Spring, 2008 CIS 610

Advanced Geometric Methods in Computer Science

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Homework 1, Corrected Version

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"A problems" are for practice only, and should not be turned in.

Problem A1. (a) Find two symmetric matrices, A and B, such that AB is not symmetric.

(b) Find two matrices, A and B, such that

$$e^A e^B \neq e^{A+B}.$$

Try

$$A = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Problem A2. (a) If $K = \mathbb{R}$ or $K = \mathbb{C}$, recall that the projective space, $\mathbf{P}(K^{n+1})$, is the set of equivalence classes of the equivalence relation, $\sim$, on $K^{n+1} - \{0\}$, defined so that, for all $u, v \in K^{n+1} - \{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in K - \{0\}.$$

The map, $p\colon (K^{n+1} - \{0\}) \to \mathbf{P}(K^{n+1})$, is the projection mapping any nonzero vector in K^{n+1} to its equivalence class modulo $\sim$. We let $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$ and $\mathbb{CP}^n = \mathbf{P}(\mathbb{C}^{n+1})$.

Prove that for any $n \geq 0$, there is a bijection between $\mathbf{P}(K^{n+1})$ and $K^n \cup \mathbf{P}(K^n)$ (which allows us to identify them).

(b) Prove that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are connected and compact.

Hint. If

$$S^n = \{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

prove that $p(S^n) = \mathbf{P}(K^{n+1})$, and recall that S^n is compact for all $n \geq 0$ and connected for $n \geq 1$. For $n = 0$, $\mathbf{P}(K)$ consists of a single point.

Problem A3. Recall that $\mathbb{R}^2$ and $\mathbb{C}$ can be identified using the bijection $(x, y) \mapsto x + iy$. Also recall that the subset $U(1) \subseteq \mathbb{C}$ consisting of all complex numbers of the form $\cos\theta + i\sin\theta$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If $c\colon U(1) \to U(1)$ is the map defined such that

$$c(z) = z^2,$$

prove that $c(z_1) = c(z_2)$ iff either $z_2 = z_1$ or $z_2 = -z_1$, and thus that c induces a bijective map $\widehat{c}\colon \mathbb{RP}^1 \to S^1$. Prove that $\widehat{c}$ is a homeomorphism (remember that $\mathbb{RP}^1$ is compact).

"B problems" must be turned in.

Problem B1 (20 pts). Let $A = (a_{i\,j})$ be a real or complex $n \times n$ matrix.

(1) If λ is an eigenvalue of A, prove that there is some eigenvector $u = (u_1, \ldots, u_n)$ of A for λ such that

$$\max_{1 \leq i \leq n} |u_i| = 1.$$

(2) If $u = (u_1, \ldots, u_n)$ is an eigenvector of A for λ as in (1), assuming that i, $1 \leq i \leq n$, is an index such that $|u_i| = 1$, prove that

$$(\lambda - a_{i\,i})u_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{i\,j}u_j,$$

and thus that

$$|\lambda - a_{i\,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i\,j}|.$$

Conclude that the eigenvalues of A are inside the union of the closed disks D_i defined such that

$$D_i = \Big\{ z \in \mathbb{C} \mid |z - a_{i\,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i\,j}| \Big\}.$$

Remark: This result is known as *Gershgorin's theorem.*

Problem B2 (10). Recall that a real $n \times n$ symmetric matrix, A, is *positive semi-definite* iff its eigenvalues, $\lambda_1, \ldots, \lambda_n$ are non-negative (i.e., $\lambda_i \geq 0$ for $i = 1, \ldots, n$) and *positive definite* iff its eigenvalues are positive (i.e., $\lambda_i > 0$ for $i = 1, \ldots, n$).

(a) Prove that a symmetric matrix, A, is positive semi-definite iff $X^\top AX \geq 0$, for all $X \neq 0$ ($X \in \mathbb{R}^n$) and positive definite iff $X^\top AX > 0$, for all $X \neq 0$ ($X \in \mathbb{R}^n$).

(b) Prove that for any two positive definite matrices, A, B, for all $\lambda, \mu \in \mathbb{R}$, with $\lambda, \mu \geq 0$ and $\lambda + \mu > 0$, the matrix $\lambda A + \mu B$ is still symmetric, positive definite. Deduce that the set of $n \times n$ symmetric positive definite matrices is convex (in fact, a cone).

Problem B3 (40 pts). (a) Given a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$, prove that there is a skew symmetric matrix B such that

$$R = (I - B)(I + B)^{-1}.$$

(b) If B is a skew symmetric $n \times n$ matrix, prove that $\lambda I_n - B$ and $\lambda I_n + B$ are invertible for all $\lambda \neq 0$, and that they commute.

(c) Prove that

$$R = (\lambda I_n - B)(\lambda I_n + B)^{-1}$$

is a rotation matrix that does not admit -1 as an eigenvalue. (Recall, a rotation is an orthogonal matrix R with positive determinant, i.e., $\det(R) = 1$.)

(d) Given any rotation matrix R that does not admit -1 as an eigenvalue, prove that there is a skew symmetric matrix B such that

$$R = (I_n - B)(I_n + B)^{-1} = (I_n + B)^{-1}(I_n - B).$$

This is known as the *Cayley representation* of rotations (Cayley, 1846).

(e) Given any rotation matrix R, prove that there is a skew symmetric matrix B such that

$$R = \left((I_n - B)(I_n + B)^{-1}\right)^2.$$

Problem B4 (60). (a) Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\begin{aligned}
\psi_1(u, v) &= \left(\frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u}{u^2 + v^2 + 1}, \frac{u^2 - v^2}{u^2 + v^2 + 1}\right), \\
\psi_2(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{u^2 - 1}{u^2 + v^2 + 1}\right), \\
\psi_3(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{1 - u^2}{u^2 + v^2 + 1}\right).
\end{aligned}$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian is never zero).

Prove that if $\psi_1(u, v) = (x, y, z, t)$, then

$$y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that u and v are solutions of the quadratic equations

$$\begin{aligned} (y^2 + z^2)u^2 - zu + z^2 &= 0 \\ (y^2 + z^2)v^2 - yv + y^2 &= 0. \end{aligned}$$

Prove that if $y^2 + z^2 \neq 0$, then

$$u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$

else

$$u = \frac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1,$$

and there are similar formulae for v. Prove that the expression giving u in terms of y and z is continuous everywhere in $\{(y, z) \mid y^2 + z^2 \leq \frac{1}{4}\}$ and similarly for the expression giving v in terms of y and z. Conclude that $\psi_1 \colon \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)$ is a homeomorphism onto its image. Therefore, $U_1 = \psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Remark: From the equations above, you can prove that $u^2 + v^2 + 1$ is a root of the equation

$$(y^2 + z^2)D^2 - D + 1 = 0.$$

Then,

$$D = \frac{1 - \sqrt{1 - 4(y^2 + z^2)}}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$

else

$$D = \frac{1 + \sqrt{1 - 4(y^2 + z^2}}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1.$$

Prove that if $\psi_2(u,v) = (x,y,z,t)$, then u and v are solutions of quadratic equations with coefficients involving x and y; find explicit formulae as for ψ_1^{-1} and conclude that $\psi_2 \colon \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)$ is a homeomorphism onto its image. The set $U_2 = \psi_2(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_3(u,v) = (x,y,z,t)$, then u and v are solutions of quadratic equations with coefficients involving x and z. As for ψ_2^{-1}, conclude that $\psi_3 \colon \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)$ is a homeomorphism onto its image. The set $U_3 = \psi_3(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that the union of the U_i's covers $\mathcal{H}(S^2)$. Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a manifold in $\mathbb{R}^4$.

Prove that if $(x,y,z,t) \in \mathcal{H}(S^2)$, then

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= xyz \\ x(z^2 - y^2) &= yzt. \end{aligned}$$

The zero locus of these equations strictly contains $\mathcal{H}(S^2)$, prove it. This is a "famous mistake" of Hilbert and Cohn-Vossen in *Geometry and the Immagination*! In an attempt to fix this bug, prove that when you express x in terms of y and z using ψ_1, you get the equation

$$x^2y^2 + x^2z^2 + y^2z^2 = xyz.$$

When you express t in terms of y and z using ψ_1, you get the equation

$$(y^2 + z^2)(z^2 - y^2 + t^2) = t(z^2 - y^2).$$

When you express t in terms of x and y using ψ_2, you get the equation

$$4(x^2 + y^2)((x^2 + y^2)t^2 + (2x^2 + y^2)^2) = (2x^2 + y^2)^2.$$

When you express t in terms of x and z using ψ_3, you get an equation similar to the previous one. Do these four equations define exactly $\mathcal{H}(S^2)$? (I suspect they do!)

(c) Investigate the surfaces in $\mathbb{R}^3$ obtained by dropping one of the four coordinates. Show that there are only two of them (the "Steiner Roman surface" and the "crosscap", up to a rigid motion).

Problem B5 (40). (a) Consider the map, $f \colon \mathbf{GL}^+(n) \to \mathbf{S}(n)$, given by

$$f(A) = A^\top A - I.$$

Check that

$$df(A)(H) = A^\top H + H^\top A,$$

for any matrix, H.

(b) Consider the map, $f \colon \mathbf{GL}(n) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of B, for any matrix B (here, I is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n)$.

(c) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n)$ is a manifold of dimension $n^2 - 1$.

(d) Let J be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n, 1)$ the group of real $(n+1) \times (n+1)$ matrices

$$\mathbf{SO}(n, 1) = \{A \in \mathbf{GL}(n+1) \mid A^\top JA = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n, 1)$ is indeed a group with the inverse of A given by $A^{-1} = JA^\top J$ (this is the *special Lorentz group*.) Consider the function $f \colon \mathbf{GL}^+(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^\top JA - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1) \times (n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top JH + H^\top JA$$

for any matrix, H. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n, 1)$ and that $\mathbf{SO}(n, 1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

Problem B6 (20 pts). (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2\cosh \omega$.

Prove that the exponential map, $\exp \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathbf{SL}(2, \mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2, \mathbb{C})$.

Problem B7 (50 pts). Recall that for any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

then the exponential map, $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$, is given by

$$\exp A = e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2,$$

if $\theta \neq k2\pi$ ($k \in \mathbb{Z}$), with $\exp(0_3) = I_3$ (Rodrigues's formula (1840)).

(a) Let $R \in \mathbf{SO}(3)$ and assume that $R \neq I$ and $\mathrm{tr}(R) \neq -1$. Then, prove that a log of R (i.e., a skew symmetric matrix, S, so that $e^S = R$) is given by

$$\log(R) = \frac{\theta}{2\sin\theta}(R - R^T),$$

where $1 + 2\cos\theta = \mathrm{tr}(R)$ and $0 < \theta < \pi$.

(b) Now, assume that $\mathrm{tr}(R) = -1$. In this case, show that R is a rotation of angle π, that R is symmetric and has eigenvalues, $-1, -1, 1$. Assuming that $e^A = R$, Rodrigues formula becomes

$$R = I + \frac{2}{\pi^2} A^2,$$

so

$$A^2 = \frac{\pi^2}{2}(R - I).$$

If we let $S = A/\pi$, we see that we need to find a skew-symmetric matrix, S, so that

$$S^2 = \frac{1}{2}(R - I) = C.$$

Observe that C is also symmetric and has eigenvalues, $-1, -1, 0$. Thus, we can diagonalize C, as

$$C = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^\top,$$

and if we let

$$S = P \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^\top,$$

check that $S^2 = C$.

(c) From (a) and (b), we know that we can compute explicity a log of a rotation matrix, although when $\theta \approx 0$, we have to be careful in computing $\frac{\sin\theta}{\theta}$; in this case, we may want to use

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots.$$

Given two rotations, $R_1, R_2 \in \mathbf{SO}(3)$, there are three natural interpolation formulae:

$$e^{(1-t)\log R_1 + t\log R_2}; \quad R_1 e^{t\log(R_1^\top R_2)}; \quad e^{t\log(R_2 R_1^\top)} R_1,$$

with $0 \leq t \leq 1$.

Write a computer program to investigate the difference between these interpolation formulae. The position of a rigid body spinning around its center of gravity is determined by a rotation matrix, $R \in \mathbf{SO}(3)$. If R_1 denotes the initial position and R_2 the final position of this rigid body, by computing interpolants of R_1 and R_2, we get a motion of the rigid body and we can create an animation of this motion by displaying several interpolants. The rigid body can be a "funny" object, for example a banana, a bottle, etc.

TOTAL: 240 points.