3.3 The Lorentz Groups $\mathbf{O}(n,1)$, $\mathbf{SO}(n,1)$ and $\mathbf{SO}_0(n,1)$

The Lorentz group provides another interesting example. Moreover, the Lorentz group $\mathbf{SO}(3,1)$ shows up in an interesting way in computer vision.

Denote the $p \times p$-identity matrix by I_p, for $p, q, \geq 1$, and define

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

If $n = p + q$, the matrix $I_{p,q}$ is associated with the non-degenerate symmetric bilinear form

$$\varphi_{p,q}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j$$

with associated quadratic form

$$\Phi_{p,q}((x_1, \ldots, x_n)) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{n} x_j^2.$$

In particular, when $p = 1$ and $q = 3$, we have the *Lorentz metric*

$$x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

In physics, x_1 is interpreted as time and written t and x_2, x_3, x_4 as coordinates in $\mathbb{R}^3$ and written x, y, z. Thus, the Lozentz metric is usually written a

$$t^2 - x^2 - y^2 - z^2,$$

although it also appears as

$$x^2 + y^2 + z^2 - t^2,$$

which is equivalent but slightly less convenient for certain purposes, as we will see later. The space $\mathbb{R}^4$ with the Lorentz metric is called *Minkowski space*. It plays an important role in Einstein's theory of special relativity.

The group $\mathbf{O}(p, q)$ is the set of all $n \times n$-matrices

$$\mathbf{O}(p, q) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q}\}.$$

This is the group of all invertible linear maps of $\mathbb{R}^n$ that preserve the quadratic form, $\Phi_{p,q}$, i.e., the group of isometries of $\Phi_{p,q}$.

Clearly, $I_{p,q}^2 = I$, so the condition $A^\top I_{p,q} A = I_{p,q}$ is equivalent to $I_{p,q} A^\top I_{p,q} A = I$, which means that

$$A^{-1} = I_{p,q} A^\top I_{p,q}.$$

Thus, $A I_{p,q} A^\top = I_{p,q}$ also holds, which shows that $\mathbf{O}(p, q)$ is closed under transposition (i.e., if $A \in \mathbf{O}(p, q)$, then $A^\top \in \mathbf{O}(p, q)$).

We have the subgroup

$$\mathbf{SO}(p, q) = \{A \in \mathbf{O}(p, q) \mid \det(A) = 1\}$$

consisting of the isometries of $(\mathbb{R}^n, \Phi_{p,q})$ with determinant $+1$. It is clear that $\mathbf{SO}(p, q)$ is also closed under transposition.

The condition $A^\top I_{p,q} A = I_{p,q}$ has an interpretation in terms of the inner product $\varphi_{p,q}$ and the columns (and rows) of A.

Indeed, if we denote the jth column of A by A_j, then

$$A^\top I_{p,q} A = (\varphi_{p,q}(A_i, A_j)),$$

so $A \in \mathbf{O}(p,q)$ iff the columns of A form an "orthonormal basis" w.r.t. $\varphi_{p,q}$, i.e.,

$$\varphi_{p,q}(A_i, A_j) = \begin{cases} \delta_{ij} & \text{if } 1 \leq i, j \leq p; \\ -\delta_{ij} & \text{if } p+1 \leq i, j \leq p+q. \end{cases}$$

The difference with the usual orthogonal matrices is that $\varphi_{p,q}(A_i, A_i) = -1$, if $p+1 \leq i \leq p+q$. As $\mathbf{O}(p,q)$ is closed under transposition, the rows of A also form an orthonormal basis w.r.t. $\varphi_{p,q}$.

It turns out that $\mathbf{SO}(p,q)$ has two connected components and the component containing the identity is a subgroup of $\mathbf{SO}(p,q)$ denoted $\mathbf{SO}_0(p,q)$.

The group $\mathbf{SO}_0(p,q)$ turns out to be homeomorphic to $\mathbf{SO}(p) \times \mathbf{SO}(q) \times \mathbb{R}^{pq}$, but this is not easy to prove. (One way to prove it is to use results on pseudo-algebraic subgroups of $\mathbf{GL}(n,\mathbb{C})$, see Knapp [**?**] or Gallier's notes on Clifford algebras (on the web)).

We will now determine the polar decomposition and the SVD decomposition of matrices in the Lorentz groups $\mathbf{O}(n,1)$ and $\mathbf{SO}(n,1)$.

Write $J = I_{n,1}$ and, given any $A \in \mathbf{O}(n,1)$, write

$$A = \begin{pmatrix} B & u \\ v^\top & c \end{pmatrix},$$

where B is an $n \times n$ matrix, u, v are (column) vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$.

We begin with the polar decomposition of matrices in the Lorentz groups $\mathbf{O}(n,1)$.

Proposition 3.3.1 *Every matrix $A \in \mathbf{O}(n,1)$ has a polar decomposition of the form*

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

or

$$A = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $Q \in \mathbf{O}(n)$ and $c = \sqrt{\|v\|^2 + 1}$.

Thus, we see that $\mathbf{O}(n,1)$ has four components corresponding to the cases:

(1) $Q \in \mathbf{O}(n)$; $\det(Q) < 0$; $+1$ as the lower right entry of the orthogonal matrix;

(2) $Q \in \mathbf{SO}(n)$; -1 as the lower right entry of the orthogonal matrix;

(3) $Q \in \mathbf{O}(n)$; $\det(Q) < 0$; -1 as the lower right entry of the orthogonal matrix;

(4) $Q \in \mathbf{SO}(n)$; $+1$ as the lower right entry of the orthogonal matrix.

Observe that $\det(A) = -1$ in cases (1) and (2) and that $\det(A) = +1$ in cases (3) and (4).

Thus, (3) and (4) correspond to the group $\mathbf{SO}(n,1)$, in which case the polar decomposition is of the form

$$A = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $Q \in \mathbf{O}(n)$, with $\det(Q) = -1$ and $c = \sqrt{\|v\|^2 + 1}$ or

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

where $Q \in \mathbf{SO}(n)$ and $c = \sqrt{\|v\|^2 + 1}$.

The components in (1) and (2) are not groups. We will show later that all four components are connected and that case (4) corresponds to a group (Proposition 3.3.7).

This group is the connected component of the identity and it is denoted $\mathbf{SO}_0(n,1)$ (see Corollary 3.4.12).

For the time being, note that $A \in \mathbf{SO}_0(n,1)$ iff $A \in \mathbf{SO}(n,1)$ and $a_{n+1\,n+1}$ $(= c) > 0$ (here, $A = (a_{i\,j})$.) In fact, we proved above that if $a_{n+1\,n+1} > 0$, then $a_{n+1\,n+1} \geq 1$.

Remark: If we let

$$\Lambda_P = \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda_T = I_{n,1},$$

where

$$I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

then we have the disjoint union

$$\begin{aligned} \mathbf{O}(n,1) = \mathbf{SO}_0(n,1) &\cup \Lambda_P \mathbf{SO}_0(n,1) \\ &\cup \Lambda_T \mathbf{SO}_0(n,1) \cup \Lambda_P \Lambda_T \mathbf{SO}_0(n,1). \end{aligned}$$

In order to determine the SVD of matrices in $\mathbf{SO}_0(n,1)$, we analyze the eigenvectors and the eigenvalues of the positive definite symmetric matrix

$$S = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

involved in Proposition 3.3.1.

Such a matrix is called a *Lorentz boost*. Observe that if $v = 0$, then $c = 1$ and $S = I_{n+1}$.

Proposition 3.3.2 *Assume $v \neq 0$. The eigenvalues of the symmetric positive definite matrix*

$$S = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $c = \sqrt{\|v\|^2 + 1}$, are 1 with multiplicity $n - 1$, and e^α and $e^{-\alpha}$ each with multiplicity 1 (for some $\alpha \geq 0$). An orthonormal basis of eigenvectors of S consists of vectors of the form

$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{v}{\sqrt{2}\|v\|} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{v}{\sqrt{2}\|v\|} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

where the $u_i \in \mathbb{R}^n$ are all orthogonal to v and pairwise orthogonal.

Corollary 3.3.3 *The singular values of any matrix $A \in \mathbf{O}(n,1)$ are 1 with multiplicity $n-1$, e^{α}, and $e^{-\alpha}$, for some $\alpha \geq 0$.*

Note that the case $\alpha = 0$ is possible, in which case, A is an orthogonal matrix of the form

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix},$$

with $Q \in \mathbf{O}(n)$. The two singular values e^{α} and $e^{-\alpha}$ tell us how much A deviates from being orthogonal.

We can now determine a convenient form for the SVD of matrices in $\mathbf{O}(n,1)$.

Theorem 3.3.4 *Every matrix $A \in \mathbf{O}(n,1)$ can be written as*

$$A = \begin{pmatrix} P & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} Q^\top & 0 \\ 0 & 1 \end{pmatrix}$$

with $\epsilon = \pm 1$, $P \in \mathbf{O}(n)$ and $Q \in \mathbf{SO}(n)$. When $A \in \mathbf{SO}(n,1)$, we have $\det(P)\epsilon = +1$, and when $A \in \mathbf{SO}_0(n,1)$, we have $\epsilon = +1$ and $P \in \mathbf{SO}(n)$, that is,

$$A = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} Q^\top & 0 \\ 0 & 1 \end{pmatrix}$$

with $P \in \mathbf{SO}(n)$ and $Q \in \mathbf{SO}(n)$.

Remark: We warn our readers about Chapter 6 of Baker's book [**?**]. Indeed, this chapter is seriously flawed.

The main two Theorems (Theorem 6.9 and Theorem 6.10) are false and as consequence, the proof of Theorem 6.11 is wrong too. Theorem 6.11 states that the exponential map $\exp\colon \mathfrak{so}(n,1) \to \mathbf{SO}_0(n,1)$ is surjective, which is correct, but known proofs are nontrivial and quite lengthy (see Section 5.5).

The proof of Theorem 6.12 is also false, although the theorem itself is correct (this is our Theorem 5.5.7, see Section 5.5).

For a thorough analysis of the eigenvalues of Lorentz isometries (and much more), one should consult Riesz [**?**] (Chapter III).

Clearly, a result similar to Theorem 3.3.4 also holds for the matrices in the groups $\mathbf{O}(1,n)$, $\mathbf{SO}(1,n)$ and $\mathbf{SO}_0(1,n)$.

For example, every matrix $A \in \mathbf{SO}_0(1, n)$ can be written as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & \cdots & 0 \\ \sinh\alpha & \cosh\alpha & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^\top \end{pmatrix},$$

where $P, Q \in \mathbf{SO}(n)$.

In the case $n = 3$, we obtain the *proper orthochronous Lorentz group*, $\mathbf{SO}_0(1, 3)$, also denoted $\mathbf{Lor}(1, 3)$. By the way, $\mathbf{O}(1, 3)$ is called the *(full) Lorentz group* and $\mathbf{SO}(1, 3)$ is the *special Lorentz group*.

Theorem 3.3.4 (really, the version for $\mathbf{SO}_0(1,n)$) shows that the Lorentz group $\mathbf{SO}_0(1,3)$ is generated by the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \quad \text{with } P \in \mathbf{SO}(3)$$

and the matrices of the form

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This fact will be useful when we prove that the homomorphism $\varphi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$ is surjective.

Remark: Unfortunately, unlike orthogonal matrices which can always be diagonalized over $\mathbb{C}$, **not** every matrix in $\mathbf{SO}(1,n)$ can be diagonalized for $n \geq 2$. This has to do with the fact that the Lie algebra $\mathfrak{so}(1,n)$ has non-zero idempotents (see Section 5.5).

It turns out that the group $\mathbf{SO}_0(1,3)$ admits another interesting characterization involving the hypersurface

$$\mathcal{H} = \{(t,x,y,z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 1\}.$$

This surface has two sheets and it is not hard to show that $\mathbf{SO}_0(1,3)$ is the subgroup of $\mathbf{SO}(1,3)$ that preserves these two sheets (does not swap them).

Actually, we will prove this fact for any n. In preparation for this we need some definitions and a few propositions.

Let us switch back to $\mathbf{SO}(n,1)$. First, as a matter of notation, we write every $u \in \mathbb{R}^{n+1}$ as $u = (\mathbf{u}, t)$, where $\mathbf{u} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, so that the Lorentz inner product can be expressed as

$$\langle u, v\rangle = \langle (\mathbf{u}, t), (\mathbf{v}, s)\rangle = \mathbf{u}\cdot\mathbf{v} - ts,$$

where $\mathbf{u}\cdot\mathbf{v}$ is the standard Euclidean inner product (the Euclidean norm of x is denoted $\|x\|$).

Then, we can classify the vectors in $\mathbb{R}^{n+1}$ as follows:

Definition 3.3.5 A nonzero vector, $u = (\mathbf{u}, t) \in \mathbb{R}^{n+1}$ is called

(a) *spacelike* iff $\langle u, u\rangle > 0$, i.e., iff $\|\mathbf{u}\|^2 > t^2$;

(b) *timelike* iff $\langle u, u\rangle < 0$, i.e., iff $\|\mathbf{u}\|^2 < t^2$;

(c) *lightlike* or *isotropic* iff $\langle u, u\rangle = 0$, i.e., iff $\|\mathbf{u}\|^2 = t^2$.

A spacelike (resp. timelike, resp. lightlike) vector is said to be *positive* iff $t > 0$ and *negative* iff $t < 0$. The set of all isotropic vectors

$$\mathcal{H}_n(0) = \{u = (\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 = t^2\}$$

is called the *light cone*. For every $r > 0$, let

$$\mathcal{H}_n(r) = \{u = (\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 - t^2 = -r\},$$

a hyperboloid of two sheets.

It is easy to check that $\mathcal{H}_n(r)$ has two connected components.

Since every Lorentz isometry, $A \in \mathbf{SO}(n, 1)$, preserves the Lorentz inner product, we conclude that A globally preserves every hyperboloid, $\mathcal{H}_n(r)$, for $r > 0$.

We claim that every $A \in \mathbf{SO}_0(n, 1)$ preserves both $\mathcal{H}_n^+(r)$ and $\mathcal{H}_n^-(r)$. This follows immediately from

Proposition 3.3.6 *If $a_{n+1\,n+1} > 0$, then every isometry, $A \in \mathbf{SO}(n, 1)$, preserves all positive (resp. negative) timelike vectors and all positive (resp. negative) lightlike vectors. Moreover, if $A \in \mathbf{SO}(n, 1)$ preserves all positive timelike vectors, then $a_{n+1\,n+1} > 0$.*

Let $\mathbf{O}^+(n, 1)$ denote the subset of $\mathbf{O}(n, 1)$ consisting of all matrices, $A = (a_{i\,j})$, such that $a_{n+1\,n+1} > 0$.

Using Proposition 3.3.6, we can now show that $\mathbf{O}^+(n, 1)$ is a subgroup of $\mathbf{O}(n, 1)$ and that $\mathbf{SO}_0(n, 1)$ is a subgroup of $\mathbf{SO}(n, 1)$. Recall that

$$\mathbf{SO}_0(n, 1) = \{A \in \mathbf{SO}(n, 1) \mid a_{n+1\,n+1} > 0\}.$$

Note that $\mathbf{SO}_0(n, 1) = \mathbf{O}^+(n, 1) \cap \mathbf{SO}(n, 1)$.

Proposition 3.3.7 *The set $\mathbf{O}^+(n, 1)$ is a subgroup of $\mathbf{O}(n, 1)$ and the set $\mathbf{SO}_0(n, 1)$ is a subgroup of $\mathbf{SO}(n, 1)$.*

Next, we wish to prove that the action $\mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ is transitive. For this, we need the next two propositions.

Proposition 3.3.8 *Let $u = (\mathbf{u}, t)$ and $v = (\mathbf{v}, s)$ be nonzero vectors in $\mathbb{R}^{n+1}$ with $\langle u, v\rangle = 0$. If u is timelike, then v is spacelike (i.e., $\langle v, v\rangle > 0$).*

Lemma 3.3.8 also holds if $u = (\mathbf{u}, t)$ is a nonzero isotropic vector and $v = (\mathbf{v}, s)$ is a nonzero vector that is not collinear with u: If $\langle u, v\rangle = 0$, then v is spacelike (i.e., $\langle v, v\rangle > 0$).

Proposition 3.3.9 *The action*
$\mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ *is transitive.*

Let us find the stabilizer of $e_{n+1} = (0, \ldots, 0, 1)$.

We must have $Ae_{n+1} = e_{n+1}$, and the polar form implies that

$$A = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with} \quad P \in \mathbf{SO}(n).$$

Therefore, the stabilizer of e_{n+1} is isomorphic to $\mathbf{SO}(n)$ and we conclude that $\mathcal{H}_n^+(1)$, as a homogeneous space, is

$$\mathcal{H}_n^+(1) \cong \mathbf{SO}_0(n, 1)/\mathbf{SO}(n).$$

We will show in Section 3.4 that $\mathbf{SO}_0(n, 1)$ is connected.

3.4 Topological Groups

Since Lie groups are topological groups (and manifolds), it is useful to gather a few basic facts about topological groups.

Definition 3.4.1 A set, G, is a *topological group* iff

(a) G is a Hausdorff topological space;

(b) G is a group (with identity 1);

(c) Multiplication, $\cdot\colon G \times G \to G$, and the inverse operation, $G \longrightarrow G\colon g \mapsto g^{-1}$, are continuous, where $G \times G$ has the product topology.

It is easy to see that the two requirements of condition (c) are equivalent to

(c′) The map $G \times G \longrightarrow G\colon (g, h) \mapsto gh^{-1}$ is continuous.

Given a topological group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Observe that $L_{a^{-1}}$ is the inverse of L_a and similarly, $R_{a^{-1}}$ is the inverse of R_a. As multiplication is continuous, we see that L_a and R_a are continuous.

Moreover, since they have a continuous inverse, they are homeomorphisms.

As a consequence, if U is an open subset of G, then so is $gU = L_g(U)$ (resp. $Ug = R_gU$), for all $g \in G$.

Therefore, the topology of a topological group (i.e., its family of open sets) is *determined* by the knowledge of the open subsets containing the identity, 1.

Given any subset, $S \subseteq G$, let $S^{-1} = \{s^{-1} \mid s \in S\}$; let $S^0 = \{1\}$ and $S^{n+1} = S^n S$, for all $n \geq 0$. Property (c) of Definition 3.4.1 has the following useful consequences:

Proposition 3.4.2 *If G is a topological group and U is any open subset containing 1, then there is some open subset, $V \subseteq U$, with $1 \in V$, so that $V = V^{-1}$ and $V^2 \subseteq U$. Furthermore, $\overline{V} \subseteq U$.*

A subset, U, containing 1 such that $U = U^{-1}$, is called *symmetric*.

Using Proposition 3.4.2, we can give a very convenient characterization of the Hausdorff separation property in a topological group.

Proposition 3.4.3 *If G is a topological group, then the following properties are equivalent:*

(1) G is Hausdorff;

(2) The set $\{1\}$ is closed;

(3) The set $\{g\}$ is closed, for every $g \in G$.

If H is a subgroup of G (not necessarily normal), we can form the set of left cosets, G/H and we have the projection, $p: G \to G/H$, where $p(g) = gH = \overline{g}$.

If G is a topological group, then G/H can be given the *quotient topology*, where a subset $U \subseteq G/H$ is open iff $p^{-1}(U)$ is open in G.

With this topology, p is continuous. The trouble is that G/H is not necessarily Hausdorff. However, we can neatly characterize when this happens.

Proposition 3.4.4 *If G is a topological group and H is a subgroup of G then the following properties hold:*

(1) The map $p: G \to G/H$ is an open map, which means that $p(V)$ is open in G/H whenever V is open in G.

(2) The space G/H is Hausdorff iff H is closed in G.

(3) If H is open, then H is closed and G/H has the discrete topology (every subset is open).

(4) The subgroup H is open iff $1 \in \overset{\circ}{H}$ (i.e., there is some open subset, U, so that $1 \in U \subseteq H$).

Proposition 3.4.5 *If G is a connected topological group, then G is generated by any symmetric neighborhood, V, of 1. In fact,*

$$G = \bigcup_{n \geq 1} V^n.$$

A subgroup, H, of a topological group G is *discrete* iff the induced topology on H is discrete, i.e., for every $h \in H$, there is some open subset, U, of G so that $U \cap H = \{h\}$.

Proposition 3.4.6 *If G is a topological group and H is discrete subgroup of G, then H is closed.*

Proposition 3.4.7 *If G is a topological group and H is any subgroup of G, then the closure, $\overline{H}$, of H is a subgroup of G.*

Proposition 3.4.8 *Let G be a topological group and H be any subgroup of G. If H and G/H are connected, then G is connected.*

Proposition 3.4.9 *Let G be a topological group and let V be any connected symmetric open subset containing 1. Then, if G_0 is the connected component of the identity, we have*

$$G_0 = \bigcup_{n \geq 1} V^n$$

and G_0 is a normal subgroup of G. Moreover, the group G/G_0 is discrete.

A topological space, X is *locally compact* iff for every point $p \in X$, there is a compact neighborhood, C of p, i.e., there is a compact, C, and an open, U, with $p \in U \subseteq C$. For example, manifolds are locally compact.

Proposition 3.4.10 *Let G be a topological group and assume that G is connected and locally compact. Then, G is countable at infinity, which means that G is the union of a countable family of compact subsets. In fact, if V is any symmetric compact neighborhood of* 1*, then*

$$G = \bigcup_{n \geq 1} V^n.$$

If a topological group, G acts on a topological space, X, and the action $\cdot \colon G \times X \to X$ is continous, we say that G acts *continuously on X*.

The following theorem gives sufficient conditions for the quotient space, G/G_x, to be homeomorphic to X.

Theorem 3.4.11 *Let G be a topological group which is locally compact and countable at infinity, X a locally compact Hausdorff topological space and assume that G acts transitively and continuously on X. Then, for any $x \in X$, the map $\varphi\colon G/G_x \to X$ is a homeomorphism.*

Proof. A proof can be found in Mneimné and Testard [**?**] (Chapter 2). $\square$

Remark: If a topological group acts continuously and transitively on a Hausdorff topological space, then for every $x \in X$, the stabilizer, G_x, is a closed subgroup of G.

This is because, as the action is continuous, the projection $\pi\colon G \longrightarrow X\colon g \mapsto g \cdot x$ is continuous, and
$G_x = \pi^{-1}(\{x\})$, with $\{x\}$ closed.

As an application of Theorem 3.4.11 and Proposition 3.4.8, we show that the Lorentz group $\mathbf{SO}_0(n,1)$ is connected.

Firstly, it is easy to check that $\mathbf{SO}_0(n,1)$ and $\mathcal{H}_n^+(1)$ satisfy the assumptions of Theorem 3.4.11 because they are both manifolds, although this notion has not been discussed yet (but will be in Chapter 4).

Also, we saw at the end of Section 3.3 that the action $\cdot\colon \mathbf{SO}_0(n,1)\times\mathcal{H}_n^+(1)\longrightarrow\mathcal{H}_n^+(1)$ of $\mathbf{SO}_0(n,1)$ on $\mathcal{H}_n^+(1)$ is transitive, so that, as topological spaces

$$\mathbf{SO}_0(n,1)/\mathbf{SO}(n)\cong\mathcal{H}_n^+(1).$$

Now, we already showed that $\mathcal{H}_n^+(1)$ is connected so, by Proposition 3.4.8, the connectivity of $\mathbf{SO}_0(n,1)$ follows from the connectivity of $\mathbf{SO}(n)$ for $n\geq 1$.

The connectivity of $\mathbf{SO}(n)$ is a consequence of the surjectivity of the exponential map (see Theorem 2.3.2) but we can also give a quick proof using Proposition 3.4.8.

Indeed, $\mathbf{SO}(n+1)$ and S^n are both manifolds and we saw in Section 3.2 that

$$\mathbf{SO}(n+1)/\mathbf{SO}(n) \cong S^n.$$

Now, S^n is connected for $n \geq 1$ and $\mathbf{SO}(1) \cong S^1$ is connected. We finish the proof by induction on n.

Corollary 3.4.12 *The Lorentz group* $\mathbf{SO}_0(n,1)$ *is connected; it is the component of the identity in* $\mathbf{O}(n,1)$.

Readers who wish to learn more about topological groups may consult Sagle and Walde [**?**] and Chevalley [**?**] for an introductory account, and Bourbaki [**?**], Weil [**?**] and Pontryagin [**?**, **?**], for a more comprehensive account (especially the last two references).