Chapter 6

Riemannian Manifolds and Connections

6.1 Riemannian Metrics

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold.

The notion of local (and global) frame plays an important technical role.

**Definition 6.1.1** Let $M$ be an $n$-dimensional smooth manifold. For any open subset, $U \subseteq M$, an $n$-tuple of vector fields, $(X_1, \ldots, X_n)$, over $U$ is called a frame over $U$ iff $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space, $T_pM$, for every $p \in U$. If $U = M$, then the $X_i$ are global sections and $(X_1, \ldots, X_n)$ is called a frame (of $M$).
The notion of a frame is due to Élie Cartan who (after Darboux) made extensive use of them under the name of moving frame (and the moving frame method).

Cartan’s terminology is intuitively clear: As a point, \( p \), moves in \( U \), the frame, \((X_1(p), \ldots, X_n(p))\), moves from fibre to fibre. Physicists refer to a frame as a choice of local gauge.

If \( \dim(M) = n \), then for every chart, \((U, \varphi)\), since \( d\varphi^{-1}_{\varphi(p)} : \mathbb{R}^n \rightarrow T_pM \) is a bijection for every \( p \in U \), the \( n \)-tuple of vector fields, \((X_1, \ldots, X_n)\), with \( X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i) \), is a frame of \( TM \) over \( U \), where \((e_1, \ldots, e_n)\) is the canonical basis of \( \mathbb{R}^n \).

The following proposition tells us when the tangent bundle is trivial (that is, isomorphic to the product, \( M \times \mathbb{R}^n \)): 
Proposition 6.1.2 The tangent bundle, $TM$, of a smooth $n$-dimensional manifold, $M$, is trivial iff it possesses a frame of global sections (vector fields defined on $M$).

As an illustration of Proposition 6.1.2 we can prove that the tangent bundle, $TS^1$, of the circle, is trivial.

Indeed, we can find a section that is everywhere nonzero, i.e. a non-vanishing vector field, namely

$$X(\cos \theta, \sin \theta) = (-\sin \theta, \cos \theta).$$

The reader should try proving that $TS^3$ is also trivial (use the quaternions).

However, $TS^2$ is nontrivial, although this not so easy to prove.

More generally, it can be shown that $TS^n$ is nontrivial for all even $n \geq 2$. It can even be shown that $S^1$, $S^3$ and $S^7$ are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.
Remark: A manifold, $M$, such that its tangent bundle, $TM$, is trivial is called parallelizable.

We now define Riemannian metrics and Riemannian manifolds.

**Definition 6.1.3** Given a smooth $n$-dimensional manifold, $M$, a *Riemannian metric on $M$ (or $TM$)* is a family, $(\langle - , - \rangle_p)_{p \in M}$, of inner products on each tangent space, $T_pM$, such that $\langle - , - \rangle_p$ depends smoothly on $p$, which means that for every chart, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, for every frame, $(X_1, \ldots, X_n)$, on $U_\alpha$, the maps

$$p \mapsto \langle X_i(p), X_j(p) \rangle_p, \quad p \in U_\alpha, \ 1 \leq i, j \leq n$$

are smooth. A smooth manifold, $M$, with a Riemannian metric is called a *Riemannian manifold*. 
If \( \dim(M) = n \), then for every chart, \((U, \varphi)\), we have the frame, \((X_1, \ldots, X_n)\), over \(U\), with \(X_i(p) = d\varphi^{-1}_\varphi(p)(e_i)\), where \((e_1, \ldots, e_n)\) is the canonical basis of \(\mathbb{R}^n\). Since every vector field over \(U\) is a linear combination, \(\sum_{i=1}^n f_i X_i\), for some smooth functions, \(f_i: U \to \mathbb{R}\), the condition of Definition 6.1.3 is equivalent to the fact that the maps,

\[
p \mapsto \langle d\varphi^{-1}_\varphi(p)(e_i), d\varphi^{-1}_\varphi(p)(e_j) \rangle_p, \quad p \in U, \ 1 \leq i, j \leq n,
\]

are smooth. If we let \(x = \varphi(p)\), the above condition says that the maps,

\[
x \mapsto \langle d\varphi^{-1}_x(e_i), d\varphi^{-1}_x(e_j) \rangle_{\varphi^{-1}(x)}, \quad x \in \varphi(U), 1 \leq i, j \leq n,
\]

are smooth.

If \(M\) is a Riemannian manifold, the metric on \(TM\) is often denoted \(g = (g_p)_{p \in M}\). In a chart, using local coordinates, we often use the notation \(g = \sum_{i,j} g_{ij} dx_i \otimes dx_j\) or simply \(g = \sum_{i,j} g_{ij} dx_i dx_j\), where

\[
g_{ij}(p) = \left\langle \left( \frac{\partial}{\partial x_i} \right)_p, \left( \frac{\partial}{\partial x_j} \right)_p \right\rangle_p.
\]
For every $p \in U$, the matrix, $(g_{ij}(p))$, is symmetric, positive definite.

The standard Euclidean metric on $\mathbb{R}^n$, namely,

$$g = dx_1^2 + \cdots + dx_n^2,$$

makes $\mathbb{R}^n$ into a Riemannian manifold.

Then, every submanifold, $M$, of $\mathbb{R}^n$ inherits a metric by restricting the Euclidean metric to $M$.

For example, the sphere, $S^{n-1}$, inherits a metric that makes $S^{n-1}$ into a Riemannian manifold. It is a good exercise to find the local expression of this metric for $S^2$ in polar coordinates.

A nontrivial example of a Riemannian manifold is the \textit{Poincaré upper half-space}, namely, the set $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$
A way to obtain a metric on a manifold, $N$, is to pull-back the metric, $g$, on another manifold, $M$, along a local diffeomorphism, $\varphi: N \to M$.

Recall that $\varphi$ is a local diffeomorphism iff

$$d\varphi_p: T_pN \to T_{\varphi(p)}M$$

is a bijective linear map for every $p \in N$.

Given any metric $g$ on $M$, if $\varphi$ is a local diffeomorphism, we define the pull-back metric, $\varphi^*g$, on $N$ induced by $g$ as follows: For all $p \in N$, for all $u, v \in T_pN,$

$$(\varphi^*g)_p(u, v) = g_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We need to check that $(\varphi^*g)_p$ is an inner product, which is very easy since $d\varphi_p$ is a linear isomorphism.

Our map, $\varphi$, between the two Riemannian manifolds $(N, \varphi^*g)$ and $(M, g)$ is a local isometry, as defined below.
Definition 6.1.4 Given two Riemannian manifolds, \((M_1, g_1)\) and \((M_2, g_2)\), a \textit{local isometry} is a smooth map, \(\varphi: M_1 \to M_2\), such that \(d\varphi_p: T_p M_1 \to T_{\varphi(p)} M_2\) is an isometry between the Euclidean spaces \((T_p M_1, (g_1)_p)\) and \((T_{\varphi(p)} M_2, (g_2)_{\varphi(p)})\), for every \(p \in M_1\), that is,

\[
(g_1)_p(u, v) = (g_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)),
\]

for all \(u, v \in T_p M_1\) or, equivalently, \(\varphi^* g_2 = g_1\). Moreover, \(\varphi\) is an \textit{isometry} iff it is a local isometry and a diffeomorphism.

The isometries of a Riemannian manifold, \((M, g)\), form a group, \(\text{Isom}(M, g)\), called the \textit{isometry group of} \((M, g)\).

An important theorem of Myers and Steenrod asserts that the isometry group, \(\text{Isom}(M, g)\), is a Lie group.
Given a map, \( \varphi: M_1 \to M_2 \), and a metric \( g_1 \) on \( M_1 \), in general, \( \varphi \) does not induce any metric on \( M_2 \).

However, if \( \varphi \) has some extra properties, it does induce a metric on \( M_2 \). This is the case when \( M_2 \) arises from \( M_1 \) as a quotient induced by some group of isometries of \( M_1 \). For more on this, see Gallot, Hulin and Lafontaine \([?]\), Chapter 2, Section 2.A.

Now, because a manifold is \textit{paracompact} (see Section 4.6), a Riemannian metric always exists on \( M \). This is a consequence of the existence of partitions of unity (see Theorem 4.6.5).

\textbf{Theorem 6.1.5} Every smooth manifold admits a Riemannian metric.
6.2 Connections on Manifolds

Given a manifold, $M$, in general, for any two points, $p, q \in M$, there is no “natural” isomorphism between the tangent spaces $T_p M$ and $T_q M$.

Given a curve, $c: [0, 1] \to M$, on $M$ as $c(t)$ moves on $M$, how does the tangent space, $T_{c(t)} M$ change as $c(t)$ moves?

If $M = \mathbb{R}^n$, then the spaces, $T_{c(t)} \mathbb{R}^n$, are canonically isomorphic to $\mathbb{R}^n$ and any vector, $v \in T_{c(0)} \mathbb{R}^n \cong \mathbb{R}^n$, is simply moved along $c$ by parallel transport, that is, at $c(t)$, the tangent vector, $v$, also belongs to $T_{c(t)} \mathbb{R}^n$.

However, if $M$ is curved, for example, a sphere, then it is not obvious how to “parallel transport” a tangent vector at $c(0)$ along a curve $c$. 
A way to achieve this is to define the notion of parallel vector field along a curve and this, in turn, can be defined in terms of the notion of covariant derivative of a vector field.

Assume for simplicity that $M$ is a surface in $\mathbb{R}^3$. Given any two vector fields, $X$ and $Y$ defined on some open subset, $U \subseteq \mathbb{R}^3$, for every $p \in U$, the directional derivative, $D_X Y(p)$, of $Y$ with respect to $X$ is defined by

$$D_X Y(p) = \lim_{t \to 0} \frac{Y(p + tX(p)) - Y(p)}{t}.$$

If $f: U \to \mathbb{R}$ is a differentiable function on $U$, for every $p \in U$, the directional derivative, $X[f](p)$ (or $X(f)(p)$), of $f$ with respect to $X$ is defined by

$$X[f](p) = \lim_{t \to 0} \frac{f(p + tX(p)) - f(p)}{t}.$$

We know that $X[f](p) = df_p(X(p))$. 
It is easily shown that $D_X Y(p)$ is $\mathbb{R}$-bilinear in $X$ and $Y$, is $C^\infty(U)$-linear in $X$ and satisfies the Leibnitz derivation rule with respect to $Y$, that is:

**Proposition 6.2.1** The directional derivative of vector fields satisfies the following properties:

\[
D_{X_1 + X_2} Y(p) = D_{X_1} Y(p) + D_{X_2} Y(p) \\
D_{fX} Y(p) = f D_X Y(p) \\
D_X (Y_1 + Y_2)(p) = D_X Y_1(p) + D_X Y_2(p) \\
D_X (fY)(p) = X[f](p) Y(p) + f(p) D_X Y(p),
\]

for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(U)$ and all $f \in C^\infty(U)$.

Now, if $p \in U$ where $U \subseteq M$ is an open subset of $M$, for any vector field, $Y$, defined on $U$ ($Y(p) \in T_p M$, for all $p \in U$), for every $X \in T_p M$, the directional derivative, $D_X Y(p)$, makes sense and it has an orthogonal decomposition,

\[
D_X Y(p) = \nabla_X Y(p) + (D_n)_X Y(p),
\]

where its **horizontal (or tangential) component** is $\nabla_X Y(p) \in T_p M$ and its normal component is $(D_n)_X Y(p)$. 

The component, $\nabla_X Y(p)$, is the \textit{covariant derivative} of $Y$ with respect to $X \in T_pM$ and it allows us to define the covariant derivative of a vector field, $Y \in \mathfrak{X}(U)$, with respect to a vector field, $X \in \mathfrak{X}(M)$, on $M$.

We easily check that $\nabla_X Y$ satisfies the four equations of Proposition 6.2.1.

In particular, $Y$, may be a vector field associated with a curve, $c : [0, 1] \to M$.

A \textit{vector field along a curve}, $c$, is a vector field, $Y$, such that $Y(c(t)) \in T_{c(t)}M$, for all $t \in [0, 1]$. We also write $Y(t)$ for $Y(c(t))$.

Then, we say that $Y$ \textit{is parallel along} $c$ \textit{iff} $\nabla_{\partial/\partial t} Y = 0$ along $c$. 

The notion of \textit{parallel transport} on a surface can be defined using parallel vector fields along curves. Let \( p, q \) be any two points on the surface \( M \) and assume there is a curve, \( c: [0, 1] \rightarrow M \), joining \( p = c(0) \) to \( q = c(1) \).

Then, using the uniqueness and existence theorem for ordinary differential equations, it can be shown that for any initial tangent vector, \( Y_0 \in T_pM \), there is a unique parallel vector field, \( Y \), along \( c \), with \( Y(0) = Y_0 \).

If we set \( Y_1 = Y(1) \), we obtain a linear map, \( Y_0 \mapsto Y_1 \), from \( T_pM \) to \( T_qM \) which is also an isometry.

As a summary, given a surface, \( M \), if we can define a notion of covariant derivative, \( \nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \), satisfying the properties of Proposition 6.2.1, then we can define the notion of parallel vector field along a curve and the notion of parallel transport, which yields a natural way of relating two tangent spaces, \( T_pM \) and \( T_qM \), using curves joining \( p \) and \( q \).
This can be generalized to manifolds using the notion of connection. We will see that the notion of connection induces the notion of curvature. Moreover, if \( M \) has a Riemannian metric, we will see that this metric induces a unique connection with two extra properties (the \textit{Levi-Civita} connection).

\textbf{Definition 6.2.2} Let \( M \) be a smooth manifold. A \textit{connection} on \( M \) is a \( \mathbb{R} \)-bilinear map, 

\[ \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M), \]

where we write \( \nabla_X Y \) for \( \nabla(X, Y) \), such that the following two conditions hold:

\[
\begin{align*}
\nabla_{fX} Y &= f \nabla_X Y \\
\nabla_X (fY) &= X[f]Y + f \nabla_X Y,
\end{align*}
\]

for all \( X, Y \in \mathfrak{X}(M) \) and all \( f \in C^\infty(M) \). The vector field, \( \nabla_X Y \), is called the \textit{covariant derivative of} \( Y \) \textit{with respect to} \( X \).

A connection on \( M \) is also known as an \textit{affine connection} on \( M \).
A basic property of $\nabla$ is that it is a \textit{local operator}.

\textbf{Proposition 6.2.3} Let $M$ be a smooth manifold and let $\nabla$ be a connection on $M$. For every open subset, $U \subseteq M$, for every vector field, $Y \in \mathfrak{X}(M)$, if $Y \equiv 0$ on $U$, then $\nabla_X Y \equiv 0$ on $U$ for all $X \in \mathfrak{X}(M)$, that is, $\nabla$ is a local operator.

Proposition 6.2.3 implies that a connection, $\nabla$, on $M$, restricts to a connection, $\nabla \upharpoonright U$, on every open subset, $U \subseteq M$.

It can also be shown that $(\nabla_X Y)(p)$ only depends on $X(p)$, that is, for any two vector fields, $X, Y \in \mathfrak{X}(M)$, if $X(p) = Y(p)$ for some $p \in M$, then

$$(\nabla_X Z)(p) = (\nabla_Y Z)(p) \quad \text{for every } Z \in \mathfrak{X}(M).$$

Consequently, for any $p \in M$, the covariant derivative, $(\nabla_u Y)(p)$, is well defined for any tangent vector, $u \in T_p M$, and any vector field, $Y$, defined on some open subset, $U \subseteq M$, with $p \in U$. 
Observe that on $U$, the $n$-tuple of vector fields, 
$$
\left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right),
$$
is a local frame.

We can write

$$
\nabla \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \right) = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k},
$$

for some unique smooth functions, $\Gamma_{ij}^{k}$, defined on $U$, called the \textit{Christoffel symbols}.

We say that a connection, $\nabla$, is \textit{flat} on $U$ iff

$$
\nabla_X \left( \frac{\partial}{\partial x_i} \right) = 0, \quad \text{for all} \quad X \in \mathfrak{X}(U), \quad 1 \leq i \leq n.
$$
Proposition 6.2.4  Every smooth manifold, $M$, possesses a connection.

Proof. We can find a family of charts, $(U_\alpha, \varphi_\alpha)$, such that $\{U_\alpha\}_\alpha$ is a locally finite open cover of $M$. If $(f_\alpha)$ is a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ and if $\nabla^\alpha$ is the flat connection on $U_\alpha$, then it is immediately verified that

$$\nabla = \sum_\alpha f_\alpha \nabla^\alpha$$

is a connection on $M$. □

Remark: A connection on $TM$ can be viewed as a linear map,

$$\nabla: \mathfrak{X}(M) \longrightarrow \text{Hom}_{C^\infty(M)}(\mathfrak{X}(M), (\mathfrak{X}(M))),$$

such that, for any fixed $Y \in \mathfrak{X}(M)$, the map, $\nabla Y : X \mapsto \nabla_X Y$, is $C^\infty(M)$-linear, which implies that $\nabla Y$ is a $(1, 1)$ tensor.
6.3 Parallel Transport

The notion of connection yields the notion of parallel transport. First, we need to define the covariant derivative of a vector field along a curve.

Definition 6.3.1 Let $M$ be a smooth manifold and let $\gamma: [a, b] \rightarrow M$ be a smooth curve in $M$. A **smooth vector field along the curve** $\gamma$ is a smooth map, $X: [a, b] \rightarrow TM$, such that $\pi(X(t)) = \gamma(t)$, for all $t \in [a, b]$ ($X(t) \in T_{\gamma(t)}M$).

Recall that the curve, $\gamma: [a, b] \rightarrow M$, is smooth iff $\gamma$ is the restriction to $[a, b]$ of a smooth curve on some open interval containing $[a, b]$. 
Proposition 6.3.2 Let \( M \) be a smooth manifold, let \( \nabla \) be a connection on \( M \) and \( \gamma: [a, b] \rightarrow M \) be a smooth curve in \( M \). There is a \( \mathbb{R} \)-linear map, \( D/dt \), defined on the vector space of smooth vector fields, \( X \), along \( \gamma \), which satisfies the following conditions:

(1) For any smooth function, \( f: [a, b] \rightarrow \mathbb{R} \),
\[
\frac{D(fX)}{dt} = \frac{df}{dt} X + f \frac{DX}{dt}
\]

(2) If \( X \) is induced by a vector field, \( Z \in \mathfrak{X}(M) \), that is, \( X(t_0) = Z(\gamma(t_0)) \) for all \( t_0 \in [a, b] \), then
\[
\frac{DX}{dt}(t_0) = (\nabla_{\gamma'(t_0)} Z)_{\gamma(t_0)}.
\]
Proof. Since $\gamma([a, b])$ is compact, it can be covered by a finite number of open subsets, $U_\alpha$, such that $(U_\alpha, \varphi_\alpha)$ is a chart. Thus, we may assume that $\gamma: [a, b] \to U$ for some chart, $(U, \varphi)$. As $\varphi \circ \gamma: [a, b] \to \mathbb{R}^n$, we can write

$$\varphi \circ \gamma(t) = (u_1(t), \ldots, u_n(t)),$$

where each $u_i = \text{pr}_i \circ \varphi \circ \gamma$ is smooth. Now, it is easy to see that

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{du_i}{dt} \left( \frac{\partial}{\partial x_i} \right)_{\gamma(t_0)}.$$

If $(s_1, \ldots, s_n)$ is a frame over $U$, we can write

$$X(t) = \sum_{i=1}^{n} X_i(t)s_i(\gamma(t)),$$

for some smooth functions, $X_i$. 
Then, conditions (1) and (2) imply that
\[
\frac{DX}{dt} = \sum_{j=1}^{n} \left( \frac{dX_{j}}{dt} s_{j}(\gamma(t)) + X_{j}(t) \nabla_{\gamma'(t)}(s_{j}(\gamma(t))) \right)
\]
and since
\[
\gamma'(t) = \sum_{i=1}^{n} \frac{du_{i}}{dt} \left( \frac{\partial}{\partial x_{i}} \right)_{\gamma(t)},
\]
there exist some smooth functions, \( \Gamma_{ij}^{k} \), so that
\[
\nabla_{\gamma'(t)}(s_{j}(\gamma(t))) = \sum_{i=1}^{n} \frac{du_{i}}{dt} \nabla_{\frac{\partial}{\partial x_{i}}}(s_{j}(\gamma(t))) = \sum_{i,k} \frac{du_{i}}{dt} \Gamma_{ij}^{k} s_{k}(\gamma(t)).
\]
It follows that
\[
\frac{DX}{dt} = \sum_{k=1}^{n} \left( \frac{dX_{k}}{dt} + \sum_{i,j} \Gamma_{ij}^{k} \frac{du_{i}}{dt} X_{j} \right) s_{k}(\gamma(t)).
\]
Conversely, the above expression defines a linear operator, \( D/dt \), and it is easy to check that it satisfies (1) and (2).
The operator, $D/dt$ is often called *covariant derivative along* $\gamma$ and it is also denoted by $\nabla_{\gamma'}(t)$ or simply $\nabla_{\gamma'}$.

**Definition 6.3.3** Let $M$ be a smooth manifold and let $\nabla$ be a connection on $M$. For every curve, $\gamma: [a, b] \to M$, in $M$, a vector field, $X$, along $\gamma$ is *parallel (along $\gamma$)* iff

$$\frac{DX}{dt} = 0.$$ 

If $M$ was embedded in $\mathbb{R}^d$, for some $d$, then to say that $X$ is parallel along $\gamma$ would mean that the directional derivative, $(D_{\gamma'}X)(\gamma(t))$, is normal to $T_{\gamma(t)}M$.

The following proposition can be shown using the existence and uniqueness of solutions of ODE’s (in our case, linear ODE’s) and its proof is omitted:
Proposition 6.3.4 Let \( M \) be a smooth manifold and let \( \nabla \) be a connection on \( M \). For every \( C^1 \) curve, \( \gamma : [a, b] \to M \), in \( M \), for every \( t \in [a, b] \) and every \( v \in T_{\gamma(t)}M \), there is a unique parallel vector field, \( X \), along \( \gamma \) such that \( X(t) = v \).

For the proof of Proposition 6.3.4 it is sufficient to consider the portions of the curve \( \gamma \) contained in some chart. In such a chart, \((U, \varphi)\), as in the proof of Proposition 6.3.2, using a local frame, \((s_1, \ldots, s_n)\), over \( U \), we have

\[
\frac{DX}{dt} = \sum_{k=1}^{n} \left( \frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt} X_j \right) s_k(\gamma(t)),
\]

with \( u_i = pr_i \circ \varphi \circ \gamma \). Consequently, \( X \) is parallel along our portion of \( \gamma \) iff the system of linear ODE’s in the unknowns, \( X_k \),

\[
\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt} X_j = 0, \quad k = 1, \ldots, n,
\]

is satisfied.
**Remark:** Proposition 6.3.4 can be extended to piecewise $C^1$ curves.

**Definition 6.3.5** Let $M$ be a smooth manifold and let $\nabla$ be a connection on $M$. For every curve, $\gamma: [a, b] \to M$, in $M$, for every $t \in [a, b]$, the parallel transport from $\gamma(a)$ to $\gamma(t)$ along $\gamma$ is the linear map from $T_{\gamma(a)}M$ to $T_{\gamma(t)}M$, which associates to any $v \in T_{\gamma(a)}M$ the vector, $X_v(t) \in T_{\gamma(t)}M$, where $X_v$ is the unique parallel vector field along $\gamma$ with $X_v(a) = v$.

The following proposition is an immediate consequence of properties of linear ODE’s:

**Proposition 6.3.6** Let $M$ be a smooth manifold and let $\nabla$ be a connection on $M$. For every $C^1$ curve, $\gamma: [a, b] \to M$, in $M$, the parallel transport along $\gamma$ defines for every $t \in [a, b]$ a linear isomorphism, $P_\gamma: T_{\gamma(a)}M \to T_{\gamma(t)}M$, between the tangent spaces, $T_{\gamma(a)}M$ and $T_{\gamma(t)}M$. 
In particular, if \( \gamma \) is a closed curve, that is, if 
\( \gamma(a) = \gamma(b) = p \), we obtain a linear isomorphism, \( P_\gamma \), of 
the tangent space, \( T_pM \), called the \textit{holonomy of } \( \gamma \). The \textit{holonomy group of } \( \nabla \) \textit{based at } \( p \), denoted \( \text{Hol}_p(\nabla) \), is 
the subgroup of \( \text{GL}(V, \mathbb{R}) \) given by 

\[
\text{Hol}_p(\nabla) = \{ P_\gamma \in \text{GL}(V, \mathbb{R}) \mid \gamma \text{ is a closed curve based at } p \}.
\]

If \( M \) is connected, then \( \text{Hol}_p(\nabla) \) depends on the base-
point \( p \in M \) up to conjugation and so \( \text{Hol}_p(\nabla) \) and 
\( \text{Hol}_q(\nabla) \) are isomorphic for all \( p, q \in M \). In this case, it 
makes sense to talk about the \textit{holonomy group of } \( \nabla \). By 
abuse of language, we call \( \text{Hol}_p(\nabla) \) the \textit{holonomy group of } \( M \).
6.4 Connections Compatible with a Metric; Levi-Civita Connections

If a Riemannian manifold, $M$, has a metric, then it is natural to define when a connection, $\nabla$, on $M$ is compatible with the metric.

Given any two vector fields, $Y, Z \in \mathfrak{X}(M)$, the smooth function, $\langle Y, Z \rangle$, is defined by

$$\langle Y, Z \rangle(p) = \langle Y_p, Z_p \rangle,$$

for all $p \in M$.

**Definition 6.4.1** Given any metric, $\langle - , - \rangle$, on a smooth manifold, $M$, a connection, $\nabla$, on $M$ is compatible with the metric, for short, a metric connection iff

$$X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

for all vector fields, $X, Y, Z \in \mathfrak{X}(M)$. 
Proposition 6.4.2 Let $M$ be a Riemannian manifold with a metric, $\langle -, - \rangle$. Then, $M$, possesses metric connections.

Proof. For every chart, $(U_\alpha, \varphi_\alpha)$, we use the Gram-Schmidt procedure to obtain an orthonormal frame over $U_\alpha$ and we let $\nabla^\alpha$ be the trivial connection over $U_\alpha$. By construction, $\nabla^\alpha$ is compatible with the metric. We finish the argument by using a partition of unity, leaving the details to the reader.

We know from Proposition 6.4.2 that metric connections on $TM$ exist. However, there are many metric connections on $TM$ and none of them seems more relevant than the others.

It is remarkable that if we require a certain kind of symmetry on a metric connection, then it is uniquely determined.
Such a connection is known as the *Levi-Civita connection*. The Levi-Civita connection can be characterized in several equivalent ways, a rather simple way involving the notion of torsion of a connection.

There are two *error terms* associated with a connection. The first one is the *curvature*,

\[
R(X, Y) = \nabla_{[X,Y]} + \nabla_Y \nabla_X - \nabla_X \nabla_Y.
\]

The second natural error term is the *torsion*, \(T(X, Y)\), of the connection, \(\nabla\), given by

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],
\]

which measures the failure of the connection to behave like the Lie bracket.
Proposition 6.4.3 (Levi-Civita, Version 1) Let $M$ be any Riemannian manifold. There is a unique, metric, torsion-free connection, $\nabla$, on $M$, that is, a connection satisfying the conditions

\[
X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle
\]
\[
\nabla_X Y - \nabla_Y X = [X, Y],
\]

for all vector fields, $X, Y, Z \in \mathfrak{X}(M)$. This connection is called the Levi-Civita connection (or canonical connection) on $M$. Furthermore, this connection is determined by the Koszul formula

\[
2\langle \nabla_X Y, Z \rangle = X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle)
\]
\[
- \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle - \langle Z, [Y, X] \rangle.
\]
\textbf{Proof}. First, we prove uniqueness. Since our metric is a non-degenerate bilinear form, it suffices to prove the Koszul formula. As our connection is compatible with the metric, we have

$$
X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\
Y(\langle X, Z \rangle) = \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle \\
-Z(\langle X, Y \rangle) = -\langle \nabla_Z X, Y \rangle - \langle X, \nabla_Z Y \rangle
$$

and by adding up the above equations, we get

$$
X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) = \\
\langle Y, \nabla_X Z - \nabla_Z X \rangle \\
+ \langle X, \nabla_Y Z - \nabla_Z Y \rangle \\
+ \langle Z, \nabla_X Y + \nabla_Y X \rangle.
$$

Then, using the fact that the torsion is zero, we get

$$
X(\langle Y, Z \rangle) + Y(\langle X, Z \rangle) - Z(\langle X, Y \rangle) = \\
\langle Y, [X, Z] \rangle + \langle X, [Y, Z] \rangle \\
+ \langle Z, [Y, X] \rangle + 2\langle Z, \nabla_X Y \rangle
$$

which yields the Koszul formula.

We will not prove existence here. The reader should consult the standard texts for a proof. □
Remark: In a chart, \((U, \varphi)\), if we set

\[
\partial_k g_{ij} = \frac{\partial}{\partial x_k} (g_{ij})
\]

then it can be shown that the Christoffel symbols are given by

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),
\]

where \((g^{kl})\) is the inverse of the matrix \((g_{kl})\).

It can be shown that a connection is torsion-free iff

\[
\Gamma^k_{ij} = \Gamma^k_{ji}, \quad \text{for all } i, j, k.
\]

We conclude this section with various useful facts about torsion-free or metric connections.

First, there is a nice characterization for the Levi-Civita connection induced by a Riemannian manifold over a submanifold.
Proposition 6.4.4 Let $M$ be any Riemannian manifold and let $N$ be any submanifold of $M$ equipped with the induced metric. If $\nabla^M$ and $\nabla^N$ are the Levi-Civita connections on $M$ and $N$, respectively, induced by the metric on $M$, then for any two vector fields, $X$ and $Y$ in $\mathfrak{X}(M)$ with $X(p), Y(p) \in T_pN$, for all $p \in N$, we have

$$\nabla^N_X Y = (\nabla^M_X Y)_\parallel,$$

where $(\nabla^M_X Y)_\parallel(p)$ is the orthogonal projection of $\nabla^M_X Y(p)$ onto $T_pN$, for every $p \in N$.

In particular, if $\gamma$ is a curve on a surface, $M \subseteq \mathbb{R}^3$, then a vector field, $X(t)$, along $\gamma$ is parallel iff $X'(t)$ is normal to the tangent plane, $T_{\gamma(t)}M$.

If $\nabla$ is a metric connection, then we can say more about the parallel transport along a curve. Recall from Section 6.3, Definition 6.3.3, that a vector field, $X$, along a curve, $\gamma$, is parallel iff

$$\frac{DX}{dt} = 0.$$
Proposition 6.4.5 Given any Riemannian manifold, $M$, and any metric connection, $\nabla$, on $M$, for every curve, $\gamma: [a, b] \to M$, on $M$, if $X$ and $Y$ are two vector fields along $\gamma$, then

$$\frac{d}{dt} \langle X(t), Y(t) \rangle = \left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle.$$ 

Using Proposition 6.4.5 we get

Proposition 6.4.6 Given any Riemannian manifold, $M$, and any metric connection, $\nabla$, on $M$, for every curve, $\gamma: [a, b] \to M$, on $M$, if $X$ and $Y$ are two vector fields along $\gamma$ that are parallel, then

$$\langle X, Y \rangle = C,$$

for some constant, $C$. In particular, $\|X(t)\|$ is constant. Furthermore, the linear isomorphism, $P_\gamma: T_{\gamma(a)} \to T_{\gamma(b)}$, is an isometry.

In particular, Proposition 6.4.6 shows that the holonomy group, $\text{Hol}_p(\nabla)$, based at $p$, is a subgroup of $O(n)$. 