$\boxed{\textbf{Spring, 2006} \quad \textbf{CIS 610}}$

Advanced Geometric Methods in Computer Science
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Homework 3

March 29, 2006; Due April 12, 2006

"A problems" are for practice only, and should not be turned in.

Problem A1. Give an example of a complex such that for some faces, σ, which is not a vertex, $\mathrm{lk}(\sigma)$ is properly contained in $\overline{\mathrm{st}(\sigma)} - \mathrm{st}(\sigma)$.

"B problems" must be turned in.

Problem B1 (20 pts). A *ray*, R, of $\mathbb{E}^d$ is any subset of the form

$$R = \{a + tu \mid t \in \mathbb{R},\ t \geq 0\},$$

for some point, $a \in \mathbb{E}^d$ and some nonzero vector, $u \in \mathbb{R}^d$. A subset, $A \subseteq \mathbb{E}^d$, is unbounded iff it is not contained in any ball.

Prove that every closed and unbounded convex set, $A \subseteq \mathbb{E}^d$, contains a ray.

Hint: Pick some point, $c \in A$, as the origin and consider the sphere, $S(c, r) \subseteq \mathbb{E}^d$, of center c and radius $r > 0$. As A is unbounded and $c \in A$, show that

$$A_r = A \cap S(c, r) \neq \emptyset$$

for all $r > 0$. As A is closed, each A_r is compact. Consider the radial projection, $\pi_r \colon S(c, r) \to S(c, 1)$, from $S(c, r)$ onto the unit sphere, $S(c, 1)$, of center c given by

$$\pi_r(a) = \frac{1}{r}a.$$

This map is invertible, the inverse being given by

$$\pi_r^{-1}(b) = rb,$$

and clearly, π is continuous. Thus, each set,

$$K_r = \pi_r(A_r)$$

is a compact subset of $S(c, 1)$. Moreover, check that $K_{r_1} \subseteq K_{r_2}$ whenever $r_2 \leq r_1$. Deduce from this and the fact that the K_r are compact that

$$\bigcap_{r>0} K_r \neq \emptyset.$$

Pick any $d \in \bigcap_{r>0} K_r$ and prove that $\{c + t\mathbf{cd} \mid t \geq 0\}$ is a ray contained in A.

Problem B2 (50 pts). A subset, $C \subseteq \mathbb{E}^d$, is a *cone* iff C is closed under positive linear combinations, that is, iff

$$\lambda_1 u_1 + \cdots + \lambda_k u_k \in C,$$

for all $u_1, \ldots, u_k \in C$ and all $\lambda_i \in \mathbb{R}$, with $\lambda_i \geq 0$ and $1 \leq i \leq k$. Note that we always $0 \in C$.

(i) Check that C is convex.

For any subset, $V \subseteq \mathbb{R}^d$, the *positive hull of* C, cone(V), is given by

$$\text{cone}(V) = \{\lambda_1 u_1 + \cdots + \lambda_k u_k \mid u_i \in V,\ \lambda_i \geq 0,\ 1 \leq i \leq k\}.$$

A cone, $C \subseteq \mathbb{E}^d$, is a $\mathcal{V}$-*cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \text{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \mathbb{R}^d$.

A cone, $C \subseteq \mathbb{E}^d$, is an $\mathcal{H}$-*cone* iff it is equal to a finite intersection of closed half spaces cut out by hyperplanes through 0.

Say that C *has* 0 *as an apex* iff there is a hyperplane, H, though 0, so that $H \cap C = \{0\}$.

(ii) Let H and H' be parallel hyperplanes with $0 \in H$. Prove that for any closed cone, C, if $C \cap H' \neq \emptyset$, then $C \cap H = \{0\}$ iff $C \cap H'$ is bounded.

Hint: If $A \cap H'$ is not bounded, use problem B1 to construct a sequence of points, a_n, along a ray in A and consider the sequence of unit vectors, $u_n = \frac{a_n}{\|a_n\|}$. Show that the u_n converge to a vector, $u \in C \cap H$, a contradiction.

(iii) Let C be a polyhedral cone of dimension $d \geq 2$. Prove that the following statements are equivalent:

(a) C has 0 as an apex.

(b) There is a hyperplane, H', not passing through 0 such that $C \cap H'$ is a polytope.

(c) There is some polytope, P, of dimension $d - 1$ such that $C = \text{cone}(P)$.

(iv) Prove that every polyhedral cone with 0 as an apex is an intersection of closed half-spaces, $\bigcap_{i=1}^{p} (H_i)_-$, with $\bigcap_{i=1}^{p} H_i = \{0\}$.

If $C = \bigcap_{i=1}^{p}(H_i)_-$ with $\bigcap_{i=1}^{p} H_i = \{0\}$, is C is $\mathcal{V}$-cone with 0 as an apex?

Problem B3 (40 pts). Let $C \subseteq \mathbb{E}^d$ be any $\mathcal{V}$-cone with nonempty interior. Pick any Ω in the interior of C, and consider the polar dual, C^*, of C w.r.t. Ω.

(i) Prove that C^* is an $\mathcal{H}$-polytope, namely if $C = \mathrm{cone}(\{u_1, \ldots, u_p\})$, then

$$C^* = (0^\dagger)_- \cap \bigcap_{i=1}^{p} (u_i^\dagger)_-,$$

where $(0^\dagger)_-$ is the closed half-space containing Ω determined by the polar hyperplane, $0^\dagger$, of 0 w.r.t. the center Ω and $(u_i^\dagger)_-$ is the closed half space defined as follows:

$$(u_i^\dagger)_- = \{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot u_i \leq 0\}.$$

Note that $\{x \in \mathbb{E}^d \mid \mathbf{\Omega x} \cdot u_i = 0\}$ is the hyperplane through Ω perpendicular to u_i, so $(u_i^\dagger)_-$ is the closed half-space on the side opposite to u_i and bounded by this hyperplane. Draw a few pictures in the case of the plane to understand what's going on.

(ii) Use (i) and the equivalence of $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes to prove that $C = C^{**}$ is an $\mathcal{H}$-cone. Therefore, any $\mathcal{V}$-cone is an $\mathcal{H}$-cone.

(iii) Use a similar argument to prove that any polyhedral set (i.e., a $\mathcal{V}$-polyhedron) is an $\mathcal{H}$-polyhedron.

Problem B4 (20 pts). Let $P \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron, $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$, where $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

(i) Check that

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$P = C(P) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$.

Conversely, prove that if $C = \mathrm{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{id+1} \geq 0$ for every $w_i \in W$, then $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

(ii) Now, let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron. Then, P is cut out by m hyperplanes, H_i, and for each H_i, there is a nonzero vector, u_i, and some $b_i \in \mathbb{R}$ so that

$$H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = b_i\}$$

and P is given by

$$P = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid u_i \cdot x \leq b_i\}.$$

If A denotes the $m \times d$ matrix whose i-th row is u_i and b is the vector $b = (b_1, \ldots, b_m)$, then we can write

$$P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}.$$

We "homogenize" $P(A, b)$ as follows: Let $C(P)$ be the subset of $\mathbb{E}^{d+1}$ defined by

$$\begin{aligned} C(P) &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax \leq x_{d+1} b,\ x_{d+1} \geq 0 \right\} \\ &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - x_{d+1} b \leq 0,\ -x_{d+1} \leq 0 \right\}. \end{aligned}$$

Thus, we see that $C(P)$ is the $\mathcal{H}$-cone given by the system of inequalities

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and that

$$P = C(P) \cap H_{d+1}.$$

Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), it is clear that $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $\mathbb{E}^d$.

Problem B4 shows that the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra reduces to the equivalence of $\mathcal{V}$-cones and $\mathcal{H}$-cones, constructively (i.e. we can go from cones to polyhedra and back algorithmically).

Problem B5 (40 pts). Let $C \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-cone. Then, C is cut out by m hyperplanes, H_i, through 0. For each H_i, there is a nonzero vector, u_i, so that

$$H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = 0\}$$

and C is given by

$$C = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid u_i \cdot x \leq 0\}.$$

If A denotes the $m \times d$ matrix whose i-th row is u_i, then we can write

$$C = P(A, 0) = \{x \in \mathbb{E}^d \mid Ax \leq 0\}.$$

Observe that $C = C_0(A) \cap H_w$, where

$$C_0(A) = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid Ax \leq w \right\}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+m}$ and

$$H_w = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid w = 0 \right\}$$

is a hyperplane in $\mathbb{E}^{d+m}$.

(i) Prove that $C_0(A)$ is a $\mathcal{V}$-cone by observing that if we write

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{d} |x_i|(\mathrm{sign}(x_i)) \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{j=1}^{m} (w_j - (Ax)_j) \begin{pmatrix} 0 \\ e_j \end{pmatrix},$$

then

$$C_0(A) = \mathrm{cone}\left(\left\{ \pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} \mid 1 \leq i \leq d \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_j \end{pmatrix} \mid 1 \leq j \leq m \right\} \right).$$

(ii) Since $C = C_0(A) \cap H_w$ is now the intersection of a $\mathcal{V}$-cone with a hyperplane, to prove that C is a $\mathcal{V}$-cone it is enough to prove that the intersection of a $\mathcal{V}$-cone with a hyperplane is also a $\mathcal{V}$-cone. For this, we use *Fourier-Motzkin elimination*. It suffices to prove the result for a hyperplane, H_k, in $\mathbb{E}^{d+m}$ of equation $y_k = 0$ $(1 \leq k \leq d+m)$.

Say $C = \mathrm{cone}(Y) \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-cone. Then, the intersection $C \cap H_k$ (where H_k is the hyperplane of equation $y_k = 0$) is a $\mathcal{V}$-cone, $C \cap H_k = \mathrm{cone}(Y^{/k})$, with

$$Y^{/k} = \{y_i \mid y_{ik} = 0\} \cup \{y_{ik} y_j - y_{jk} y_i \mid y_{ik} > 0,\ y_{jk} < 0\},$$

the set of vectors obtained from Y by "eliminating the k-th coordinate". Here, each y_i is a vector in $\mathbb{R}^d$.

Hint. The only nontrivial direction is to prove that $C \cap H_k \subseteq \mathrm{cone}(Y^{/k})$. For this, consider any $v = \sum_{i=1}^{d} t_i y_i \in C \cap H_k$, with $t_i \geq 0$ and $v_k = 0$. Such a v can be written

$$v = \sum_{i \mid y_{ik}=0} t_i y_i + \sum_{i \mid y_{ik}>0} t_i y_i + \sum_{j \mid y_{jk}<0} t_j y_j$$

and as $v_k = 0$, we have

$$\sum_{i \mid y_{ik}>0} t_i y_{ik} + \sum_{j \mid y_{jk}<0} t_j y_{jk} = 0.$$

If $t_i y_{ik} = 0$ for $i = 1, \ldots, d$, we are done. Otherwise, we can write

$$\Lambda = \sum_{i \mid y_{ik}>0} t_i y_{ik} = \sum_{j \mid y_{jk}<0} -t_j y_{jk} > 0.$$

Then,

$$v = \sum_{i \mid y_{ik}=0} t_i y_i + \frac{1}{\Lambda} \sum_{i \mid y_{ik}>0} \left(\sum_{j \mid y_{jk}<0} -t_j y_{jk} \right) t_i y_i + \frac{1}{\Lambda} \sum_{j \mid y_{jk}<0} \left(\sum_{i \mid y_{ik}>0} t_i y_{ik} \right) t_j y_j.$$

Conclude that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

(iii) Use Problem B4 to prove that if P is an $\mathcal{H}$-polyhedron then it is a $\mathcal{V}$-polyhedron.

Problem B6 (20 pts). Prove that Farkas Lemma, version III implies Farkas Lemma, version II (from the notes).

TOTAL: 190 points.