Spring, 2006 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 2

February 20, 2006; Due March 13, 2006

"A problems" are for practice only, and should not be turned in.

Problem A1. Let $(e_1, \ldots, e_n)$ be an orthonormal basis for E. If X and Y are arbitrary $n \times n$ matrices, denoting as usual the jth column of X by X_j, and similarly for Y, show that

$$X^\top Y = (X_i \cdot Y_j)_{1 \le i,j \le n}.$$

Use this to prove that

$$A^\top A = A\,A^\top = I_n$$

iff the column vectors $(A_1, \ldots, A_n)$ form an orthonormal basis. Show that the conditions $A\,A^\top = I_n$, $A^\top A = I_n$, and $A^{-1} = A^\top$ are equivalent.

Problem A2. Compute the real Fourier coefficients of the function $id(x) = x$ over $[-\pi, \pi]$ and prove that

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right).$$

What is the value of the Fourier series at $\pm\pi$? What is the value of the Fourier near $\pm\pi$? Do you find this surprising?

Problem A3. Prove Lemma 6.2.2 from my book.

"B problems" must be turned in.

Problem B1 (30 pts). (a) Prove that the dual C^* of the cube $C = [-1, 1]^n$ is the convex hull of the $2n$ points $\{e_i, -e_i \mid 1 \le i \le n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the ith vector in the standard basis. The dual of a cube is called a *cross-polytope*. Check that the cube C has 2^n vertices and $2n$ faces, whereas its dual C^* has $2n$ vertices and 2^n faces. Draw C^* for $n = 3$.

What is the dual of an n-simplex?

(b) Consider in $\mathbb{E}^3$ the polyhedron I defined as follows. If $\tau = (\sqrt{5}+1)/2$, then the vertices of I are the twelve points

$$(0,\ \pm\tau,\ \pm 1), \quad (\pm 1,\ 0,\ \pm\tau), \quad (\pm\tau,\ \pm 1,\ 0).$$

This polyhedron is called an *icosahedron*. Check that the icosahedron has 20 faces. Draw an icosahedron (or better, make a cardboard model).

Prove that the dual D of the icosahedron is a convex polyhedron whose twenty vertices are

$$(\pm 1,\ \pm 1,\ \pm 1), \quad (0,\ \pm 1/\tau,\ \pm\tau), \quad (\pm\tau,\ 0,\ \pm 1/\tau), \quad (\pm 1/\tau,\ \pm\tau,\ 0).$$

This polyhedron D is called a *dodecahedron*. Observe that it is "built up" on the cube $[-1,1]^3$. Can you explain how? Check that the dodecahedron has 12 faces. Draw a dodecahedron (or better, make a cardboard model).

Problem B2 (40 pts). Let A be a nonempty convex subset of $\mathbb{A}^n$. A function $f\colon A \to \mathbb{R}$ is *convex* if

$$f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b)$$

for all $a, b \in A$ and for all $\lambda \in [0,1]$.

(a) If f is convex, prove that

$$f\Big(\sum_{i\in I} \lambda_i a_i\Big) \leq \sum_{i\in I} \lambda_i f(a_i)$$

for every finite convex combination in A, i.e., any finite family $(a_i)_{i\in I}$ of points in A and any family $(\lambda_i)_{i\in I}$ with $\sum_{i\in I} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i \in I$.

(b) Let $f\colon A \to \mathbb{R}$ be a convex function and assume that A is convex and compact and that f is continuous. Prove that f achieves its maximum in some extremal point of A.

Problem B3 (30 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a bilinear form on a real vector space E of finite dimension n. Given any basis $(e_1, \ldots, e_n)$ of E, let $A = (\alpha_{i\,j})$ be the matrix defined such that

$$\alpha_{i\,j} = \varphi(e_i, e_j),$$

$1 \leq i, j \leq n$. We call A *the matrix of* φ *w.r.t. the basis* $(e_1, \ldots, e_n)$.

(a) For any two vectors x and y, if X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^\top A Y.$$

(b) Recall that A is a *symmetric* matrix if $A = A^\top$. Prove that φ is symmetric if A is a symmetric matrix.

(c) If $(f_1, \ldots, f_n)$ is another basis of E and P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of φ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^\top AP.$$

The common rank of all matrices representing φ is called the *rank* of φ.

Problem B4 (100 pts). (a) Let A be any subset of $\mathbb{A}^n$. Prove that if A is compact, then its convex hull $\mathcal{C}(A)$ is also compact.

(b) Give a proof of the following version of Helly's theorem using Corollary 1.10 of the notes on convex sets (Convex sets: A deeper look):

Given any affine space E of dimension m, for every family $\{K_1, \ldots, K_n\}$ of n convex and compact subsets of E, if $n \geq m+2$ and the intersection $\bigcap_{i \in I} K_i$ of any $m+1$ of the K_i is nonempty (where $I \subseteq \{1, \ldots, n\}$, $|I| = m+1$), then $\bigcap_{i=1}^n K_i$ is nonempty.

Hint: First, prove that the general case can be reduced to the case where $n = m+2$.

(c) Use (b) to prove Helly's theorem without the assumption that the K_i are compact.

You will need to construct some nonempty compacts $C_i \subseteq K_i$. For this, you will need to prove that the convex hull of finitely many points is compact.

(d) Prove that Helly's theorem holds even if the family $(K_i)_{I \in I}$ is infinite, provided that the K_i are convex and compact.

Problem B5 (30 pts). In $\mathbb{E}^3$, consider the closed convex set (cone), A, defined by the inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq xy,$$

and let D be the line given by $x = 0, z = 1$. Prove that $D \cap A = \emptyset$, both A and D are convex and closed, yet every plane containing D meets A. Therefore, A and D give another counter-example to the Hahn-Banach theorem where A is closed (one cannot relax the hypothesis that A is open).

Problem B6 (50 pts). (a) Let C be a circle of radius R and center O, and let P be any point in the Euclidean plane $\mathbb{E}^2$. Consider the lines Δ through P that intersect the circle C, generally in two points A and B. Prove that for all such lines,

$$\mathbf{PA} \cdot \mathbf{PB} = \|\mathbf{PO}\|^2 - R^2.$$

Hint. If P is not on C, let B' be the antipodal of B (i.e., $\mathbf{OB'} = -\mathbf{OB}$). Then $\mathbf{AB} \cdot \mathbf{AB'} = 0$ and

$$\mathbf{PA} \cdot \mathbf{PB} = \mathbf{PB'} \cdot \mathbf{PB} = (\mathbf{PO} - \mathbf{OB}) \cdot (\mathbf{PO} + \mathbf{OB}) = \|\mathbf{PO}\|^2 - R^2.$$

The quantity $\|\mathbf{PO}\|^2 - R^2$ is called the *power of P w.r.t. C*, and it is denoted by $\mathcal{P}(P, C)$.

Show that if Δ is tangent to C, then $A = B$ and

$$\|\mathbf{PA}\|^2 = \|\mathbf{PO}\|^2 - R^2.$$

Show that P is inside C iff $\mathcal{P}(P, C) < 0$, on C iff $\mathcal{P}(P, C) = 0$, outside C if $\mathcal{P}(P, C) > 0$.

If the equation of C is

$$x^2 + y^2 - 2ax - 2by + c = 0,$$

prove that the power of $P = (x, y)$ w.r.t. C is given by

$$\mathcal{P}(P, C) = x^2 + y^2 - 2ax - 2by + c.$$

(b) Given two nonconcentric circles C and C', show that the set of points having equal power w.r.t. C and C' is a line orthogonal to the line through the centers of C and C'. If the equations of C and C' are

$$x^2 + y^2 - 2ax - 2by + c = 0 \quad \text{and} \quad x^2 + y^2 - 2a'x - 2b'y + c' = 0,$$

show that the equation of this line is

$$2(a - a')x + 2(b - b')y + c' - c = 0.$$

This line is called the *radical axis* of C and C'.

(c) Given three distinct nonconcentric circles C, C', and C'', prove that either the three pairwise radical axes of these circles are parallel or that they intersect in a single point ω that has equal power w.r.t. C, C', and C''. In the first case, the centers of C, C', and C'' are collinear. In the second case, if the power of ω is positive, prove that ω is the center of a circle Γ orthogonal to C, C', and C'', and if the power of ω is negative, ω is inside C, C', and C''.

(d) Given any $k \in \mathbb{R}$ with $k \neq 0$ and any point a, recall that an *inversion of pole a and power k* is a map $h\colon (\mathbb{E}^n - \{a\}) \to \mathbb{E}^n$ defined such that for every $x \in \mathbb{E}^n - \{a\}$,

$$h(x) = a + k\frac{\mathbf{ax}}{\|\mathbf{ax}\|^2}.$$

For example, when $n = 2$, chosing any orthonormal frame with origin a, h is defined by the map

$$(x,\, y) \mapsto \left(\frac{kx}{x^2 + y^2},\, \frac{ky}{x^2 + y^2}\right).$$

When the centers of C, C' and C'' are not collinear and the power of ω is positive, prove that by a suitable inversion, C, C' and C'' are mapped to three circles whose centers are collinear.

Prove that if three distinct nonconcentric circles C, C', and C'' have collinear centers, then there are at most eight circles simultaneously tangent to C, C', and C'', and at most two for those exterior to C, C', and C''.

(e) Prove that an inversion in $\mathbb{E}^3$ maps a sphere to a sphere or to a plane. Prove that inversions preserve tangency and orthogonality of planes and spheres.

TOTAL: 280 points.