) \mid B \in \mathfrak{so}(n-1) \right\}$$

and the reader should check that

$$\mathfrak{so}(n, 1) = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}.$$

We also have

$$[\mathfrak{m}, \mathfrak{n}] \subseteq \mathfrak{n},$$

so $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{so}(n, 1)$.

The group, M, associated with $\mathfrak{m}$ is isomorphic to $\mathbf{SO}(n-1)$ and it can be shown that $B = MAN$ is a subgroup of $\mathbf{SO}_0(n, 1)$. In fact,

$$\begin{aligned}\mathbf{SO}_0(n, 1)/(MAN) &= KAN/MAN = K/M \\ &= \mathbf{SO}(n)/\mathbf{SO}(n-1) = S^{n-1}.\end{aligned}$$

It is customary to denote the subalgebra $\mathfrak{m} \oplus \mathfrak{a}$ by $\mathfrak{g}_0$, the algebra $\mathfrak{n}$ by $\mathfrak{g}_1$ and $\overline{\mathfrak{n}}$ by $\mathfrak{g}_{-1}$, so that $\mathfrak{so}(n, 1) = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$ is also written

$$\mathfrak{so}(n, 1) = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

By the way, if $N \in \mathfrak{n}$, then

$$e^N = I + N + \frac{1}{2}N^2,$$

and since $N + \frac{1}{2}N^2$ is also nilpotent, e^N can't be diagonalized when $N \neq 0$.

This provides a simple example of matrices in $\mathbf{SO}_0(n,1)$ that can't be diagonalized.

Combining Proposition 3.3.1 and Proposition 5.5.2, we have the corollary:

Corollary 5.5.3 *Every matrix* $A \in \mathbf{O}(n,1)$ *can be written as*

$$A = \begin{pmatrix} Q & 0 \\ 0 & \epsilon \end{pmatrix} e^{\begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}}$$

where $Q \in \mathbf{O}(n)$, $\epsilon = \pm 1$ *and* $u \in \mathbb{R}^n$.

Observe that Corollary 5.5.3 proves that every matrix, $A \in \mathbf{SO}_0(n,1)$, can be written as

$$A = Pe^S, \quad \text{with } P \in K \cong \mathbf{SO}(n) \text{ and } S \in \mathfrak{p},$$

i.e.,

$$\mathbf{SO}_0(n,1) = K \exp(\mathfrak{p}),$$

a version of the polar decomposition for $\mathbf{SO}_0(n,1)$.

Now, it is known that the exponential map, $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$, is surjective. So, when $A \in \mathbf{SO}_0(n,1)$, since then $Q \in \mathbf{SO}(n)$ and $\epsilon = +1$, the matrix

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

is the exponential of some skew symmetric matrix

$$C = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{so}(n,1),$$

and we can write $A = e^C e^Z$, with $C \in \mathfrak{k}$ and $Z \in \mathfrak{p}$.

Unfortunately, C and Z generally don't commute, so it is generally not true that $A = e^{C+Z}$. Thus, we don't get an "easy" proof of the surjectivity of the exponential $\exp\colon \mathfrak{so}(n,1) \to \mathbf{SO}_0(n,1)$.

This is not too surprising because, to the best of our knowledge, proving surjectivity for all n is not a simple matter.

One proof is due to Nishikawa [**?**] (1983). Nishikawa's paper is rather short, but this is misleading. Indeed, Nishikawa relies on a classic paper by Djokovic [**?**], which itself relies heavily on another fundamental paper by Burgoyne and Cushman [**?**], published in 1977.

Burgoyne and Cushman determine the conjugacy classes for some linear Lie groups and their Lie algebras, where the linear groups arise from an inner product space (real or complex). This inner product is nondegenerate, symmetric, or hermitian or skew-symmetric of skew-hermitian. Altogether, one has to read over 40 pages to fully understand the proof of surjectivity.

In his introduction, Nishikawa states that he is not aware of any other proof of the surjectivity of the exponential for $\mathbf{SO}_0(n, 1)$.

However, such a proof was also given by Marcel Riesz as early as 1957, in some lectures notes that he gave while visiting the University of Maryland in 1957-1958. These notes were probably not easily available until 1993, when they were published in book form, with commentaries, by Bolinder and Lounesto [**?**].

Interestingly, these two proofs use very different methods. The Nishikawa–Djokovic–Burgoyne and Cushman proof makes heavy use of methods in Lie groups and Lie algebra, although not far beyond linear algebra.

Riesz's proof begins with a deep study of the structure of the minimal polynomial of a Lorentz isometry (Chapter III). This is a beautiful argument that takes about 10 pages.

The story is not over, as it takes most of Chapter IV (some 40 pages) to prove the surjectivity of the exponential (actually, Riesz proves other things along the way). In any case, the reader can see that both proofs are quite involved.

It is worth noting that Milnor (1969) also uses techniques very similar to those used by Riesz (in dealing with minimal polynomials of isometries) in his paper on isometries of inner product spaces [**?**].

What we will do to close this section is to give a relatively simple proof that the exponential map,
$\exp\colon \mathfrak{so}(1,3) \to \mathbf{SO}_0(1,3)$, is surjective.

In the case of $\mathbf{SO}_0(1,3)$, we can use the fact that $\mathbf{SL}(2,\mathbb{C})$ is a two-sheeted covering space of $\mathbf{SO}_0(1,3)$, which means that there is a homomorphism, $\varphi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$, which is surjective and that $\mathrm{Ker}\, \varphi = \{-I, I)$.

Then, the small miracle is that, although the exponential, $\exp\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$, is *not* surjective, for every $A \in \mathbf{SL}(2,\mathbb{C})$, *either A or $-A$ is in the image of the exponential!*

Proposition 5.5.4 *Given any matrix*

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}),$$

let ω be any of the two complex roots of a^2+bc. If $\omega \neq 0$, then

$$e^B = \cosh\omega\, I + \frac{\sinh\,\omega}{\omega}\, B,$$

and $e^B = I + B$, if $a^2+bc = 0$. Furthermore, every matrix $A \in \mathbf{SL}(2,\mathbb{C})$ is in the image of the exponential map, unless $A = -I + N$, where N is a nonzero nilpotent (i.e., $N^2 = 0$ with $N \neq 0$). Consequently, for any $A \in \mathbf{SL}(2,\mathbb{C})$, either A or $-A$ is of the form e^B, for some $B \in \mathfrak{sl}(2,\mathbb{C})$.

Remark: If we restrict our attention to $\mathbf{SL}(2,\mathbb{R})$, then we have the following proposition that can be used to prove that the exponential map $\exp\colon \mathfrak{so}(1,2) \to \mathbf{SO}_0(1,2)$ is surjective:

Proposition 5.5.5 *Given any matrix*

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

if $a^2 + b > 0$*, then let* $\omega = \sqrt{a^2 + bc} > 0$ *and if* $a^2 + b < 0$*, then let* $\omega = \sqrt{-(a^2 + bc)} > 0$ *(i.e.,* $\omega^2 = -(a^2 + bc)$*). In the first case* ($a^2 + bc > 0$)*, we have*

$$e^B = \cosh \omega \, I + \frac{\sinh \, \omega}{\omega} \, B,$$

and in the second case ($a^2 + bc < 0$)*, we have*

$$e^B = \cos \omega \, I + \frac{\sin \, \omega}{\omega} \, B.$$

If $a^2 + bc = 0$*, then* $e^B = I + B$*. Furthermore, every matrix* $A \in \mathbf{SL}(2, \mathbb{R})$ *whose trace satisfies* $\mathrm{tr}(A) \geq -2$ *in the image of the exponential map. Consequently, for any* $A \in \mathbf{SL}(2, \mathbb{R})$*, either* A *or* $-A$ *is of the form* e^B*, for some* $B \in \mathfrak{sl}(2, \mathbb{R})$*.*

We now return to the relationship between $\mathbf{SL}(2, \mathbb{C})$ and $\mathbf{SO}_0(1, 3)$.

In order to define a homomorphism $\varphi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$, we begin by defining a linear bijection, h, between $\mathbb{R}^4$ and $\mathbf{H}(2)$, the set of complex 2×2 Hermitian matrices, by

$$(t,x,y,z) \mapsto \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}.$$

Those familiar with quantum physics will recognize a linear combination of the Pauli matrices! The inverse map is easily defined and we leave it as an exercise.

For instance, given a Hermitian matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$t = \frac{a+d}{2},\; x = \frac{a-d}{2}, \quad \text{etc.}$$

Next, for any $A \in \mathbf{SL}(2, \mathbb{C})$, we define a map, $l_A \colon \mathbf{H}(2) \to \mathbf{H}(2)$, *via*

$$S \mapsto ASA^*.$$

(Here, $A^* = \overline{A}^\top$.)

Using the linear bijection $h \colon \mathbb{R}^4 \to \mathbf{H}(2)$ and its inverse, we obtain a map $\mathrm{lor}_A \colon \mathbb{R}^4 \to \mathbb{R}^4$, where

$$\mathrm{lor}_A = h^{-1} \circ l_A \circ h.$$

As ASA^* is hermitian, we see that l_A is well defined. It is obviously linear and since $\det(A) = 1$ (recall, $A \in \mathbf{SL}(2, \mathbb{C})$) and

$$\det \begin{pmatrix} t + x & y - iz \\ y + iz & t - x \end{pmatrix} = t^2 - x^2 - y^2 - z^2,$$

we see that lor_A preserves the Lorentz metric!

Furthermore, it is not hard to prove that $\mathbf{SL}(2, \mathbb{C})$ is connected (use the polar form or analyze the eigenvalues of a matrix in $\mathbf{SL}(2, \mathbb{C})$, for example, as in Duistermatt and Kolk [**?**] (Chapter 1, Section 1.2)) and that the map

$$\varphi \colon A \mapsto \mathrm{lor}_A$$

is a continuous group homomorphism. Thus, the range of φ is a connected subgroup of $\mathbf{SO}_0(1, 3)$.

This shows that $\varphi \colon \mathbf{SL}(2, \mathbb{C}) \to \mathbf{SO}_0(1, 3)$ is indeed a homomorphism. It remains to prove that it is surjective and that its kernel is $\{I, -I\}$.

Proposition 5.5.6 *The homomorphism,* $\varphi \colon \mathbf{SL}(2, \mathbb{C}) \to \mathbf{SO}_0(1, 3)$*, is surjective and its kernel is* $\{I, -I\}$*.*

Remark: The group $\mathbf{SL}(2, \mathbb{C})$ is isomorphic to the group $\mathbf{Spin}(1, 3)$, which is a (simply-connected) double-cover of $\mathbf{SO}_0(1, 3)$.

This is a standard result of Clifford algebra theory, see Bröcker and tom Dieck [**?**] or Fulton and Harris [**?**]. What we just did is to provide a direct proof of this fact.

We just proved that there is an isomorphism

$$\mathbf{SL}(2, \mathbb{C})/\{I, -I\} \cong \mathbf{SO}_0(1, 3).$$

However, the reader may recall that $\mathbf{SL}(2, \mathbb{C})/\{I, -I\} = \mathbf{PSL}(2, \mathbb{C}) \cong \mathbf{M\ddot{o}b}^+$. Therefore, the Lorentz group is isomorphic to the Möbius group.

We now have all the tools to prove that the exponential map, $\exp\colon \mathfrak{so}(1, 3) \to \mathbf{SO}_0(1, 3)$, is surjective.

Theorem 5.5.7 *The exponential map,*
$\exp\colon \mathfrak{so}(1,3) \to \mathbf{SO}_0(1,3)$, *is surjective.*

Proof. First, recall from Proposition 5.2.5 that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{SL}(2,\mathbb{C}) & \xrightarrow{\varphi} & \mathbf{SO}_0(1,3)\,. \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{sl}(2,\mathbb{C}) & \xrightarrow[d\varphi_1]{} & \mathfrak{so}(1,3) \end{array}$$

Pick any $A \in \mathbf{SO}_0(1,3)$. By Proposition 5.5.6, the homomorphism φ is surjective and as $\mathrm{Ker}\,\varphi = \{I, -I\}$, there exists some $B \in \mathbf{SL}(2,\mathbb{C})$ so that

$$\varphi(B) = \varphi(-B) = A.$$

Now, by Proposition 5.5.4, for any $B \in \mathbf{SL}(2,\mathbb{C})$, either B or $-B$ is of the form e^C, for some $C \in \mathfrak{sl}(2,\mathbb{C})$. By the commutativity of the diagram, if we let
$D = d\varphi_1(C) \in \mathfrak{so}(1,3)$, we get

$$A = \varphi(\pm e^C) = e^{d\varphi_1(C)} = e^D,$$

with $D \in \mathfrak{so}(1,3)$, as required. $\square$

Remark: We can restrict the bijection $h\colon \mathbb{R}^4 \to \mathbf{H}(2)$ defined earlier to a bijection between $\mathbb{R}^3$ and the space of real symmetric matrices of the form

$$\begin{pmatrix} t+x & y \\ y & t-x \end{pmatrix}.$$

Then, if we also restrict ourselves to $\mathbf{SL}(2,\mathbb{R})$, for any $A \in \mathbf{SL}(2,\mathbb{R})$ and any symmetric matrix, S, as above, we get a map

$$S \mapsto ASA^{\top}.$$

The reader should check that these transformations correspond to isometries in $\mathbf{SO}_0(1,2)$ and we get a homomorphism, $\varphi\colon \mathbf{SL}(2,\mathbb{R}) \to \mathbf{SO}_0(1,2)$.

Then, we have a version of Proposition 5.5.6 for $\mathbf{SL}(2,\mathbb{R})$ and $\mathbf{SO}_0(1,2)$:

Proposition 5.5.8 *The homomorphism,*
$\varphi\colon \mathbf{SL}(2, \mathbb{R}) \to \mathbf{SO}_0(1, 2)$, *is surjective and its kernel is* $\{I, -I\}$.

Using Proposition 5.5.8 and Proposition 5.5.5, we get a version of Theorem 5.5.7 for $\mathbf{SO}_0(1, 2)$:

Theorem 5.5.9 *The exponential map,*
$\exp\colon \mathfrak{so}(1, 2) \to \mathbf{SO}_0(1, 2)$, *is surjective.*

Also observe that $\mathbf{SO}_0(1, 1)$ consists of the matrices of the form

$$A = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

and a direct computation shows that

$$e^{\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}.$$

Thus, we see that the map $\exp\colon \mathfrak{so}(1,1) \to \mathbf{SO}_0(1,1)$ is also surjective.

Therefore, we have proved that $\exp\colon \mathfrak{so}(1,n) \to \mathbf{SO}_0(1,n)$ is surjective for $n = 1, 2, 3$.

This actually holds for all $n \geq 1$, but the proof is much more involved, as we already discussed earlier.