Chapter 5

Review of Groups and Group Actions

5.1 Groups

Definition 5.1.1 A *group* is a set, G, equipped with an operation, $\cdot\colon G \times G \to G$, having the following properties: $\cdot$ is *associative*, has an *identity element*, $e \in G$, and every element in G is *invertible* (w.r.t. $\cdot$). More explicitly, this means that the following equations hold for all $a, b, c \in G$:

(G1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$ (associativity);

(G2) $a \cdot e = e \cdot a = a.$ (identity);

(G3) For every $a \in G$, there is some $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (inverse).

A group G is *abelian* (or *commutative*) if

$$a \cdot b = b \cdot a$$

for all $a, b \in G$.

Observe that a group is never empty, since $e \in G$.

Some examples of groups are given below:

Example 5.1

1. The set $\mathbb{Z} = \{\ldots, -n, \ldots, -1, 0, 1, \ldots, n \ldots\}$ of integers is a group under addition, with identity element 0. However, $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ is not a group under multiplication.

2. The set $\mathbb{Q}$ of rational numbers is a group under addition, with identity element 0. The set $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ is also a group under multiplication, with identity element 1.

3. Similarly, the sets $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers are groups under addition (with identity element 0), and $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$ are groups under multiplication (with identity element 1).

4. The sets $\mathbb{R}^n$ and $\mathbb{C}^n$ of n-tuples of real or complex numbers are groups under componentwise addition:

$$(x_1, \ldots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_n, \ldots, x_n + y_n),$$

with identity element $(0, \ldots, 0)$. All these groups are abelian.

5. Given any nonempty set S, the set of bijections $f\colon S \to S$, also called *permutations of* S, is a group under function composition (i.e., the multiplication of f and g is the composition $g \circ f$), with identity element the identity function id_S. This group is not abelian as soon as S has more than two elements.

6. The set of $n \times n$ matrices with real (or complex) coefficients is a group under addition of matrices, with identity element the null matrix. It is denoted by $\mathrm{M}_n(\mathbb{R})$ (or $\mathrm{M}_n(\mathbb{C})$).

7. The set $\mathbb{R}[X]$ of polynomials in one variable with real coefficients is a group under addition of polynomials.

8. The set of $n \times n$ invertible matrices with real (or complex) coefficients is a group under matrix multiplication, with identity element the identity matrix I_n. This group is called the *general linear group* and is usually denoted by $\mathbf{GL}(n, \mathbb{R})$ (or $\mathbf{GL}(n, \mathbb{C})$).

9. The set of $n \times n$ invertible matrices with real (or complex) coefficients and determinant $+1$ is a group under matrix multiplication, with identity element the identity matrix I_n. This group is called the *special linear group* and is usually denoted by $\mathbf{SL}(n, \mathbb{R})$ (or $\mathbf{SL}(n, \mathbb{C})$).

10. The set of $n \times n$ invertible matrices with real coefficients such that $RR^\top = I_n$ and of determinant $+1$ is a group called the *orthogonal group* and is usually denoted by $\mathbf{SO}(n)$ (where $R^\top$ is the *transpose* of the matrix R, i.e., the rows of $R^\top$ are the columns of R). It corresponds to the rotations in $\mathbb{R}^n$.

11. Given an open interval $]a, b[$, the set $C(]a, b[)$ of continuous functions $f\colon]a, b[\to \mathbb{R}$ is a group under the operation $f + g$ defined such that
$$(f + g)(x) = f(x) + g(x)$$
for all $x \in]a, b[$.

Given a group, G, for any two subsets $R, S \subseteq G$, we let
$$RS = \{r \cdot s \mid r \in R,\ s \in S\}.$$
In particular, for any $g \in G$, if $R = \{g\}$, we write
$$gS = \{g \cdot s \mid s \in S\}$$
and similarly, if $S = \{g\}$, we write
$$Rg = \{r \cdot g \mid r \in R\}.$$

From now on, we will drop the multiplication sign and write $g_1 g_2$ for $g_1 \cdot g_2$.

Definition 5.1.2 Given a group, G, a subset, H, of G is a *subgroup of* G iff

(1) The identity element, e, of G also belongs to H ($e \in H$);

(2) For all $h_1, h_2 \in H$, we have $h_1 h_2 \in H$;

(3) For all $h \in H$, we have $h^{-1} \in H$.

It is easily checked that a subset, $H \subseteq G$, is a subgroup of G iff H is nonempty and whenever $h_1, h_2 \in H$, then $h_1 h_2^{-1} \in H$.

If H is a subgroup of G and $g \in G$ is any element, the sets of the form gH are called *left cosets of* H *in* G and the sets of the form Hg are called *right cosets of* H *in* G.

The left cosets (resp. right cosets) of H induce an equivalence relation, $\sim$, defined as follows: For all $g_1, g_2 \in G$,

$$g_1 \sim g_2 \quad \text{iff} \quad g_1 H = g_2 H$$

(resp. $g_1 \sim g_2$ iff $Hg_1 = Hg_2$).

Obviously, $\sim$ is an equivalence relation. Now, it is easy to see that $g_1 H = g_2 H$ iff $g_2^{-1} g_1 \in H$, so the equivalence class of an element $g \in G$ is the coset gH (resp. Hg).

The set of left cosets of H in G (which, in general, is **not** a group) is denoted G/H. The "points" of G/H are obtained by "collapsing" all the elements in a coset into a single element.

It is tempting to define a multiplication operation on left cosets (or right cosets) by setting

$$(g_1H)(g_2H) = (g_1g_2)H,$$

but this operation is not well defined in general, unless the subgroup H possesses a special property.

This property is typical of the kernels of group homomorphisms, so we are led to

Definition 5.1.3 Given any two groups, G, G', a function $\varphi\colon G \to G'$ is a *homomorphism* iff

$$\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2), \quad \text{for all } g_1, g_2 \in G.$$

Taking $g_1 = g_2 = e$ (in G), we see that

$$\varphi(e) = e',$$

and taking $g_1 = g$ and $g_2 = g^{-1}$, we see that

$$\varphi(g^{-1}) = \varphi(g)^{-1}.$$

If $\varphi\colon G \to G'$ and $\psi\colon G' \to G''$ are group homomorphisms, then $\psi \circ \varphi\colon G \to G''$ is also a homomorphism.

If $\varphi\colon G \to G'$ is a homomorphism of groups and $H \subseteq G$ and $H' \subseteq G'$ are two subgroups, then it is easily checked that

$$\text{Im } H = \varphi(H) = \{\varphi(g) \mid g \in H\} \quad \text{is a subgroup of } G'$$

(Im H is called the *image of H by φ*) and

$$\varphi^{-1}(H') = \{g \in G \mid \varphi(g) \in H'\} \quad \text{is a subgroup of } G.$$

In particular, when $H' = \{e'\}$, we obtain the *kernel*, Ker φ, of φ. Thus,

$$\text{Ker } \varphi = \{g \in G \mid \varphi(g) = e'\}.$$

It is immediately verified that $\varphi\colon G \to G'$ is injective iff Ker $\varphi = \{e\}$. (We also write Ker $\varphi = (0)$.)

We say that φ is an *isomorphism* if there is a homomorphism, $\psi: G' \to G$, so that

$$\psi \circ \varphi = \mathrm{id}_G \quad \text{and} \quad \varphi \circ \psi = \mathrm{id}_{G'}.$$

In this case, ψ is unique and it is denoted φ^{-1}.

When φ is an isomorphism we say the the groups G and G' are *isomorphic*. When $G' = G$, a group isomorphism is called an *automorphism*.

We claim that $H = \mathrm{Ker}\, \varphi$ satisfies the following property:

$$gH = Hg, \quad \text{for all } g \in G. \tag{$*$}$$

First, note that $(*)$ is equivalent to

$$gHg^{-1} = H, \quad \text{for all } g \in G,$$

and the above is equivalent to

$$gHg^{-1} \subseteq H, \quad \text{for all } g \in G. \tag{$**$}$$

Definition 5.1.4 For any group, G, a subgroup, $N \subseteq G$, is a *normal subgroup* of G iff

$$gNg^{-1} = N, \quad \text{for all } g \in G.$$

This is denoted by $N \lhd G$.

If N is a normal subgroup of G, the equivalence relation induced by left cosets is the same as the equivalence induced by right cosets.

Furthermore, this equivalence relation, $\sim$, is a *congruence*, which means that: For all $g_1, g_2, g_1', g_2' \in G$,

(1) If $g_1N = g_1'N$ and $g_2N = g_2'N$, then $g_1g_2N = g_1'g_2'N$, and

(2) If $g_1N = g_2N$, then $g_1^{-1}N = g_2^{-1}N$.

As a consequence, we can define a group structure on the set $G/\sim$ of equivalence classes modulo $\sim$, by setting

$$(g_1N)(g_2N) = (g_1g_2)N.$$

This group is denoted G/N. The equivalence class, gN, of an element $g \in G$ is also denoted $\overline{g}$. The map $\pi: G \to G/N$, given by

$$\pi(g) = \overline{g} = gN,$$

is clearly a group homomorphism called the *canonical projection*.

Given a homomorphism of groups, $\varphi: G \to G'$, we easily check that the groups $G/\mathrm{Ker}\ \varphi$ and $\mathrm{Im}\ \varphi = \varphi(G)$ are isomorphic.

5.2 Group Actions and Homogeneous Spaces, I

If X is a set (usually, some kind of geometric space, for example, the sphere in $\mathbb{R}^3$, the upper half-plane, etc.), the "symmetries" of X are often captured by the action of a group, G, on X.

In fact, if G is a Lie group and the action satisfies some simple properties, the set X can be given a manifold structure which makes it a projection (quotient) of G, a so-called "homogeneous space".

Definition 5.2.1 Given a set, X, and a group, G, a *left action of G on X* (for short, an *action of G on X*) is a function, $\varphi\colon G \times X \to X$, such that

(1) For all $g, h \in G$ and all $x \in X$,

$$\varphi(g, \varphi(h, x)) = \varphi(gh, x),$$

(2) For all $x \in X$,

$$\varphi(1, x) = x,$$

where $1 \in G$ is the identity element of G.

To alleviate the notation, we usually write $g \cdot x$ or even gx for $\varphi(g, x)$, in which case, the above axioms read:

(1) For all $g, h \in G$ and all $x \in X$,

$$g \cdot (h \cdot x) = gh \cdot x,$$

(2) For all $x \in X$,

$$1 \cdot x = x.$$

The set X is called a *(left) G-set*.

The action φ is *faithful* or *effective* iff for any g, if $g \cdot x = x$ for all $x \in X$, then $g = 1$; the action φ is *transitive* iff for any two elements $x, y \in X$, there is some $g \in G$ so that $g \cdot x = y$.

Given an action, $\varphi\colon G \times X \to X$, for every $g \in G$, we have a function, $\varphi_g\colon X \to X$, defined by

$$\varphi_g(x) = g \cdot x, \quad \text{for all } x \in X.$$

Observe that φ_g has $\varphi_{g^{-1}}$ as inverse, since

$$\begin{aligned}\varphi_{g^{-1}}(\varphi_g(x)) = \varphi_{g^{-1}}(g \cdot x) &= g^{-1} \cdot (g \cdot x) \\ &= (g^{-1}g) \cdot x \\ &= 1 \cdot x = x,\end{aligned}$$

and similarly, $\varphi_g \circ \varphi_{g^{-1}} = \mathrm{id}$.

Therefore, φ is a bijection of X, i.e., a permutation of X.

Moreover, we check immediately that

$$\varphi_g \circ \varphi_h = \varphi_{gh},$$

so, the map $g \mapsto \varphi_g$ is a group homomorphism from G to $\mathfrak{S}_X$, the group of permutations of X. With a slight abuse of notation, this group homomorphism $G \longrightarrow \mathfrak{S}_X$ is also denoted φ.

Conversely, it is easy to see that any group homomorphism, $\varphi\colon G \to \mathfrak{S}_X$, yields a group action, $\cdot\colon G \times X \longrightarrow X$, by setting

$$g \cdot x = \varphi(g)(x).$$

Observe that an action, φ, is faithful iff the group homomorphism, $\varphi\colon G \to \mathfrak{S}_X$, is injective. Also, we have $g \cdot x = y$ iff $g^{-1} \cdot y = x$.

Definition 5.2.2 Given two G-sets, X and Y, a function, $f\colon X \to Y$, is said to be *equivariant*, or a *G-map* iff for all $x \in X$ and all $g \in G$, we have

$$f(g \cdot x) = g \cdot f(x).$$

Remark: We can also define a *right action*,

$\cdot\colon X \times G \to X$, of a group G on a set X, as a map satisfying the conditions

(1) For all $g, h \in G$ and all $x \in X$,

$$(x \cdot g) \cdot h = x \cdot gh,$$

(2) For all $x \in X$,

$$x \cdot 1 = x.$$

Every notion defined for left actions is also defined for right actions, in the obvious way.

Here are some examples of (left) group actions.

Example 1: The unit sphere S^2 (more generally, S^{n-1}).

Recall that for any $n \geq 1$, the *(real) unit sphere*, S^{n-1}, is the set of points in $\mathbb{R}^n$ given by

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

In particular, S^2 is the usual sphere in $\mathbb{R}^3$.

Since the group $\mathbf{SO}(3) = \mathbf{SO}(3, \mathbb{R})$ consists of (orientation preserving) linear isometries, i.e., *linear* maps that are distance preserving (and of determinant $+1$), and every linear map leaves the origin fixed, we see that any rotation maps S^2 into itself.

Beware that this would be false if we considered the group of *affine* isometries, $\mathbf{SE}(3)$, of $\mathbb{E}^3$. For example, a screw motion does *not* map S^2 into itself, even though it is distance preserving, because the origin is translated.

Thus, we have an action, $\cdot\colon \mathbf{SO}(3) \times S^2 \to S^2$, given by

$$R \cdot x = Rx.$$

The verification that the above is indeed an action is trivial. This action is transitive.

Similarly, for any $n \geq 1$, we get an action, $\cdot\colon \mathbf{SO}(n) \times S^{n-1} \to S^{n-1}$. It is easy to show that this action is transitive.

Analogously, we can define the *(complex) unit sphere*, Σ^{n-1}, as the set of points in $\mathbb{C}^n$ given by

$$\Sigma^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1\overline{z}_1 + \cdots + z_n\overline{z}_n = 1\}.$$

If we write $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$, then

$$\begin{aligned}\Sigma^{n-1} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \\ x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 = 1\}.\end{aligned}$$

Therefore, we can view the complex sphere, Σ^{n-1} (in $\mathbb{C}^n$), as the real sphere, S^{2n-1} (in $\mathbb{R}^{2n}$).

By analogy with the real case, we can define an action, $\cdot\colon \mathbf{SU}(n) \times \Sigma^{n-1} \to \Sigma^{n-1}$, of the group, $\mathbf{SU}(n)$, of *linear* maps of $\mathbb{C}^n$ preserving the hermitian inner product (and the origin, as all linear maps do) and this action is transitive.

One should not confuse the unit sphere, Σ^{n-1}, with the hypersurface, $S^{n-1}_{\mathbb{C}}$, given by

$$S^{n-1}_{\mathbb{C}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 = 1\}.$$

For instance, one should check that a line, L, through the origin intersects Σ^{n-1} in a circle, whereas it intersects $S^{n-1}_{\mathbb{C}}$ in exactly two points!

Example 2: The upper half-plane.

The *upper half-plane*, H, is the open subset of $\mathbb{R}^2$ consisting of all points, $(x, y) \in \mathbb{R}^2$, with $y > 0$.

It is convenient to identify H with the set of complex numbers, $z \in \mathbb{C}$, such that $\Im z > 0$. Then, we can define an action, $\cdot\colon \mathbf{SL}(2, \mathbb{R}) \times H \to H$, as follows: For any $z \in H$, for any $A \in \mathbf{SL}(2, \mathbb{R})$,

$$A \cdot z = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$.

It is easily verified that $A \cdot z$ is indeed always well defined and in H when $z \in H$. This action is transitive (check this).

Maps of the form
$$z \mapsto \frac{az+b}{cz+d},$$
where $z \in \mathbb{C}$ and $ad - bc = 1$, are called *Möbius transformations*.

Here, $a, b, c, d \in \mathbb{R}$, but in general, we allow $a, b, c, d \in \mathbb{C}$. Actually, these transformations are not necessarily defined everywhere on $\mathbb{C}$, for example, for $z = -d/c$ if $c \neq 0$.

To fix this problem, we add a "point at infinity", ∞, to $\mathbb{C}$ and define Möbius transformations as functions $\mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$. If $c = 0$, the Möbius transformation sends ∞ to itself, otherwise, $-d/c \mapsto \infty$ and $\infty \mapsto a/c$.

The space $\mathbb{C} \cup \{\infty\}$ can be viewed as the plane, $\mathbb{R}^2$, extended with a point at infinity. Using a stereographic projection from the sphere S^2 to the plane, (say from the north pole to the equatorial plane), we see that there is a bijection between the sphere, S^2, and $\mathbb{C} \cup \{\infty\}$.

More precisely, the *stereographic projection* of the sphere S^2 from the north pole, $N = (0, 0, 1)$, to the plane $z = 0$ (extended with the point at infinity, ∞) is given by

$$(x, y, z) \in S^2 - \{(0, 0, 1)\} \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \frac{x + iy}{1 - z},$$

with $(0, 0, 1) \mapsto \infty$.

The inverse stereographic projection is given by

$$(x, y) \mapsto \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right),$$

with $\infty \mapsto (0, 0, 1)$.

Intuitively, the inverse stereographic projection "wraps" the equatorial plane around the sphere. The space $\mathbb{C} \cup \{\infty\}$ is known as the *Riemann sphere*.

We will see shortly that $\mathbb{C} \cup \{\infty\} \cong S^2$ is also the complex projective line, $\mathbb{CP}^1$.

In summary, Möbius transformations are bijections of the Riemann sphere. It is easy to check that these transformations form a group under composition for all $a, b, c, d \in \mathbb{C}$, with $ad - bc = 1$. This is the *Möbius group*, denoted $\mathbf{Möb}^+$.

The Möbius transformations corresponding to the case $a, b, c, d \in \mathbb{R}$, with $ad - bc = 1$ form a subgroup of $\mathbf{Möb}^+$ denoted $\mathbf{Möb}^+_{\mathbb{R}}$.

The map from $\mathbf{SL}(2, \mathbb{C})$ to $\mathbf{Möb}^+$ that sends $A \in \mathbf{SL}(2, \mathbb{C})$ to the corresponding Möbius transformation is a surjective group homomorphism and one checks easily that its kernel is $\{-I, I\}$ (where I is the 2×2 identity matrix).

Therefore, the Möbius group $\mathbf{M\ddot{o}b}^+$ is isomorphic to the quotient group $\mathbf{SL}(2,\mathbb{C})/\{-I, I\}$, denoted $\mathbf{PSL}(2,\mathbb{C})$.

This latter group turns out to be the group of projective transformations of the projective space $\mathbb{CP}^1$.

The same reasoning shows that the subgroup $\mathbf{M\ddot{o}b}^+_{\mathbb{R}}$ is isomorphic to $\mathbf{SL}(2,\mathbb{R})/\{-I, I\}$, denoted $\mathbf{PSL}(2,\mathbb{R})$.

The group $\mathbf{SL}(2,\mathbb{C})$ acts on $\mathbb{C}\cup\{\infty\} \cong S^2$ the same way that $\mathbf{SL}(2,\mathbb{R})$ acts on H, namely: For any $A \in \mathbf{SL}(2,\mathbb{C})$, for any $z \in \mathbb{C}\cup\{\infty\}$,

$$A \cdot z = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1.$$

This action is clearly transitive.

One may recall from complex analysis that the (complex) Möbius transformation

$$z \mapsto \frac{z - i}{z + i}$$

is a biholomorphic isomorphism between the upper half plane, H, and the open unit disk,

$$D = \{z \in \mathbb{C} \mid |z| < 1\}.$$

As a consequence, it is possible to define a transitive action of $\mathbf{SL}(2, \mathbb{R})$ on D. This can be done in a more direct fashion, using a group isomorphic to $\mathbf{SL}(2, \mathbb{R})$, namely, $\mathbf{SU}(1, 1)$ (a group of complex matrices), but we don't want to do this right now.

Example 3: The set of $n \times n$ symmetric, positive, definite matrices, $\mathbf{SPD}(n)$.

The group $\mathbf{GL}(n) = \mathbf{GL}(n, \mathbb{R})$ acts on $\mathbf{SPD}(n)$ as follows: For all $A \in \mathbf{GL}(n)$ and all $S \in \mathbf{SPD}(n)$,

$$A \cdot S = ASA^{\top}.$$

It is easily checked that $ASA^{\top}$ is in $\mathbf{SPD}(n)$ if S is in $\mathbf{SPD}(n)$. This action is transitive because every SPD matrix, S, can be written as $S = AA^{\top}$, for some invertible matrix, A (prove this as an exercise).

Example 4: The projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

The *(real) projective space*, $\mathbb{RP}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{R}^{n+1}$ (where $n \geq 0$).

Since a one-dimensional subspace, $L \subseteq \mathbb{R}^{n+1}$, is spanned by any nonzero vector, $u \in L$, we can view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

In terms of this definition, there is a projection, $pr \colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$, given by $pr(u) = [u]_\sim$, the equivalence class of u modulo $\sim$.

Write $[u]$ for the line defined by the nonzero vector, u.

Since every line, L, in $\mathbb{R}^{n+1}$ intersects the sphere S^n in two antipodal points, we can view $\mathbb{RP}^n$ as the quotient of the sphere S^n by identification of antipodal points. We write

$$S^n/\{I, -I\} \cong \mathbb{RP}^n.$$

We define an action of $\mathbf{SO}(n+1)$ on $\mathbb{RP}^n$ as follows: For any line, $L = [u]$, for any $R \in \mathbf{SO}(n+1)$,

$$R \cdot L = [Ru].$$

Since R is linear, the line $[Ru]$ is well defined, i.e., does not depend on the choice of $u \in L$. It is clear that this action is transitive.

The *(complex) projective space*, $\mathbb{CP}^n$, is defined analogously as the set of all lines through the origin in $\mathbb{C}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$ (where $n \geq 0$).

This time, we can view $\mathbb{CP}^n$ as the set of equivalence classes of vectors in $\mathbb{C}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{C}.$$

We have the projection, $pr\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$, given by $pr(u) = [u]_\sim$, the equivalence class of u modulo $\sim$.

Again, write $[u]$ for the line defined by the nonzero vector, u.

Remark: Algebraic geometers write $\mathbb{P}^n_{\mathbb{R}}$ for $\mathbb{RP}^n$ and $\mathbb{P}^n_{\mathbb{C}}$ (or even $\mathbb{P}^n$) for $\mathbb{CP}^n$.

Recall that $\Sigma^n \subseteq \mathbb{C}^{n+1}$, the unit sphere in $\mathbb{C}^{n+1}$, is defined by

$$\Sigma^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1\overline{z}_1 + \cdots + z_{n+1}\overline{z}_{n+1} = 1\}.$$

For any line, $L = [u]$, where $u \in \mathbb{C}^{n+1}$ is a nonzero vector, writing $u = (u_1, \ldots, u_{n+1})$, a point $z \in \mathbb{C}^{n+1}$ belongs to L iff $z = \lambda(u_1, \ldots, u_{n+1})$, for some $\lambda \in \mathbb{C}$.

Therefore, the intersection, $L \cap \Sigma^n$, of the line L and the sphere Σ^n is given by

$$\begin{aligned} L \cap \Sigma^n = \{ & \lambda(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} \mid \\ & \lambda \in \mathbb{C},\ \lambda\overline{\lambda}(u_1\overline{u}_1 + \cdots + u_{n+1}\overline{u}_{n+1}) = 1\}, \end{aligned}$$

i.e.,

$$\begin{aligned} L \cap \Sigma^n = \Big\{ & \lambda(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} \mid \lambda \in \mathbb{C}, \\ & |\lambda| = \frac{1}{\sqrt{|u_1|^2 + \cdots + |u_{n+1}|^2}} \Big\}. \end{aligned}$$

Thus, we see that there is a bijection between $L \cap \Sigma^n$ and the circle, S^1, i.e., geometrically, $L \cap \Sigma^n$ is a circle.

Moreover, since any line, L, through the origin is determined by just one other point, we see that for any two lines L_1 and L_2 through the origin,

$$L_1 \neq L_2 \quad \text{iff} \quad (L_1 \cap \Sigma^n) \cap (L_2 \cap \Sigma^n) = \emptyset.$$

However, Σ^n is the sphere S^{2n+1} in $\mathbb{R}^{2n+2}$.

It follows that $\mathbb{CP}^n$ is the quotient of S^{2n+1} by the equivalence relation, $\sim$, defined such that

$$y \sim z \quad \text{iff} \quad y, z \in L \cap \Sigma^n,$$

for some line, L, through the origin.

Therefore, we can write

$$S^{2n+1}/S^1 \cong \mathbb{CP}^n.$$

The case $n = 1$ is particularly interesting, as it turns out that

$$S^3/S^1 \cong S^2.$$

This is the famous *Hopf fibration*. To show this, proceed as follows: As

$$S^3 \cong \Sigma^1 = \{(z, z') \in \mathbb{C}^2 \mid |z|^2 + |z'|^2 = 1\},$$

define a map, $\mathrm{HF}\colon S^3 \to S^2$, by

$$\mathrm{HF}((z, z')) = (2z\overline{z'}, |z|^2 - |z'|^2).$$

We leave as a homework exercise to prove that this map has range S^2 and that

$$\mathrm{HF}((z_1, z_1')) = \mathrm{HF}((z_2, z_2'))$$

iff

$$(z_1, z_1') = \lambda(z_2, z_2'), \quad \text{for some } \lambda \text{ with } |\lambda| = 1.$$

In other words, for any point, $p \in S^2$, the inverse image, $\mathrm{HF}^{-1}(p)$ (also called *fibre* over p), is a circle on S^3.

Consequently, S^3 can be viewed as the union of a family of disjoint circles. This is the *Hopf fibration*.

It is possible to visualize the Hopf fibration using the stereographic projection from S^3 onto $\mathbb{R}^3$. This is a beautiful and puzzling picture. For example, see Berger [**?**]. Therefore, HF induces a bijection from $\mathbb{CP}^1$ to S^2, and it is a homeomorphism.

We define an action of $\mathbf{SU}(n+1)$ on $\mathbb{CP}^n$ as follows: For any line, $L = [u]$, for any $R \in \mathbf{SU}(n+1)$,

$$R \cdot L = [Ru].$$

Again, this action is well defined and it is transitive.

Example 5: Affine spaces.

If E is any (real) vector space and X is any set, a transitive and faithful action, $\cdot\colon E \times X \to X$, of the additive group of E on X makes X into an *affine space*. The intuition is that the members of E are translations.

Those familiar with affine spaces as in Gallier [**?**] (Chapter 2) or Berger [**?**] will point out that if X is an affine space, then, not only is the action of E on X transitive, but more is true: For any two points, $a, b \in E$, there is a *unique* vector, $u \in E$, such that $u \cdot a = b$.

By the way, the action of E on X is usually considered to be a right action and is written additively, so $u \cdot a$ is written $a + u$ (the result of translating a by u).

Thus, it would seem that we have to require more of our action. However, this is not necessary because E (under addition) is *abelian*. More precisely, we have the proposition

Proposition 5.2.3 *If G is an abelian group acting on a set X and the action $\cdot\colon G \times X \to X$ is transitive and faithful, then for any two elements $x, y \in X$, there is a unique $g \in G$ so that $g \cdot x = y$ (the action is simply transitive).*

More examples will be considered later.

The subset of group elements that leave some given element $x \in X$ fixed plays an important role.

Definition 5.2.4 Given an action, $\cdot\colon G \times X \to X$, of a group G on a set X, for any $x \in X$, the group G_x (also denoted $\mathrm{Stab}_G(x)$), called the *stabilizer* of x or *isotropy group at* x is given by

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

We have to verify that G_x is indeed a subgroup of G, but this is easy.

In general, G_x is **not** a normal subgroup.

Observe that

$$G_{g\cdot x} = gG_xg^{-1},$$

for all $g \in G$ and all $x \in X$.

Therefore, the stabilizers of x and $g \cdot x$ are conjugate of each other.

When the action of G on X is transitive, for any fixed $x \in G$, the set X is a quotient (as set, not as group) of G by G_x. Indeed, we can define the map, $\pi_x \colon G \to X$, by

$$\pi_x(g) = g \cdot x, \quad \text{for all } g \in G.$$

Observe that

$$\pi_x(gG_x) = (gG_x) \cdot x = g \cdot (G_x \cdot x) = g \cdot x = \pi_x(g).$$

This shows that $\pi_x \colon G \to X$ induces a quotient map, $\overline{\pi}_x \colon G/G_x \to X$, from the set, G/G_x, of (left) cosets of G_x to X, defined by

$$\overline{\pi}_x(gG_x) = g \cdot x.$$

Since

$$\pi_x(g) = \pi_x(h) \quad \text{iff} \quad g \cdot x = h \cdot x \quad \text{iff} \quad g^{-1}h \cdot x = x$$

iff

$$g^{-1}h \in G_x \quad \text{iff} \quad gG_x = hG_x,$$

we deduce $\overline{\pi}_x \colon G/G_x \to X$ is injective.

However, since our action is assumed to be transitive, for every $y \in X$, there is some $g \in G$ so that $g \cdot x = y$ and so, $\overline{\pi}_x(gG_x) = g \cdot x = y$, i.e., the map $\overline{\pi}_x$ is also surjective.

Therefore, the map $\overline{\pi}_x\colon G/G_x \to X$ is a bijection (of sets, not groups). The map $\pi_x\colon G \to X$ is also surjective. Let us record this important fact as

Proposition 5.2.5 *If $\cdot\colon G \times X \to X$ is a transitive action of a group G on a set X, for every fixed $x \in X$, the surjection, $\pi\colon G \to X$, given by*

$$\pi(g) = g \cdot x$$

induces a bijection

$$\overline{\pi}\colon G/G_x \to X,$$

where G_x is the stabilizer of x.

The map $\pi\colon G \to X$ (corresponding to a fixed $x \in X$) is sometimes called a *projection* of G onto X. Proposition 5.2.5 shows that for every $y \in X$, the subset, $\pi^{-1}(y)$, (called the *fibre above* y) is equal to some coset, gG_x, of G and thus, is in bijection with the group G_x itself.

We can think of G as a moving family of fibres, G_x, parametrized by X.

This point of view of viewing a space as a moving family of simpler spaces is typical in (algebraic) geometry, and underlies the notion of (principal) fibre bundle.

Note that if the action $\cdot\colon G \times X \to X$ is transitive, then the stabilizers G_x and G_y of any two elements $x, y \in X$ are isomorphic, as they as conjugates. Thus, in this case, it is enough to compute one of these stabilizers for a "convenient" x.

As the situation of Proposition 5.2.5 is of particular interest, we make the following definition:

Definition 5.2.6 A set, X, is said to be a *homogeneous space* if there is a transitive action, $\cdot\colon G \times X \to X$, of some group, G, on X.

We see that all the spaces of Example 1–5 are homogeneous spaces.

Another example that will play an important role when we deal with Lie groups is the situation where we have a group, G, a subgroup, H, of G (not necessarily normal) and where $X = G/H$, the set of left cosets of G modulo H.

The group G acts on G/H by left multiplication:

$$a \cdot (gH) = (ag)H,$$

where $a, g \in G$. This action is clearly transitive and one checks that the stabilizer of gH is gHg^{-1}.

If G is a topological group and H is a closed subgroup of G (see later for an explanation), it turns out that G/H is Hausdorff (Recall that a topological space, X, is *Hausdorff* iff for any two distinct points $x \neq y \in X$, there exists two disjoint open subsets, U and V, with $x \in U$ and $y \in V$.)

If G is a Lie group, we obtain a manifold.

Even if G and X are topological spaces and the action, $\cdot \colon G \times X \to X$, is continuous, the space G/G_x under the quotient topology is, in general, **not** homeomorphic to X.

We will give later sufficient conditions that insure that X is indeed a topological space or even a manifold. In particular, X will be a manifold when G is a Lie group.

In general, an action $\cdot \colon G \times X \to X$ is not transitive on X, but for every $x \in X$, it is transitive on the set

$$O(x) = G \cdot x = \{g \cdot x \mid g \in G\}.$$

Such a set is called the *orbit* of x. The orbits are the equivalence classes of the following equivalence relation:

Definition 5.2.7 Given an action, $\cdot\colon G \times X \to X$, of some group, G, on X, the equivalence relation, $\sim$, on X is defined so that, for all $x, y \in X$,

$$x \sim y \quad \text{iff} \quad y = g \cdot x, \quad \text{for some } g \in G.$$

For every $x \in X$, the equivalence class of x is the *orbit of* x, denoted $O(x)$ or $\mathrm{Orb}_G(x)$, with

$$O(x) = \{g \cdot x \mid g \in G\}.$$

The set of orbits is denoted X/G.

The orbit space, X/G, is obtained from X by an identification (or merging) process: For every orbit, all points in that orbit are merged into a single point.

For example, if $X = S^2$ and G is the group consisting of the restrictions of the two linear maps I and $-I$ of $\mathbb{R}^3$ to S^2 (where $-I(x, y, z) = (-x, -y, -z)$), then

$$X/G = S^2/\{I, -I\} \cong \mathbb{RP}^2.$$

Many manifolds can be obtained in this fashion, including the torus, the Klein bottle, the Möbius band, etc.

Since the action of G is transitive on $O(x)$, by Proposition 5.2.5, we see that for every $x \in X$, we have a bijection

$$O(x) \cong G/G_x.$$

As a corollary, if both X and G are finite, for any set, $A \subseteq X$, of representatives from every orbit, we have the *orbit formula*:

$$|X| = \sum_{a \in A} [G\colon G_x] = \sum_{a \in A} |G|/|G_x|.$$

Even if a group action, $\cdot\colon G \times X \to X$, is not transitive, when X is a manifold, we can consider the set of orbits, X/G, and if the action of G on X satisfies certain conditions, X/G is actually a manifold.

Manifolds arising in this fashion are often called *orbifolds*.

In summary, we see that manifolds arise in at least two ways from a group action:

(1) As homogeneous spaces, G/G_x, if the action is transitive.

(2) As orbifolds, X/G.

Of course, in both cases, the action must satisfy some additional properties.

Let us now determine some stabilizers for the actions of Examples 1–4, and for more examples of homogeneous spaces.

(a) Consider the action, $\cdot\colon \mathbf{SO}(n) \times S^{n-1} \to S^{n-1}$, of $\mathbf{SO}(n)$ on the sphere S^{n-1} ($n \geq 1$) defined in Example 1. Since this action is transitive, we can determine the stabilizer of any convenient element of S^{n-1}, say $e_1 = (1, 0, \ldots, 0)$.

In order for any $R \in \mathbf{SO}(n)$ to leave e_1 fixed, the first column of R must be e_1, so R is an orthogonal matrix of the form

$$R = \begin{pmatrix} 1 & U \\ 0 & S \end{pmatrix}, \quad \text{with} \quad \det(S) = 1.$$

As the rows of R must be unit vector, we see that $U = 0$ and $S \in \mathbf{SO}(n-1)$.

Therefore, the stabilizer of e_1 is isomorphic to $\mathbf{SO}(n-1)$, and we deduce the bijection

$$\mathbf{SO}(n)/\mathbf{SO}(n-1) \cong S^{n-1}.$$

Strictly speaking, $\mathbf{SO}(n-1)$ is not a subgroup of $\mathbf{SO}(n)$ and in all rigor, we should consider the subgroup, $\widetilde{\mathbf{SO}}(n-1)$, of $\mathbf{SO}(n)$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad \det(S) = 1$$

and write

$$\mathbf{SO}(n)/\widetilde{\mathbf{SO}}(n-1) \cong S^{n-1}.$$

However, it is common practice to identify $\mathbf{SO}(n-1)$ with $\widetilde{\mathbf{SO}}(n-1)$.

When $n = 2$, as $\mathbf{SO}(1) = \{1\}$, we find that $\mathbf{SO}(2) \cong S^1$, a circle, a fact that we already knew.

When $n = 3$, we find that $\mathbf{SO}(3)/\mathbf{SO}(2) \cong S^2$. This says that $\mathbf{SO}(3)$ is somehow the result of glueing circles to the surface of a sphere (in $\mathbb{R}^3$), in such a way that these circles do not intersect. This is hard to visualize!

A similar argument for the complex unit sphere, Σ^{n-1}, shows that

$$\mathbf{SU}(n)/\mathbf{SU}(n-1) \cong \Sigma^{n-1} \cong S^{2n-1}.$$

Again, we identify $\mathbf{SU}(n-1)$ with a subgroup of $\mathbf{SU}(n)$, as in the real case. In particular, when $n = 2$, as $\mathbf{SU}(1) = \{1\}$, we find that

$$\mathbf{SU}(2) \cong S^3,$$

i.e., the group $\mathbf{SU}(2)$ is topologically the sphere S^3! Actually, this is not surprising if we remember that $\mathbf{SU}(2)$ is in fact the group of unit quaternions.

(b) We saw in Example 2 that the action, $\cdot\colon \mathbf{SL}(2, \mathbb{R}) \times H \to H$, of the group $\mathbf{SL}(2, \mathbb{R})$ on the upper half plane is transitive. Let us find out what the stabilizer of $z = i$ is. We should have

$$\frac{ai+b}{ci+d} = i,$$

that is, $ai + b = -c + di$, i.e.,

$$(d-a)i = b + c.$$

Since a, b, c, d are real, we must have $d = a$ and $b = -c$. Moreover, $ad - bc = 1$, so we get $a^2 + b^2 = 1$. We conclude that a matrix in $\mathbf{SL}(2, \mathbb{R})$ fixes i iff it is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with} \quad a^2 + b^2 = 1.$$

Clearly, these are the rotation matrices in $\mathbf{SO}(2)$ and so, the stabilizer of i is $\mathbf{SO}(2)$. We conclude that

$$\mathbf{SL}(2, \mathbb{R})/\mathbf{SO}(2) \cong H.$$

This time, we can view $\mathbf{SL}(2, \mathbb{R})$ as the result of glueing circles to the upper half plane. This is not so easy to visualize.

There is a better way to visualize the topology of $\mathbf{SL}(2, \mathbb{R})$ by making it act on the open disk, D. We will return to this action in a little while.

Now, consider the action of $\mathbf{SL}(2, \mathbb{C})$ on $\mathbb{C} \cup \{\infty\} \cong S^2$. As it is transitive, let us find the stabilizer of $z = 0$. We must have

$$\frac{b}{d} = 0,$$

and as $ad - bc = 1$, we must have $b = 0$ and $ad = 1$. Thus, the stabilizer of 0 is the subgroup, $\mathbf{SL}(2, \mathbb{C})_0$, of $\mathbf{SL}(2, \mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad \text{where} \quad a \in \mathbb{C} - \{0\} \quad \text{and} \quad c \in \mathbb{C}.$$

We get

$$\mathbf{SL}(2, \mathbb{C})/\mathbf{SL}(2, \mathbb{C})_0 \cong \mathbb{C} \cup \{\infty\} \cong S^2,$$

but this is not very illuminating.

(c) In Example 3, we considered the action,
$\cdot \colon \mathbf{GL}(n) \times \mathbf{SPD}(n) \to \mathbf{SPD}(n)$, of $\mathbf{GL}(n)$ on $\mathbf{SPD}(n)$, the set of symmetric positive definite matrices.

As this action is transitive, let us find the stabilizer of I. For any $A \in \mathbf{GL}(n)$, the matrix A stabilizes I iff

$$AIA^{\top} = AA^{\top} = I.$$

Therefore, the stabilizer of I is $\mathbf{O}(n)$ and we find that

$$\mathbf{GL}(n)/\mathbf{O}(n) = \mathbf{SPD}(n).$$

Observe that if $\mathbf{GL}^{+}(n)$ denotes the subgroup of $\mathbf{GL}(n)$ consisting of all matrices with a strictly positive determinant, then we have an action
$\cdot \colon \mathbf{GL}^{+}(n) \times \mathbf{SPD}(n) \to \mathbf{SPD}(n)$.

This action is transtive and we find that the stabilizer of I is $\mathbf{SO}(n)$; consequently, we get

$$\mathbf{GL}^{+}(n)/\mathbf{SO}(n) = \mathbf{SPD}(n).$$

(d) In Example 4, we considered the action, $\cdot \colon \mathbf{SO}(n+1) \times \mathbb{RP}^n \to \mathbb{RP}^n$, of $\mathbf{SO}(n+1)$ on the (real) projective space, $\mathbb{RP}^n$. As this action is transitive, let us find the stabilizer of the line, $L = [e_1]$, where $e_1 = (1, 0, \ldots, 0)$.

We find that the stabilizer of $L = [e_1]$ is isomorphic to the group $\mathbf{O}(n)$ and so,

$$\mathbf{SO}(n+1)/\mathbf{O}(n) \cong \mathbb{RP}^n.$$

Strictly speaking, $\mathbf{O}(n)$ is not a subgroup of $\mathbf{SO}(n+1)$, so the above equation does not make sense. We should write

$$\mathbf{SO}(n+1)/\widetilde{\mathbf{O}}(n) \cong \mathbb{RP}^n,$$

where $\widetilde{\mathbf{O}}(n)$ is the subgroup of $\mathbf{SO}(n+1)$ consisting of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad S \in \mathbf{O}(n),\ \ \alpha = \pm 1, \quad \det(S) = \alpha.$$

However, the common practice is to write $\mathbf{O}(n)$ instead of $\widetilde{\mathbf{O}}(n)$.

We should mention that $\mathbb{RP}^3$ and $\mathbf{SO}(3)$ are homeomorphic spaces. This is shown using the quaternions, for example, see Gallier [**?**], Chapter 8.

A similar argument applies to the action, $\cdot\colon \mathbf{SU}(n+1) \times \mathbb{CP}^n \to \mathbb{CP}^n$, of $\mathbf{SU}(n+1)$ on the (complex) projective space, $\mathbb{CP}^n$. We find that

$$\mathbf{SU}(n+1)/\mathbf{U}(n) \cong \mathbb{CP}^n.$$

Again, the above is a bit sloppy as $\mathbf{U}(n)$ is not a subgroup of $\mathbf{SU}(n+1)$. To be rigorous, we should use the subgroup, $\widetilde{\mathbf{U}}(n)$, consisting of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad S \in \mathbf{U}(n),\ |\alpha| = 1, \quad \det(S) = \overline{\alpha}.$$

The common practice is to write $\mathbf{U}(n)$ instead of $\widetilde{\mathbf{U}}(n)$. In particular, when $n = 1$, we find that

$$\mathbf{SU}(2)/\mathbf{U}(1) \cong \mathbb{CP}^1.$$

But, we know that $\mathbf{SU}(2) \cong S^3$ and, clearly, $\mathbf{U}(1) \cong S^1$. So, again, we find that $S^3/S^1 \cong \mathbb{CP}^1$ (but we know, more, namely, $S^3/S^1 \cong S^2 \cong \mathbb{CP}^1$.)

(e) We now consider a generalization of projective spaces (real and complex). First, consider the real case.

Given any $n \geq 1$, for any k, with $0 \leq k \leq n$, let $G(k, n)$ be the set of all linear k-dimensional subspaces of $\mathbb{R}^n$ (also called *k-planes*).

Any k-dimensional subspace, U, of $\mathbb{R}$ is spanned by k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; write $U = \mathrm{span}(u_1, \ldots, u_k)$.

We can define an action, $\cdot \colon \mathbf{O}(n) \times G(k, n) \to G(k, n)$, as follows: For any $R \in \mathbf{O}(n)$, for any $U = \mathrm{span}(u_1, \ldots, u_k)$, let

$$R \cdot U = \mathrm{span}(Ru_1, \ldots, Ru_k).$$

We have to check that the above is well defined but this is not hard.

It is also easy to see that this action is transitive. Thus, it is enough to find the stabilizer of any k-plane.

We can show that the stabilizer of U is isomorphic to $\mathbf{O}(k) \times \mathbf{O}(n-k)$ and we find that

$$\mathbf{O}(n)/(\mathbf{O}(k) \times \mathbf{O}(n-k)) \cong G(k,n).$$

It turns out that this makes $G(k,n)$ into a smooth manifold of dimension $k(n-k)$ called a *Grassmannian*.

If we recall the projection $pr\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$, by definition, a *$k$-plane* in $\mathbb{RP}^n$ is the image under pr of any $(k+1)$-plane in $\mathbb{R}^{n+1}$.

So, for example, a line in $\mathbb{RP}^n$ is the image of a 2-plane in $\mathbb{R}^{n+1}$, and a hyperplane in $\mathbb{RP}^n$ is the image of a hyperplane in $\mathbb{R}^{n+1}$.

The advantage of this point of view is that the k-planes in $\mathbb{RP}^n$ are arbitrary, i.e., they do not have to go through "the origin" (which does not make sense, anyway!).

Then, we see that we can interpret the Grassmannian, $G(k+1, n+1)$, as a space of "parameters" for the k-planes in $\mathbb{RP}^n$. For example, $G(2, n+1)$ parametrizes the lines in $\mathbb{RP}^n$. In this viewpoint, $G(k+1, n+1)$ is usually denoted $\mathbb{G}(k, n)$.

It can be proved (using some exterior algebra) that $G(k, n)$ can be embedded in $\mathbb{RP}^{\binom{n}{k}-1}$.

Much more is true. For example, $G(k, n)$ is a projective variety, which means that it can be defined as a subset of $\mathbb{RP}^{\binom{n}{k}-1}$ equal to the zero locus of a set of homogeneous equations.

There is even a set of quadratic equations, known as the *Plücker equations*, defining $G(k, n)$. In particular, when $n = 4$ and $k = 2$, we have $G(2, 4) \subseteq \mathbb{RP}^5$ and $G(2, 4)$ is defined by a single equation of degree 2.

The Grassmannian $G(2,4) = \mathbb{G}(1,3)$ is known as the *Klein quadric*. This hypersurface in $\mathbb{RP}^5$ parametrizes the lines in $\mathbb{RP}^3$. It play an important role in computer vision.

Complex Grassmannians are defined in a similar way, by replacing $\mathbb{R}$ by $\mathbb{C}$ throughout. The complex Grassmannian, $G_{\mathbb{C}}(k,n)$, is a complex manifold as well as a real manifold and we have

$$\mathbf{U}(n)/(\mathbf{U}(k) \times \mathbf{U}(n-k)) \cong G_{\mathbb{C}}(k,n).$$

We now return to case (b) to give a better picture of $\mathbf{SL}(2,\mathbb{R})$. Instead of having $\mathbf{SL}(2,\mathbb{R})$ act on the upper half plane we define an action of $\mathbf{SL}(2,\mathbb{R})$ on the open unit disk, D.

Technically, it is easier to consider the group, $\mathbf{SU}(1,1)$, which is isomorphic to $\mathbf{SL}(2,\mathbb{R})$, and to make $\mathbf{SU}(1,1)$ act on D.

The group $\mathbf{SU}(1,1)$ is the group of 2×2 complex matrices of the form

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{with} \quad a\overline{a} - b\overline{b} = 1.$$

The reader should check that if we let

$$g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

then the map from $\mathbf{SL}(2,\mathbb{R})$ to $\mathbf{SU}(1,1)$ given by

$$A \mapsto gAg^{-1}$$

is an isomorphism. Observe that the Möbius transformation associated with g is

$$z \mapsto \frac{z-i}{z+1},$$

which is the holomorphic isomorphism mapping H to D mentionned earlier!

Now, we can define a bijection between $\mathbf{SU}(1,1)$ and $S^1 \times D$ given by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto (a/|a|, b/a).$$

We conclude that $\mathbf{SL}(2,\mathbb{R}) \cong \mathbf{SU}(1,1)$ is topologically an open solid torus (i.e., with the surface of the torus removed). It is possible to further classify the elements of $\mathbf{SL}(2,\mathbb{R})$ into three categories and to have geometric interpretations of these as certain regions of the torus. For details, the reader should consult Carter, Segal and Macdonald [**?**] or Duistermatt and Kolk [**?**] (Chapter 1, Section 1.2).

The group $\mathbf{SU}(1,1)$ acts on D by interpreting any matrix in $\mathbf{SU}(1,1)$ as a Möbius tranformation, i.e.,

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} \right).$$

The reader should check that these transformations preserve D.

Both the upper half-plane and the open disk are models of Lobachevsky's non-Euclidean geometry (where the parallel postulate fails).

They are also models of hyperbolic spaces (Riemannian manifolds with constant negative curvature, see Gallot, Hulin and Lafontaine [**?**], Chapter III).

According to Dubrovin, Fomenko, and Novikov [**?**] (Chapter 2, Section 13.2), the open disk model is due to Poincaré and the upper half-plane model to Klein, although Poincaré was the first to realize that the upper half-plane is a hyperbolic space.