Advanced Geometric Methods in Computer Science
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Homework 3

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“A problems” are for practice only, and should not be turned in.

Problem A1. Give an example of a complex such that for some faces, \( \sigma \), which is not a vertex, \( \text{lk}(\sigma) \) is properly contained in \( \text{st}(\sigma) - \text{st}(\sigma) \).

“B problems” must be turned in.

Problem B1 (20 pts). A ray, \( R \), of \( \mathbb{E}^d \) is any subset of the form

\[
R = \{ a + tu \mid t \in \mathbb{R}, t \geq 0 \},
\]

for some point, \( a \in \mathbb{E}^d \) and some nonzero vector, \( u \in \mathbb{R}^d \). A subset, \( A \subseteq \mathbb{E}^d \), is unbounded iff it is not contained in any ball.

Prove that every closed and unbounded convex set, \( A \subseteq \mathbb{E}^d \), contains a ray.

\textit{Hint:} Pick some point, \( c \in A \), as the origin and consider the sphere, \( S(c, r) \subseteq \mathbb{E}^d \), of center \( c \) and radius \( r > 0 \). As \( A \) is unbounded and \( c \in A \), show that

\[
A_r = A \cap S(c, r) \neq \emptyset
\]

for all \( r > 0 \). As \( A \) is closed, each \( A_r \) is compact. Consider the radial projection,

\[
\pi_r : S(c, r) \to S(c, 1),
\]

from \( S(c, r) \) onto the unit sphere, \( S(c, 1) \), of center \( c \) given by

\[
\pi_r(a) = \frac{1}{r} a.
\]

This map is invertible, the inverse being given by

\[
\pi_r^{-1}(b) = rb,
\]

and clearly, \( \pi \) is continuous. Thus, each set,

\[
K_r = \pi_r(A_r)
\]
is a compact subset of \( S(c, 1) \). Moreover, check that \( K_{r_1} \subseteq K_{r_2} \) whenever \( r_2 \leq r_1 \). Deduce from this and the fact that the \( K_r \) are compact that

\[
\bigcap_{r > 0} K_r \neq \emptyset.
\]

Pick any \( d \in \bigcap_{r > 0} K_r \) and prove that \( \{ c + tcd \mid t \geq 0 \} \) is a ray contained in \( A \).

**Problem B2 (50 pts).** A subset, \( C \subseteq \mathbb{E}^d \), is a cone iff \( C \) is closed under positive linear combinations, that is, iff

\[
\lambda_1 u_1 + \cdots + \lambda_k u_k \in C,
\]

for all \( u_1, \ldots, u_k \in C \) and all \( \lambda_i \in \mathbb{R} \), with \( \lambda_i \geq 0 \) and \( 1 \leq i \leq k \). Note that we always \( 0 \in C \).

(i) Check that \( C \) is convex.

For any subset, \( V \subseteq \mathbb{R}^d \), the positive hull of \( C \), \( \text{cone}(V) \), is given by

\[
\text{cone}(V) = \{ \lambda_1 u_1 + \cdots + \lambda_k u_k \mid u_i \in V, \lambda_i \geq 0, 1 \leq i \leq k \}.
\]

A cone, \( C \subseteq \mathbb{E}^d \), is a \( \mathcal{V} \)-cone or polyhedral cone if \( C \) is the positive hull of a finite set of vectors, that is,

\[
C = \text{cone}\{ u_1, \ldots, u_p \},
\]

for some vectors, \( u_1, \ldots, u_p \in \mathbb{R}^d \).

A cone, \( C \subseteq \mathbb{E}^d \), is an \( \mathcal{H} \)-cone iff it is equal to a finite intersection of closed half spaces cut out by hyperplanes through 0.

Say that \( C \) has 0 as an apex iff there is a hyperplane, \( H \), though 0, so that \( H \cap C = \{0\} \).

(ii) Let \( H \) and \( H' \) be parallel hyperplanes with \( 0 \in H \). Prove that for any closed cone, \( C \), if \( C \cap H' \neq \emptyset \), then \( C \cap H = \{0\} \) iff \( C \cap H' \) is bounded.

*Hint:* If \( A \cap H' \) is not bounded, use problem B1 to construct a sequence of points, \( a_n \), along a ray in \( A \) and consider the sequence of unit vectors, \( u_n = \frac{a_n}{\|a_n\|} \). Show that the \( u_n \) converge to a vector, \( u \in C \cap H \), a contradiction.

(iii) Let \( C \) be a polyhedral cone of dimension \( d \geq 2 \). Prove that the following statements are equivalent:

(a) \( C \) has 0 as an apex.

(b) There is a hyperplane, \( H' \), not passing through 0 such that \( C \cap H' \) is a polytope.

(c) There is some polytope, \( P \), of dimension \( d - 1 \) such that \( C = \text{cone}(P) \).

(iv) Prove that every polyhedral cone with 0 as an apex is an intersection of closed half-spaces, \( \bigcap_{i=1}^n (H_i)_- \), with \( \bigcap_{i=1}^n H_i = \{0\} \).
If $C = \bigcap_{i=1}^{p}(H_i)_-$ with $\bigcap_{i=1}^{p} H_i = \{0\}$, is $C$ is $\mathcal{V}$-cone with 0 as an apex?

**Problem B3 (40 pts).** Let $C \subseteq \mathbb{E}^d$ be any $\mathcal{V}$-cone with nonempty interior. Pick any $\Omega$ in the interior of $C$, and consider the polar dual, $C^*$, of $C$ w.r.t. $\Omega$.

(i) Prove that $C^*$ is an $\mathcal{H}$-polytope, namely if $C = \text{cone} \{u_1, \ldots, u_p\}$, then

$$C^* = (0^\dagger)_- \cap \bigcap_{i=1}^{p}(u_i^\dagger)_-,$$

where $(0^\dagger)_-$ is the closed half-space containing $\Omega$ determined by the polar hyperplane, $0^\dagger$, of 0 w.r.t. the center $\Omega$ and $(u_i^\dagger)_-$ is the closed half space defined as follows:

$$(u_i^\dagger)_- = \{x \in \mathbb{E}^d \mid \Omega x \cdot u_i \leq 0\}.$$

Note that $\{x \in \mathbb{E}^d \mid \Omega x \cdot u_i = 0\}$ is the hyperplane through $\Omega$ perpendicular to $u_i$, so $(u_i^\dagger)_-$ is the closed half-space on the side opposite to $u_i$ and bounded by this hyperplane. Draw a few pictures in the case of the plane to understand what’s going on.

(ii) Use (i) and the equivalence of $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes to prove that $C = C^{**}$ is an $\mathcal{H}$-cone. Therefore, any $\mathcal{V}$-cone is an $\mathcal{H}$-cone.

(iii) Use a similar argument to prove that any polyhedral set (i.e., a $\mathcal{V}$-polyhedron) is an $\mathcal{H}$-polyhedron.

**Problem B4 (20 pts).** Let $P \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron, $P = \text{conv}(Y) + \text{cone}(V)$, where $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Define $\hat{Y} = \{\hat{y}_1, \ldots, \hat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and $\hat{V} = \{\hat{v}_1, \ldots, \hat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\hat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \quad \hat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

(i) Check that

$$C(P) = \text{cone} \{\hat{Y} \cup \hat{V}\}$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$P = C(P) \cap H_{d+1},$$

where $H_{d+1}$ is the hyperplane of equation $x_{d+1} = 1$.

Conversely, prove that if $C = \text{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i(d+1)} \geq 0$ for every $w_i \in W$, then $P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

(ii) Now, let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron. Then, $P$ is cut out by $m$ hyperplanes, $H_i$, and for each $H_i$, there is a nonzero vector, $u_i$, and some $b_i \in \mathbb{R}$ so that

$$H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = b_i\}.$$
and $P$ is given by

$$P = \bigcap_{i=1}^{m} \{ x \in \mathbb{E}^d \mid u_i \cdot x \leq b_i \}.$$ 

If $A$ denotes the $m \times d$ matrix whose $i$-th row is $u_i$ and $b$ is the vector $b = (b_1, \ldots, b_m)$, then we can write

$$P = P(A, b) = \{ x \in \mathbb{E}^d \mid Ax \leq b \}.$$ 

We “homogenize” $P(A, b)$ as follows: Let $C(P)$ be the subset of $\mathbb{E}^{d+1}$ defined by

$$C(P) = \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \mid Ax \leq x_{d+1}b, x_{d+1} \geq 0 \right\} = \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - x_{d+1}b \leq 0, -x_{d+1} \leq 0 \right\}.$$ 

Thus, we see that $C(P)$ is the $\mathcal{H}$-cone given by the system of inequalities

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and that

$$P = C(P) \cap H_{d+1}.$$ 

Conversely, if $Q$ is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), it is clear that $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $\mathbb{E}^d$.

Problem B4 shows that the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra reduces to the equivalence of $\mathcal{V}$-cones and $\mathcal{H}$-cones, constructively (i.e. we can go from cones to polyhedra and back algorithmically).

**Problem B5 (40 pts).** Let $C \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-cone. Then, $C$ is cut out by $m$ hyperplanes, $H_i$, through 0. For each $H_i$, there is a nonzero vector, $u_i$, so that

$$H_i = \{ x \in \mathbb{E}^d \mid u_i \cdot x = 0 \}$$

and $C$ is given by

$$C = \bigcap_{i=1}^{m} \{ x \in \mathbb{E}^d \mid u_i \cdot x \leq 0 \}.$$ 

If $A$ denotes the $m \times d$ matrix whose $i$-th row is $u_i$, then we can write

$$C = P(A, 0) = \{ x \in \mathbb{E}^d \mid Ax \leq 0 \}.$$ 

Observe that $C = C_0(A) \cap H_w$, where

$$C_0(A) = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid Ax \leq w \right\}.$$
is an $H$-cone in $\mathbb{E}^{d+m}$ and
\[ H_w = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid w = 0 \right\} \]
is a hyperplane in $\mathbb{E}^{d+m}$.

(i) Prove that $C_0(A)$ is a $V$-cone by observing that if we write
\[
\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{d} |x_i| (\text{sign}(x_i)) \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{j=1}^{m} (w_j - (Ax)_j) \begin{pmatrix} 0 \\ e_j \end{pmatrix},
\]
then
\[ C_0(A) = \text{cone} \left( \left\{ \pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} \mid 1 \leq i \leq d \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_j \end{pmatrix} \mid 1 \leq j \leq m \right\} \right). \]

(ii) Since $C = C_0(A) \cap H_w$ is now the intersection of a $V$-cone with a hyperplane, to prove that $C$ is a $V$-cone it is enough to prove that the intersection of a $V$-cone with a hyperplane is also a $V$-cone. For this, we use Fourier-Motzkin elimination. It suffices to prove the result for a hyperplane, $H_k$, in $\mathbb{E}^{d+m}$ of equation $y_k = 0$ ($1 \leq k \leq d + m$).

Say $C = \text{cone}(Y) \subseteq \mathbb{R}^d$ is a $V$-cone. Then, the intersection $C \cap H_k$ (where $H_k$ is the hyperplane of equation $y_k = 0$) is a $V$-cone, $C \cap H_k = \text{cone}(Y^*/k)$, with $Y^*/k = \{ y_i \mid y_{ik} = 0 \} \cup \{ y_{ik}y_j - y_{jk}y_i \mid y_{ik} > 0, y_{jk} < 0 \}$, the set of vectors obtained from $Y$ by “eliminating the $k$-th coordinate”. Here, each $y_i$ is a vector in $\mathbb{R}^d$.

**Hint.** The only nontrivial direction is to prove that $C \cap H_k \subseteq \text{cone}(Y^*/k)$. For this, consider any $v = \sum_{i=1}^{d} t_i y_i \in C \cap H_k$, with $t_i \geq 0$ and $v_k = 0$. Such a $v$ can be written
\[
v = \sum_{i \mid y_{ik} = 0} t_i y_i + \sum_{i \mid y_{ik} > 0} t_i y_i + \sum_{j \mid y_{jk} < 0} t_j y_j
\]
and as $v_k = 0$, we have
\[
\sum_{i \mid y_{ik} > 0} t_i y_{ik} + \sum_{j \mid y_{jk} < 0} t_j y_{jk} = 0.
\]
If $t_i y_{ik} = 0$ for $i = 1, \ldots, d$, we are done. Otherwise, we can write
\[
\Lambda = \sum_{i \mid y_{ik} > 0} t_i y_{ik} = \sum_{j \mid y_{jk} < 0} -t_j y_{jk} > 0.
\]
Then,
\[
v = \sum_{i \mid y_{ik} = 0} t_i y_i + \frac{1}{\Lambda} \sum_{i \mid y_{ik} > 0} \left( \sum_{j \mid y_{jk} < 0} -t_j y_{jk} \right) t_i y_i + \frac{1}{\Lambda} \sum_{j \mid y_{jk} < 0} \left( \sum_{i \mid y_{ik} > 0} t_i y_{ik} \right) t_j y_j.
\]
Conclude that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

(iii) Use Problem B4 to prove that if $P$ is an $\mathcal{H}$-polyhedron then it is a $\mathcal{V}$-polyhedron.

**Problem B6 (20 pts).** Prove that Farkas Lemma, version III implies Farkas Lemma, version II (from the notes).

**TOTAL: 190 points.**