Homework 2

February 20, 2006; Due March 13, 2006

“A problems” are for practice only, and should not be turned in.

**Problem A1.** Let \((e_1, \ldots, e_n)\) be an orthonormal basis for \(E\). If \(X\) and \(Y\) are arbitrary \(n \times n\) matrices, denoting as usual the \(j\)th column of \(X\) by \(X_j\), and similarly for \(Y\), show that

\[
X^\top Y = (X_i \cdot Y_j)_{1 \leq i, j \leq n}.
\]

Use this to prove that

\[
A^\top A = A A^\top = I_n
\]

iff the column vectors \((A_1, \ldots, A_n)\) form an orthonormal basis. Show that the conditions \(A^\top A = I_n, A A^\top = I_n,\) and \(A^{-1} = A^\top\) are equivalent.

**Problem A2.** Compute the real Fourier coefficients of the function \(id(x) = x\) over \([-\pi, \pi]\) and prove that

\[
x = 2 \left( \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right).
\]

What is the value of the Fourier series at \(\pm \pi\)? What is the value of the Fourier near \(\pm \pi\)? Do you find this surprising?

**Problem A3.** Prove Lemma 6.2.2 from my book.

“B problems” must be turned in.

**Problem B1 (30 pts).** (a) Prove that the dual \(C^*\) of the cube \(C = [-1, 1]^n\) is the convex hull of the \(2n\) points \(\{e_i, -e_i \mid 1 \leq i \leq n\}\), where \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\), the \(i\)th vector in the standard basis. The dual of a cube is called a cross-polytope. Check that the cube \(C\) has \(2^n\) vertices and \(2n\) faces, whereas its dual \(C^*\) has \(2n\) vertices and \(2^n\) faces. Draw \(C^*\) for \(n = 3\).

What is the dual of an \(n\)-simplex?
(b) Consider in \( \mathbb{R}^3 \) the polyhedron \( I \) defined as follows. If \( \tau = \frac{\sqrt{5} + 1}{2} \), then the vertices of \( I \) are the twelve points
\[
(0, \pm \tau, \pm 1), \quad (\pm 1, 0, \pm \tau), \quad (\pm \tau, \pm 1, 0).
\]
This polyhedron is called an icosahedron. Check that the icosahedron has 20 faces. Draw an icosahedron (or better, make a cardboard model).

Prove that the dual \( D \) of the icosahedron is a convex polyhedron whose twenty vertices are
\[
(\pm 1, \pm 1, \pm 1), \quad (0, \pm 1/\tau, \pm \tau), \quad (\pm \tau, 0, \pm 1/\tau), \quad (\pm 1/\tau, \pm \tau, 0).
\]
This polyhedron \( D \) is called a dodecahedron. Observe that it is “built up” on the cube \([-1, 1]^3\). Can you explain how? Check that the dodecahedron has 12 faces. Draw a dodecahedron (or better, make a cardboard model).

**Problem B2 (40 pts).** Let \( A \) be a nonempty convex subset of \( \mathbb{R}^n \). A function \( f: A \to \mathbb{R} \) is convex if
\[
f((1 - \lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b)
\]
for all \( a, b \in A \) and for all \( \lambda \in [0, 1] \).

(a) If \( f \) is convex, prove that
\[
f\left(\sum_{i \in I} \lambda_i a_i\right) \leq \sum_{i \in I} \lambda_i f(a_i)
\]
for every finite convex combination in \( A \), i.e., any finite family \( (a_i)_{i \in I} \) of points in \( A \) and any family \( (\lambda_i)_{i \in I} \) with \( \sum_{i \in I} \lambda_i = 1 \) and \( \lambda_i \geq 0 \) for all \( i \in I \).

(b) Let \( f: A \to \mathbb{R} \) be a convex function and assume that \( A \) is convex and compact and that \( f \) is continuous. Prove that \( f \) achieves its maximum in some extremal point of \( A \).

**Problem B3 (30 pts).** Let \( \varphi: E \times E \to \mathbb{R} \) be a bilinear form on a real vector space \( E \) of finite dimension \( n \). Given any basis \( (e_1, \ldots, e_n) \) of \( E \), let \( A = (\alpha_{i,j}) \) be the matrix defined such that
\[
\alpha_{i,j} = \varphi(e_i, e_j),
\]
\( 1 \leq i, j \leq n \). We call \( A \) the matrix of \( \varphi \) w.r.t. the basis \( (e_1, \ldots, e_n) \).

(a) For any two vectors \( x \) and \( y \), if \( X \) and \( Y \) denote the column vectors of coordinates of \( x \) and \( y \) w.r.t. the basis \( (e_1, \ldots, e_n) \), prove that
\[
\varphi(x, y) = X^\top A Y.
\]

(b) Recall that \( A \) is a symmetric matrix if \( A = A^\top \). Prove that \( \varphi \) is symmetric if \( A \) is a symmetric matrix.
(c) If \((f_1, \ldots, f_n)\) is another basis of \(E\) and \(P\) is the change of basis matrix from \((e_1, \ldots, e_n)\) to \((f_1, \ldots, f_n)\), prove that the matrix of \(\varphi\) w.r.t. the basis \((f_1, \ldots, f_n)\) is
\[
P^\top A P.
\]
The common rank of all matrices representing \(\varphi\) is called the rank of \(\varphi\).

**Problem B4 (100 pts).** (a) Let \(A\) be any subset of \(\mathbb{A}^n\). Prove that if \(A\) is compact, then its convex hull \(C(A)\) is also compact.

(b) Give a proof of the following version of Helly’s theorem using Corollary 1.10 of the notes on convex sets (Convex sets: A deeper look):

Given any affine space \(E\) of dimension \(m\), for every family \(\{K_1, \ldots, K_n\}\) of \(n\) convex and compact subsets of \(E\), if \(n \geq m + 2\) and the intersection \(\bigcap_{i \in I} K_i\) of any \(m + 1\) of the \(K_i\) is nonempty (where \(I \subseteq \{1, \ldots, n\}\), \(|I| = m + 1\)), then \(\bigcap_{i=1}^{n} K_i\) is nonempty.

*Hint:* First, prove that the general case can be reduced to the case where \(n = m + 2\).

(c) Use (b) to prove Helly’s theorem without the assumption that the \(K_i\) are compact.

You will need to construct some nonempty compacts \(C_i \subseteq K_i\). For this, you will need to prove that the convex hull of finitely many points is compact.

(d) Prove that Helly’s theorem holds even if the family \((K_i)_{i \in I}\) is infinite, provided that the \(K_i\) are convex and compact.

**Problem B5 (30 pts).** In \(\mathbb{E}^3\), consider the closed convex set (cone), \(A\), defined by the inequalities
\[
x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq xy,
\]
and let \(D\) be the line given by \(x = 0, z = 1\). Prove that \(D \cap A = \emptyset\), both \(A\) and \(D\) are convex and closed, yet every plane containing \(D\) meets \(A\). Therefore, \(A\) and \(D\) give another counter-example to the Hahn-Banach theorem where \(A\) is closed (one cannot relax the hypothesis that \(A\) is open).

**Problem B6 (50 pts).** (a) Let \(C\) be a circle of radius \(R\) and center \(O\), and let \(P\) be any point in the Euclidean plane \(\mathbb{E}^2\). Consider the lines \(\Delta\) through \(P\) that intersect the circle \(C\), generally in two points \(A\) and \(B\). Prove that for all such lines,
\[
PA \cdot PB = \|PO\|^2 - R^2.
\]
*Hint.* If \(P\) is not on \(C\), let \(B'\) be the antipodal of \(B\) (i.e., \(OB' = -OB\)). Then \(AB \cdot AB' = 0\) and
\[
PA \cdot PB = PB' \cdot PB = (PO - OB) \cdot (PO + OB) = \|PO\|^2 - R^2.
\]

The quantity \(\|PO\|^2 - R^2\) is called the power of \(P\) w.r.t. \(C\), and it is denoted by \(\mathcal{P}(P, C)\).
Show that if \( \Delta \) is tangent to \( C \), then \( A = B \) and
\[
\|PA\|^2 = \|PO\|^2 - R^2.
\]
Show that \( P \) is inside \( C \) iff \( \mathcal{P}(P, C) < 0 \), on \( C \) iff \( \mathcal{P}(P, C) = 0 \), outside \( C \) if \( \mathcal{P}(P, C) > 0 \).

If the equation of \( C \) is
\[
x^2 + y^2 - 2ax - 2by + c = 0,
\]
prove that the power of \( P = (x, y) \) w.r.t. \( C \) is given by
\[
\mathcal{P}(P, C) = x^2 + y^2 - 2ax - 2by + c.
\]

(b) Given two nonconcentric circles \( C \) and \( C' \), show that the set of points having equal power w.r.t. \( C \) and \( C' \) is a line orthogonal to the line through the centers of \( C \) and \( C' \). If the equations of \( C \) and \( C' \) are
\[
x^2 + y^2 - 2ax - 2by + c = 0 \quad \text{and} \quad x^2 + y^2 - 2a'x - 2b'y + c' = 0,
\]
show that the equation of this line is
\[
2(a - a')x + 2(b - b')y + c' - c = 0.
\]
This line is called the \textit{radical axis} of \( C \) and \( C' \).

(c) Given three distinct nonconcentric circles \( C \), \( C' \), and \( C'' \), prove that either the three pairwise radical axes of these circles are parallel or that they intersect in a single point \( \omega \) that has equal power w.r.t. \( C \), \( C' \), and \( C'' \). In the first case, the centers of \( C \), \( C' \), and \( C'' \) are collinear. In the second case, if the power of \( \omega \) is positive, prove that \( \omega \) is the center of a circle \( \Gamma \) orthogonal to \( C \), \( C' \), and \( C'' \), and if the power of \( \omega \) is negative, \( \omega \) is inside \( C \), \( C' \), and \( C'' \).

(d) Given any \( k \in \mathbb{R} \) with \( k \neq 0 \) and any point \( a \), recall that an \textit{inversion of pole} \( a \) and \textit{power} \( k \) is a map \( h: (\mathbb{E}^n - \{a\}) \to \mathbb{E}^n \) defined such that for every \( x \in \mathbb{E}^n - \{a\} \),
\[
h(x) = a + k \frac{ax}{\|ax\|^2}.
\]
For example, when \( n = 2 \), chosing any orthonormal frame with origin \( a \), \( h \) is defined by the map
\[
(x, y) \mapsto \left( \frac{kx}{x^2 + y^2}, \frac{ky}{x^2 + y^2} \right).
\]
When the centers of \( C \), \( C' \) and \( C'' \) are not collinear and the power of \( \omega \) is positive, prove that by a suitable inversion, \( C \), \( C' \) and \( C'' \) are mapped to three circles whose centers are collinear.

Prove that if three distinct nonconcentric circles \( C \), \( C' \), and \( C'' \) have collinear centers, then there are at most eight circles simultaneously tangent to \( C \), \( C' \), and \( C'' \), and at most two for those exterior to \( C \), \( C' \), and \( C'' \).

(e) Prove that an inversion in \( \mathbb{E}^3 \) maps a sphere to a sphere or to a plane. Prove that inversions preserve tangency and orthogonality of planes and spheres.

\textbf{TOTAL: 280 points.}