Spring, 2006 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 1

January 23, 2006; Due February 8, 2006

"A problems" are for practice only, and should not be turned in.

Problem A1. (a) Given a tetrahedron (a, b, c, d), given any two distinct points $x, y \in \{a, b, c, d\}$, let let $m_{x,y}$ be the middle of the edge (x, y). Prove that the barycenter g of the weighted points $(a, 1/4)$, $(b, 1/4)$, $(c, 1/4)$, and $(d, 1/4)$, is the common intersection of the line segments $(m_{a,b}, m_{c,d})$, $(m_{a,c}, m_{b,d})$, and $(m_{a,d}, m_{b,c})$. Show that if g_d is the barycenter of the weighted points $(a, 1/3), (b, 1/3), (c, 1/3)$ then g is the barycenter of $(d, 1/4)$ and $(g_d, 3/4)$.

Problem A2. Given any two affine spaces E and F, for any affine map $f\colon E \to F$, for any convex set U in E and any convex set V in F, prove that $f(U)$ is convex and that $f^{-1}(V)$ is convex. Recall that

$$f(U) = \{b \in F \mid \exists a \in U,\, b = f(a)\}$$

is the *direct image of* U *under* f, and that

$$f^{-1}(V) = \{a \in E \mid \exists b \in V,\, b = f(a)\}$$

is the *inverse image of* V *under* f.

Problem A3. Let E be a nonempty set and $\overrightarrow{E}$ be a vector space and assume that there is a function $\Phi\colon E \times E \to \overrightarrow{E}$, such that if we denote $\Phi(a, b)$ by $\mathbf{ab}$, the following properties hold:

(1) $\mathbf{ab} + \mathbf{bc} = \mathbf{ac}$, for all $a, b, c \in E$;

(2) For every $a \in E$, the map $\Phi_a\colon E \to \overrightarrow{E}$ defined such that for every $b \in E$, $\Phi_a(b) = \mathbf{ab}$, is a bijection.

Let $\Psi_a\colon \overrightarrow{E} \to E$ be the inverse of $\Phi_a\colon E \to \overrightarrow{E}$.

Prove that the function $+\colon E \times \overrightarrow{E} \to E$ defined such that

$$a + u = \Psi_a(u)$$

for all $a \in E$ and all $u \in \overrightarrow{E}$ makes $(E, \overrightarrow{E}, +)$ into an affine space.

Note: We showed in class that an affine space $(E, \overrightarrow{E}, +)$ satisfies the properties stated above. Thus, we obtain an equivalent characterization of affine spaces.

"B problems" must be turned in.

Problem B1 (30 pts). Given any two distinct points a, b in $\mathbb{A}^2$ of barycentric coordinates (a_0, a_1, a_2) and (b_0, b_1, b_2) with respect to any given affine frame, show that the equation of the line $\langle a, b\rangle$ determined by a and b is

$$\begin{vmatrix} a_0 & b_0 & x \\ a_1 & b_1 & y \\ a_2 & b_2 & z \end{vmatrix} = 0,$$

or equivalently

$$(a_1b_2 - a_2b_1)x + (a_2b_0 - a_0b_2)y + (a_0b_1 - a_1b_0)z = 0,$$

where (x, y, z) are the barycentric coordinates of the generic point on the line $\langle a, b\rangle$.

Prove that the equation of a line in barycentric coordinates is of the form

$$ux + vy + wz = 0,$$

where $u \neq v$, or $v \neq w$, or $u \neq w$. Show that two equations

$$ux + vy + wz = 0 \quad \text{and} \quad u'x + v'y + w'z = 0$$

represent the same line in barycentric coordinates iff $(u', v', w') = \lambda(u, v, w)$ for some $\lambda \in \mathbb{R}$ (with $\lambda \neq 0$).

A triple (u, v, w) where $u \neq v$, or $v \neq w$, or $u \neq w$, is called a system of *tangential coordinates* of the line defined by the equation

$$ux + vy + wz = 0.$$

Problem B2 (30 pts). Given two lines D and D' in $\mathbb{A}^2$ defined by tangential coordinates (u, v, w) and (u', v', w') (as defined in problem B1), let

$$d = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ 1 & 1 & 1 \end{vmatrix} = vw' - wv' + wu' - uw' + uv' - vu'.$$

(a) Prove that D and D' have a unique intersection point iff $d \neq 0$, and that when it exists, the barycentric coordinates of this intersection point are

$$\frac{1}{d}(vw' - wv',\, wu' - uw',\, uv' - vu').$$

(b) Letting (O, i, j) be any affine frame for $\mathbb{A}^2$, recall that when $x+y+z=0$, for any point a, the vector

$$x\mathbf{aO} + y\mathbf{ai} + z\mathbf{aj}$$

is independent of a and equal to

$$y\mathbf{Oi} + z\mathbf{Oj} = (y, z).$$

The triple (x, y, z) such that $x+y+z=0$ is called the *barycentric coordinates* of the vector $y\mathbf{Oi} + z\mathbf{Oj}$ w.r.t. the affine frame (O, i, j).

Given any affine frame (O, i, j), prove that for $u \neq v$, or $v \neq w$, or $u \neq w$, the line of equation

$$ux + vy + wz = 0$$

in barycentric coordinates (x, y, z) (where $x+y+z=1$) has for direction the set of vectors of barycentric coordinates (x, y, z) such that

$$ux + vy + wz = 0$$

(where $x+y+z=0$).

Prove that D and D' are parallel iff $d=0$. In this case, if $D \neq D'$, show that the common direction of D and D' is defined by the vector of barycentric coordinates

$$(vw' - wv',\, wu' - uw',\, uv' - vu').$$

(c) Given three lines D, D', and D'', at least two of which are distinct, and defined by tangential coordinates (u, v, w), (u', v', w'), and (u'', v'', w''), prove that D, D', and D'' are parallel or have a unique intersection point iff

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = 0.$$

Problem B3 (40 pts). This problem uses notions and results from Problems B1 and B2. In view of (a) and (b) of Problem B2, it is natural to extend the notion of barycentric coordinates of a point in $\mathbb{A}^2$ as follows. Given any affine frame (a, b, c) in $\mathbb{A}^2$, we will say that the barycentric coordinates (x, y, z) of a point M, where $x+y+z=1$, are the *normalized barycentric coordinates* of M. Then, any triple (x, y, z) such that $x+y+z \neq 0$ is also called a system of barycentric coordinates for the point of normalized barycentric coordinates

$$\frac{1}{x+y+z}\,(x, y, z).$$

With this convention, the intersection of the two lines D and D' is either a point or a vector, in both cases of barycentric coordinates

$$(vw' - wv',\ wu' - uw',\ uv' - vu').$$

When the above is a vector, we can think of it as a point at infinity (in the direction of the line defined by that vector).

Let (D_0, D_0'), (D_1, D_1'), and (D_2, D_2') be three pairs of six distinct lines, such that the four lines belonging to any union of two of the above pairs are neither parallel nor concurrent (have a common intersection point). If D_0 and D_0' have a unique intersection point, let M be this point, and if D_0 and D_0' are parallel, let M denote a nonnull vector defining the common direction of D_0 and D_0'. In either case, let (m, m', m'') be the barycentric coordinates of M, as explained at the beginning of the problem. We call M the *intersection* of D_0 and D_0'. Similarly, define $N = (n, n', n'')$ as the intersection of D_1 and D_1', and $P = (p, p', p'')$ as the intersection of D_2 and D_2'.

Prove that

$$\begin{vmatrix} m & n & p \\ m' & n' & p' \\ m'' & n'' & p'' \end{vmatrix} = 0$$

iff either

(i) (D_0, D_0'), (D_1, D_1'), and (D_2, D_2') are pairs of parallel lines; or

(ii) the lines of some pair (D_i, D_i') are parallel, each pair (D_j, D_j') (with $j \neq i$) has a unique intersection point, and these two intersection points are distinct and determine a line parallel to the lines of the pair (D_i, D_i'); or

(iii) each pair (D_i, D_i') $(i = 0, 1, 2)$ has a unique intersection point, and these points M, N, P are distinct and collinear.

Problem B4 (40 pts). The purpose of this problem is to prove *Pascal's Theorem* for the nondegenerate conics. In the affine plane $\mathbb{A}^2$, a *conic* is the set of points of coordinates (x, y) such that

$$\alpha x^2 + \beta y^2 + 2\gamma xy + 2\delta x + 2\lambda y + \mu = 0,$$

where $\alpha \neq 0$ or $\beta \neq 0$ or $\gamma \neq 0$. We can write the equation of the conic as

$$(x, y, 1) \begin{pmatrix} \alpha & \gamma & \delta \\ \gamma & \beta & \lambda \\ \delta & \lambda & \mu \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

If we now use barycentric coordinates (x, y, z) (where $x + y + z = 1$), we can write

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} \alpha & \gamma & \delta \\ \gamma & \beta & \lambda \\ \delta & \lambda & \mu \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) Letting $A = C^\top BC$, prove that the equation of the conic becomes

$$X^\top AX = 0.$$

Prove that A is symmetric, that $\det(A) = \det(B)$, and that $X^\top AX$ is homogeneous of degree 2. The equation $X^\top AX = 0$ is called the *homogeneous equation* of the conic.

We say that a conic of homogeneous equation $X^\top AX = 0$ is *nondegenerate* if $\det(A) \neq 0$, and *degenerate* if $\det(A) = 0$. Show that this condition does not depend on the choice of the affine frame.

(b) Given an affine frame (A, B, C), prove that any conic passing through A, B, C has an equation of the form

$$ayz + bxz + cxy = 0.$$

Prove that a conic containing more than one point is degenerate iff it contains three distinct collinear points. In this case, the conic is the union of two lines.

(c) Prove *Pascal's Theorem*. Given any six distinct points A, B, C, A', B', C', if no three of the above points are collinear, then a nondegenerate conic passes through these six points iff the intersection points M, N, P (in the sense of Problem B2) of the pairs of lines (BC', CB'), (CA', AC') and (AB', BA') are collinear in the sense of Problem B3.

Hint. Use the affine frame (A, B, C), and let (a, a', a''), (b, b', b''), and (c, c', c'') be the barycentric coordinates of A', B', C' respectively, and show that M, N, P have barycentric coordinates

$$(bc, cb', c''b), \quad (c'a, c'a', c''a'), \quad (ab'', a''b', a''b'').$$

Problem B5 (20 pts). (a) Let E be an affine space over $\mathbb{R}$, and let $(a_1, \ldots, a_n)$ be any $n \geq 3$ points in E. Let $(\lambda_1, \ldots, \lambda_n)$ be any n scalars in $\mathbb{R}$, with $\lambda_1 + \cdots + \lambda_n = 1$. Show that there must be some i, $1 \leq i \leq n$, such that $\lambda_i \neq 1$. To simplify the notation, assume that $\lambda_1 \neq 1$. Show that the barycenter $\lambda_1 a_1 + \cdots + \lambda_n a_n$ can be obtained by first determining the barycenter b of the $n-1$ points $a_2, \ldots, a_n$ assigned some appropriate weights, and then the barycenter of a_1 and b assigned the weights λ_1 and $\lambda_2 + \cdots + \lambda_n$. From this, show that the barycenter of any $n \geq 3$ points can be determined by repeated computations of barycenters of two points. Deduce from the above that a nonempty subset V of E is an affine subspace iff whenever V contains any two points $x, y \in V$, then V contains the entire line $(1-\lambda)x + \lambda y$, $\lambda \in \mathbb{R}$.

(b) Assume that K is a field such that $2 = 1 + 1 \neq 0$, and let E be an affine space over K. In the case where $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i = 1$, for $1 \leq i \leq n$ and $n \geq 3$, show that the barycenter $a_1 + a_2 + \cdots + a_n$ can still be computed by repeated computations of barycenters of two points.

Finally, assume that the field K contains at least three elements (thus, there is some $\mu \in K$ such that $\mu \neq 0$ and $\mu \neq 1$, but $2 = 1 + 1 = 0$ is possible). Prove that the barycenter of any $n \geq 3$ points can be determined by repeated computations of barycenters of two points. Prove that a nonempty subset V of E is an affine subspace iff whenever V contains any two points $x, y \in V$, then V contains the entire line $(1-\lambda)x + \lambda y$, $\lambda \in K$.

Hint. When $2 = 0$, $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i = 1$, for $1 \leq i \leq n$, show that n must be odd, and that the problem reduces to computing the barycenter of three points in two steps involving two barycenters. Since there is some $\mu \in K$ such that $\mu \neq 0$ and $\mu \neq 1$, note that μ^{-1} and $(1-\mu)^{-1}$ both exist, and use the fact that

$$\frac{-\mu}{1-\mu} + \frac{1}{1-\mu} = 1.$$

Problem B6 (30 pts). (i) Let (a, b, c) be three points in $\mathbb{A}^2$, and assume that (a, b, c) are not collinear. For any point $x \in \mathbb{A}^2$, if $x = \lambda_0 a + \lambda_1 b + \lambda_2 c$, where $(\lambda_0, \lambda_1, \lambda_2)$ are the barycentric coordinates of x with respect to (a, b, c), show that

$$\lambda_0 = \frac{\det(\mathbf{xb}, \mathbf{bc})}{\det(\mathbf{ab}, \mathbf{ac})}, \qquad \lambda_1 = \frac{\det(\mathbf{ax}, \mathbf{ac})}{\det(\mathbf{ab}, \mathbf{ac})}, \qquad \lambda_2 = \frac{\det(\mathbf{ab}, \mathbf{ax})}{\det(\mathbf{ab}, \mathbf{ac})}.$$

Conclude that $\lambda_0, \lambda_1, \lambda_2$ are certain signed ratios of the areas of the triangles (a, b, c), (x, a, b), (x, a, c), and (x, b, c).

(ii) Let (a, b, c) be three points in $\mathbb{A}^3$, and assume that (a, b, c) are not collinear. For any point x in the plane determined by (a, b, c), if $x = \lambda_0 a + \lambda_1 b + \lambda_2 c$, where $(\lambda_0, \lambda_1, \lambda_2)$ are the barycentric coordinates of x with respect to (a, b, c), show that

$$\lambda_0 = \frac{\mathbf{xb} \times \mathbf{bc}}{\mathbf{ab} \times \mathbf{ac}}, \qquad \lambda_1 = \frac{\mathbf{ax} \times \mathbf{ac}}{\mathbf{ab} \times \mathbf{ac}}, \qquad \lambda_2 = \frac{\mathbf{ab} \times \mathbf{ax}}{\mathbf{ab} \times \mathbf{ac}}.$$

Given any point O not in the plane of the triangle (a, b, c), prove that

$$\lambda_1 = \frac{\det(\mathbf{Oa}, \mathbf{Ox}, \mathbf{Oc})}{\det(\mathbf{Oa}, \mathbf{Ob}, \mathbf{Oc})}, \quad \lambda_2 = \frac{\det(\mathbf{Oa}, \mathbf{Ob}, \mathbf{Ox})}{\det(\mathbf{Oa}, \mathbf{Ob}, \mathbf{Oc})},$$

and

$$\lambda_0 = \frac{\det(\mathbf{Ox}, \mathbf{Ob}, \mathbf{Oc})}{\det(\mathbf{Oa}, \mathbf{Ob}, \mathbf{Oc})}.$$

(iii) Let (a, b, c, d) be four points in $\mathbb{A}^3$, and assume that (a, b, c, d) are not coplanar. For any point $x \in \mathbb{A}^3$, if $x = \lambda_0 a + \lambda_1 b + \lambda_2 c + \lambda_3 d$, where $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of x with respect to (a, b, c, d), show that

$$\lambda_1 = \frac{\det(\mathbf{ax}, \mathbf{ac}, \mathbf{ad})}{\det(\mathbf{ab}, \mathbf{ac}, \mathbf{ad})}, \quad \lambda_2 = \frac{\det(\mathbf{ab}, \mathbf{ax}, \mathbf{ad})}{\det(\mathbf{ab}, \mathbf{ac}, \mathbf{ad})}, \quad \lambda_3 = \frac{\det(\mathbf{ab}, \mathbf{ac}, \mathbf{ax})}{\det(\mathbf{ab}, \mathbf{ac}, \mathbf{ad})},$$

and

$$\lambda_0 = \frac{\det(\mathbf{xb}, \mathbf{bc}, \mathbf{bd})}{\det(\mathbf{ab}, \mathbf{ac}, \mathbf{ad})}.$$

Conclude that $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are certain signed ratios of the volumes of the five tetrahedra (a, b, c, d), (x, a, b, c), (x, a, b, d), (x, a, c, d), and (x, b, c, d).

(iv) Let $(a_0, \ldots, a_m)$ be $m+1$ points in $\mathbb{A}^m$, and assume that they are affinely independent. For any point $x \in \mathbb{A}^m$, if $x = \lambda_0 a_0 + \cdots + \lambda_m a_m$, where $(\lambda_0, \ldots, \lambda_m)$ are the barycentric coordinates of x with respect to $(a_0, \ldots, a_m)$, show that

$$\lambda_i = \frac{\det(\mathbf{a_0a_1}, \ldots, \mathbf{a_0a_{i-1}}, \mathbf{a_0x}, \mathbf{a_0a_{i+1}}, \ldots, \mathbf{a_0a_m})}{\det(\mathbf{a_0a_1}, \ldots, \mathbf{a_0a_{i-1}}, \mathbf{a_0a_i}, \mathbf{a_0a_{i+1}}, \ldots, \mathbf{a_0a_m})}$$

for every i, $1 \leq i \leq m$, and

$$\lambda_0 = \frac{\det(\mathbf{xa_1}, \mathbf{a_1a_2}, \ldots, \mathbf{a_1a_m})}{\det(\mathbf{a_0a_1}, \ldots, \mathbf{a_0a_i}, \ldots, \mathbf{a_0a_m})}.$$

Conclude that λ_i is the signed ratio of the volumes of the simplexes $(a_0, \ldots, x, \ldots a_m)$ and $(a_0, \ldots, a_i, \ldots a_m)$, where $0 \leq i \leq m$.

Problem B7 (20 pts). Let S be any nonempty subset of an affine space E. Given some point $a \in S$, we say that S is *star-shaped with respect to* a iff the line segment $[a, x]$ is contained in S for every $x \in S$, i.e. $(1 - \lambda)a + \lambda x \in S$ for all λ such that $0 \leq \lambda \leq 1$. We say that S is *star-shaped* iff it is star-shaped w.r.t. to some point $a \in S$.

(1) Prove that every nonempty convex set is star-shaped.

(2) Show that there are star-shaped subsets that are not convex. Show that there are nonempty subsets that are not star-shaped (give an example in $\mathbb{A}^n$, $n = 1, 2, 3$).

(3) Given a star-shaped subset S of E, let $N(S)$ be the set of all points $a \in S$ such that S is star-shaped with respect to a. Prove that $N(S)$ is convex.

Problem B8 (50 pts). (a) Let E be a vector space, and let U and V be two subspaces of E so that they form a direct sum $E = U \oplus V$. Recall that this means that every vector $x \in E$ can be written as $x = u + v$, for some unique $u \in U$ and some unique $v \in V$. Define the function $p_U \colon E \to U$ (resp. $p_V \colon E \to V$) so that $p_U(x) = u$ (resp. $p_V(x) = v$), where $x = u + v$, as explained above. Check that that p_U and p_V are linear.

(b) Now assume that E is an affine space (nontrivial), and let U and V be affine subspaces such that $\overrightarrow{E} = \overrightarrow{U} \oplus \overrightarrow{V}$. Pick any $\Omega \in V$, and define $q_U \colon E \to \overrightarrow{U}$ (resp. $q_V \colon E \to \overrightarrow{V}$, with $\Omega \in U$) so that

$$q_U(a) = p_{\overrightarrow{U}}(\mathbf{\Omega a}) \quad \text{(resp.} \quad q_V(a) = p_{\overrightarrow{V}}(\mathbf{\Omega a})), \quad \text{for every } a \in E.$$

Prove that q_U does not depend on the choice of $\Omega \in V$ (resp. q_V does not depend on the choice of $\Omega \in U$). Define the map $p_U\colon E \to U$ (resp. $p_V\colon E \to V$) so that

$$p_U(a) = a - q_V(a) \quad (\text{resp.} \quad p_V(a) = a - q_U(a)), \quad \text{for every } a \in E.$$

Prove that p_U (resp. p_V) is affine.

The map p_U (resp. p_V) is called the *projection onto U parallel to V (resp. projection onto V parallel to U).*

(c) Let $(a_0, \ldots, a_n)$ be $n+1$ affinely independent points in $\mathbb{A}^n$, and let $\Delta(a_0, \ldots, a_n)$ denote the convex hull of $(a_0, \ldots, a_n)$ (an n-simplex). Prove that if $f\colon \mathbb{A}^n \to \mathbb{A}^n$ is an affine map sending $\Delta(a_0, \ldots, a_n)$ inside itself, i.e.,

$$f(\Delta(a_0, \ldots, a_n)) \subseteq \Delta(a_0, \ldots, a_n),$$

then, f has some fixed point $b \in \Delta(a_0, \ldots, a_n)$, i.e.,

$$f(b) = b.$$

Hint: Proceed by induction on n. First, treat the case $n = 1$. The affine map is determined by $f(a_0)$ and $f(a_1)$, which are affine combinations of a_0 and a_1. There is an explicit formula for some fixed point of f. For the induction step, compose f with some suitable projections.

TOTAL: 260 points.