Advanced Geometric Methods in Computer Science
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Homework 1, Corrected Version
February 18, 2008; Due March 5, 2008

“A problems” are for practice only, and should not be turned in.

Problem A1. (a) Find two symmetric matrices, $A$ and $B$, such that $AB$ is not symmetric.
(b) Find two matrices, $A$ and $B$, such that $e^A e^B \neq e^{A+B}$.

Try

$$A = \frac{\pi}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\pi}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$ 

Problem A2. (a) If $K = \mathbb{R}$ or $K = \mathbb{C}$, recall that the projective space, $P(K^{n+1})$, is the set of equivalence classes of the equivalence relation, $\sim$, on $K^{n+1} - \{0\}$, defined so that, for all $u, v \in K^{n+1} - \{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in K - \{0\}.$$ 

The map, $p: (K^{n+1} - \{0\}) \rightarrow P(K^{n+1})$, is the projection mapping any nonzero vector in $K^{n+1}$ to its equivalence class modulo $\sim$. We let $\mathbb{RP}^n = P(\mathbb{R}^{n+1})$ and $\mathbb{CP}^n = P(\mathbb{C}^{n+1})$.

Prove that for any $n \geq 0$, there is a bijection between $P(K^{n+1})$ and $K^n \cup P(K^n)$ (which allows us to identify them).

(b) Prove that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are connected and compact.

Hint. If

$$S^n = \{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

prove that $p(S^n) = P(K^{n+1})$, and recall that $S^n$ is compact for all $n \geq 0$ and connected for $n \geq 1$. For $n = 0$, $P(K)$ consists of a single point.

Problem A3. Recall that $\mathbb{R}^2$ and $\mathbb{C}$ can be identified using the bijection $(x, y) \mapsto x + iy$. Also recall that the subset $U(1) \subseteq \mathbb{C}$ consisting of all complex numbers of the form $\cos \theta + i \sin \theta$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If $c: U(1) \rightarrow U(1)$ is the map defined such that

$$c(z) = z^2,$$
prove that \( c(z_1) = c(z_2) \) iff either \( z_2 = z_1 \) or \( z_2 = -z_1 \), and thus that \( c \) induces a bijective map \( \widehat{c}: \mathbb{RP}^1 \to S^1 \). Prove that \( \widehat{c} \) is a homeomorphism (remember that \( \mathbb{RP}^1 \) is compact).

“B problems” must be turned in.

**Problem B1 (20 pts).** Let \( A = (a_{ij}) \) be a real or complex \( n \times n \) matrix.

(1) If \( \lambda \) is an eigenvalue of \( A \), prove that there is some eigenvector \( u = (u_1, \ldots, u_n) \) of \( A \) for \( \lambda \) such that

\[
\max_{1 \leq i \leq n} |u_i| = 1.
\]

(2) If \( u = (u_1, \ldots, u_n) \) is an eigenvector of \( A \) for \( \lambda \) as in (1), assuming that \( i, 1 \leq i \leq n \), is an index such that \( |u_i| = 1 \), prove that

\[
(\lambda - a_{ii}) u_i = \sum_{j=1, j \neq i}^{n} a_{ij} u_j,
\]

and thus that

\[
|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|.
\]

Conclude that the eigenvalues of \( A \) are inside the union of the closed disks \( D_i \) defined such that

\[
D_i = \left\{ z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}| \right\}.
\]

**Remark:** This result is known as **Gershgorin’s theorem.**

**Problem B2 (10).** Recall that a real \( n \times n \) symmetric matrix, \( A \), is **positive semi-definite** iff its eigenvalues, \( \lambda_1, \ldots, \lambda_n \) are non-negative (i.e., \( \lambda_i \geq 0 \) for \( i = 1, \ldots, n \)) and **positive definite** iff its eigenvalues are positive (i.e., \( \lambda_i > 0 \) for \( i = 1, \ldots, n \)).

(a) Prove that a symmetric matrix, \( A \), is positive semi-definite iff \( X^TAX \geq 0 \), for all \( X \neq 0 \) (\( X \in \mathbb{R}^n \)) and positive definite iff \( X^TAX > 0 \), for all \( X \neq 0 \) (\( X \in \mathbb{R}^n \)).

(b) Prove that for any two positive definite matrices, \( A, B \), for all \( \lambda, \mu \in \mathbb{R} \), with \( \lambda, \mu \geq 0 \) and \( \lambda + \mu > 0 \), the matrix \( \lambda A + \mu B \) is still symmetric, positive definite. Deduce that the set of \( n \times n \) symmetric positive definite matrices is convex (in fact, a cone).

**Problem B3 (40 pts).** (a) Given a rotation matrix

\[
R = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
\]

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where $0 < \theta < \pi$, prove that there is a skew symmetric matrix $B$ such that
\[ R = (I - B)(I + B)^{-1}. \]

(b) If $B$ is a skew symmetric $n \times n$ matrix, prove that $\lambda I_n - B$ and $\lambda I_n + B$ are invertible for all $\lambda \neq 0$, and that they commute.

(c) Prove that
\[ R = (\lambda I_n - B)(\lambda I_n + B)^{-1} \]
is a rotation matrix that does not admit $-1$ as an eigenvalue. (Recall, a rotation is an orthogonal matrix $R$ with positive determinant, i.e., $\det(R) = 1$.)

(d) Given any rotation matrix $R$ that does not admit $-1$ as an eigenvalue, prove that there is a skew symmetric matrix $B$ such that
\[ R = (I_n - B)(I_n + B)^{-1} = (I_n + B)^{-1}(I_n - B). \]
This is known as the Cayley representation of rotations (Cayley, 1846).

(e) Given any rotation matrix $R$, prove that there is a skew symmetric matrix $B$ such that
\[ R = ((I_n - B)(I_n + B)^{-1})^2. \]

Problem B4 (60). (a) Consider the map $H: \mathbb{R}^3 \to \mathbb{R}^4$ defined such that
\[ (x, y, z) \mapsto (xy, yz, xz, x^2 - y^2). \]
Prove that when it is restricted to the sphere $S^2$ (in $\mathbb{R}^3$), we have $H(x, y, z) = H(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $H(S^2)$ consists of two antipodal points.

Prove that the map $H$ induces an injective map from the projective plane onto $H(S^2)$, and that it is a homeomorphism.

(b) The map $H$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by
\[
\psi_1(u, v) = \left( \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u}{u^2 + v^2 + 1}, \frac{u^2 - v^2}{u^2 + v^2 + 1} \right), \\
\psi_2(u, v) = \left( \frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{u^2 - 1}{u^2 + v^2 + 1} \right), \\
\psi_3(u, v) = \left( \frac{u}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{1 - u^2}{u^2 + v^2 + 1} \right).
\]
Observe that $\psi_1$ is the composition $H \circ \alpha_1$, where $\alpha_1: \mathbb{R}^2 \to S^2$ is given by
\[ (u, v) \mapsto \left( \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}} \right). \]
that $\psi_2$ is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2: \mathbb{R}^2 \rightarrow S^2$ is given by
\[(u, v) \mapsto \left( \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right),\]
and $\psi_3$ is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3: \mathbb{R}^2 \rightarrow S^2$ is given by
\[(u, v) \mapsto \left( \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right),\]
Prove that each $\psi_i$ is injective, continuous and nonsingular (i.e., the Jacobian is never zero).

Prove that if $\psi_1(u, v) = (x, y, z, t)$, then
\[y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \iff u^2 + v^2 = 1.\]

Prove that $u$ and $v$ are solutions of the quadratic equations
\[(y^2 + z^2)u^2 - zu + z^2 = 0\]
\[(y^2 + z^2)v^2 - yv + y^2 = 0.\]

Prove that if $y^2 + z^2 \neq 0$, then
\[u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,\]
else
\[u = \frac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1,\]
and there are similar formulae for $v$. Prove that the expression giving $u$ in terms of $y$ and $z$ is continuous everywhere in $\{(y, z) \mid y^2 + z^2 \leq \frac{1}{4}\}$ and similarly for the expression giving $v$ in terms of $y$ and $z$. Conclude that $\psi_1: \mathbb{R}^2 \rightarrow \psi_1(\mathbb{R}^2)$ is a homeomorphism onto its image.

Therefore, $U_1 = \psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

**Remark:** From the equations above, you can prove that $u^2 + v^2 + 1$ is a root of the equation
\[(y^2 + z^2)D^2 - D + 1 = 0.\]

Then,
\[D = \frac{1 - \sqrt{1 - 4(y^2 + z^2)}}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,\]
else
\[D = \frac{1 + \sqrt{1 - 4(y^2 + z^2)}}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1.\]
Prove that if \( \psi_2(u,v) = (x,y,z,t) \), then \( u \) and \( v \) are solutions of quadratic equations with coefficients involving \( x \) and \( y \); find explicit formulae as for \( \psi_1^{-1} \) and conclude that \( \psi_2: \mathbb{R}^2 \to \psi_3(\mathbb{R}^2) \) is a homeomorphism onto its image. The set \( U_2 = \psi_2(\mathbb{R}^2) \) is an open subset of \( \mathcal{H}(S^2) \).

Prove that if \( \psi_3(u,v) = (x,y,z,t) \), then \( u \) and \( v \) are solutions of quadratic equations with coefficients involving \( x \) and \( z \). As for \( \psi_2^{-1} \), conclude that \( \psi_3: \mathbb{R}^2 \to \psi_3(\mathbb{R}^2) \) is a homeomorphism onto its image. The set \( U_3 = \psi_3(\mathbb{R}^2) \) is an open subset of \( \mathcal{H}(S^2) \).

Prove that the union of the \( U_i \)'s covers \( \mathcal{H}(S^2) \). Conclude that \( \psi_1, \psi_2, \psi_3 \) are parametrizations of \( \mathbb{R}P^2 \) as a manifold in \( \mathbb{R}^4 \).

The zero locus of these equations strictly contains \( \mathcal{H}(S^2) \), prove it. This is a “famous mistake” of Hilbert and Cohn-Vossen in *Geometry and the Immagination*! In an attempt to fix this bug, prove that when you express \( x \) in terms of \( y \) and \( z \) using \( \psi_1 \), you get the equation

\[ x^2y^2 + x^2z^2 + y^2z^2 = xyz. \]

When you express \( t \) in terms of \( y \) and \( z \) using \( \psi_1 \), you get the equation

\[ (y^2 + z^2)(z^2 - y^2 + t^2) = t(z^2 - y^2). \]

When you express \( t \) in terms of \( x \) and \( y \) using \( \psi_2 \), you get the equation

\[ 4(x^2 + y^2)((x^2 + y^2)t^2 + (2x^2 + y^2)^2) = (2x^2 + y^2)^2. \]

When you express \( t \) in terms of \( x \) and \( z \) using \( \psi_3 \), you get an equation similar to the previous one. Do these four equations define exactly \( \mathcal{H}(S^2) \)? (I suspect they do!)

(c) Investigate the surfaces in \( \mathbb{R}^3 \) obtained by dropping one of the four coordinates. Show that there are only two of them (the “Steiner Roman surface” and the “crosscap”, up to a rigid motion).

**Problem B5 (40).** (a) Consider the map, \( f: \text{GL}^+(n) \to S(n) \), given by

\[ f(A) = A^\top A - I. \]

Check that

\[ df(A)(H) = A^\top H + H^\top A, \]

for any matrix, \( H \).

(b) Consider the map, \( f: \text{GL}(n) \to \mathbb{R} \), given by

\[ f(A) = \det(A). \]
Prove that \( df(I)(B) = \text{tr}(B) \), the trace of \( B \), for any matrix \( B \) (here, \( I \) is the identity matrix). Then, prove that

\[
df(A)(B) = \det(A)\text{tr}(A^{-1}B),
\]

where \( A \in \text{GL}(n) \).

(c) Use the map \( A \mapsto \det(A) - 1 \) to prove that \( \text{SL}(n) \) is a manifold of dimension \( n^2 - 1 \).

(d) Let \( J \) be the \((n+1) \times (n+1)\) diagonal matrix

\[
J = \begin{pmatrix}
I_n & 0 \\
0 & -1
\end{pmatrix}.
\]

We denote by \( \text{SO}(n,1) \) the group of real \((n+1) \times (n+1)\) matrices

\[
\text{SO}(n,1) = \{ A \in \text{GL}(n+1) \mid A^\top JA = J \text{ and } \det(A) = 1 \}.
\]

Check that \( \text{SO}(n,1) \) is indeed a group with the inverse of \( A \) given by \( A^{-1} = JA^\top J \) (this is the special Lorentz group.) Consider the function \( f: \text{GL}^+(n+1) \to \text{S}(n+1) \), given by

\[
f(A) = A^\top JA - J,
\]

where \( \text{S}(n+1) \) denotes the space of \((n+1) \times (n+1)\) symmetric matrices. Prove that

\[
df(A)(H) = A^\top JH + H^\top JA
\]

for any matrix, \( H \). Prove that \( df(A) \) is surjective for all \( A \in \text{SO}(n,1) \) and that \( \text{SO}(n,1) \) is a manifold of dimension \( \frac{n(n+1)}{2} \).

**Problem B6 (20 pts).** (a) Given any matrix

\[
B = \begin{pmatrix}
a & b \\
c & -a
\end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),
\]

if \( \omega^2 = a^2 + bc \) and \( \omega \) is any of the two complex roots of \( a^2 + bc \), prove that if \( \omega \neq 0 \), then

\[
e^B = \cosh \omega I + \frac{\sinh \omega}{\omega} B,
\]

and \( e^B = I + B \), if \( a^2 + bc = 0 \). Observe that \( \text{tr}(e^B) = 2 \cosh \omega \).

Prove that the exponential map, \( \exp: \mathfrak{sl}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C}) \), is *not* surjective. For instance, prove that

\[
\begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix}
\]

is not the exponential of any matrix in \( \mathfrak{sl}(2, \mathbb{C}) \).
Problem B7 (50 pts). Recall that for any matrix

\[
A = \begin{pmatrix}
0 & -c & b \\
c & 0 & -a \\
-b & a & 0
\end{pmatrix},
\]

if we let \( \theta = \sqrt{a^2 + b^2 + c^2} \) and

\[
B = \begin{pmatrix}
a^2 & ab & ac \\
ab & b^2 & bc \\
ac & bc & c^2
\end{pmatrix},
\]

then the exponential map, \( \exp: \mathfrak{so}(3) \to \text{SO}(3) \), is given by

\[
\exp A = e^A = \cos \theta I_3 + \frac{\sin \theta}{\theta} A + \frac{1 - \cos \theta}{\theta^2} B,
\]
or, equivalently, by

\[
e^A = I_3 + \frac{\sin \theta}{\theta} A + \frac{1 - \cos \theta}{\theta^2} A^2,
\]

if \( \theta \neq k2\pi \) (\( k \in \mathbb{Z} \)), with \( \exp(0) = I_3 \) (Rodrigues’s formula (1840)).

(a) Let \( R \in \text{SO}(3) \) and assume that \( R \neq I \) and \( \text{tr}(R) \neq -1 \). Then, prove that a log of \( R \) (i.e., a skew symmetric matrix, \( S \), so that \( e^S = R \)) is given by

\[
\log(R) = \frac{\theta}{2\sin \theta}(R - R^T),
\]

where \( 1 + 2\cos \theta = \text{tr}(R) \) and \( 0 < \theta < \pi \).

(b) Now, assume that \( \text{tr}(R) = -1 \). In this case, show that \( R \) is a rotation of angle \( \pi \), that \( R \) is symmetric and has eigenvalues, \(-1, -1, 1\). Assuming that \( e^A = R \), Rodrigues formula becomes

\[
R = I + \frac{2}{\pi^2} A^2,
\]

so

\[
A^2 = \frac{\pi^2}{2}(R - I).
\]

If we let \( S = A/\pi \), we see that we need to find a skew-symmetric matrix, \( S \), so that

\[
S^2 = \frac{1}{2}(R - I) = C.
\]

Observe that \( C \) is also symmetric and has eigenvalues, \(-1, -1, 0\). Thus, we can diagonalize \( C \), as

\[
C = P \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} P^T,
\]
and if we let
\[ S = P \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^\top, \]
check that \( S^2 = C \).

(c) From (a) and (b), we know that we can compute explicitly a log of a rotation matrix, although when \( \theta \approx 0 \), we have to be careful in computing \( \frac{\sin \theta}{\theta} \); in this case, we may want to use
\[ \sin \frac{\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots. \]

Given two rotations, \( R_1, R_2 \in SO(3) \), there are three natural interpolation formulae:
\[
e^{(1-t) \log R_1 + t \log R_2}; \quad R_1 e^{t \log (R_1^\top R_2)}; \quad e^{t \log (R_2 R_1^\top)} R_1,
\]
with \( 0 \leq t \leq 1 \).

Write a computer program to investigate the difference between these interpolation formulae. The position of a rigid body spinning around its center of gravity is determined by a rotation matrix, \( R \in SO(3) \). If \( R_1 \) denotes the initial position and \( R_2 \) the final position of this rigid body, by computing interpolants of \( R_1 \) and \( R_2 \), we get a motion of the rigid body and we can create an animation of this motion by displaying several interpolants. The rigid body can be a “funny” object, for example a banana, a bottle, etc.

**TOTAL:** 240 points.