Manifolds, Lie Groups, Lie Algebras, with Applications

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CIS610, Spring 2005

Motivations and Goals

1. **Motivations**

Observation: Often, the set of all objects having some common properties has some topological structure, i.e., it makes sense to say when two objects are close to each other.

If one is lucky, a notion of distance also makes sense. This can be useful when we need to compare and classify objects.

Sometimes, we are lucky and the set of objects forms a vector space (isomorphic to $\mathbb{R}^n$).

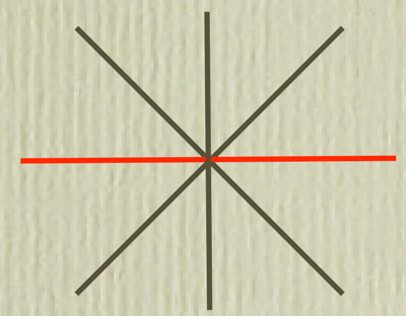
Unfortunately, often, this is not the case. However, in many cases, locally, the set of objects looks like $\mathbb{R}^n$. Even better, some sort of tangent space is defined at every point. The set of objects has the structure of a (smooth) manifold.

If we are lucky, there is also a way to multiply these objects and so, not only do we have a manifold structure but also a group.

Furthermore, the manifold and the group structure may be nicely compatible and we have a Lie group!

Let us consider some examples:

1. The set of all lines in the plane through the origin.



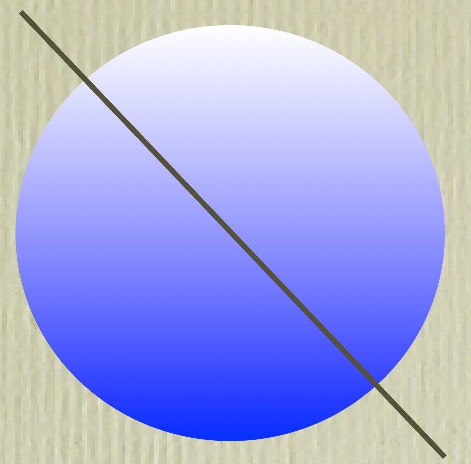
If we draw a circle centered at the origin, we see that every line intersects this circle in two (antipodal) points. Better, if we consider a half circle (say, the upper half circle), we see that every line except the red line intersects the circle in a single point.

The red line intersects the circle in exactly two points. If we identify these two antipodal points, we obtain another circle and so, the set of lines in the plane is in bijection with a circle.

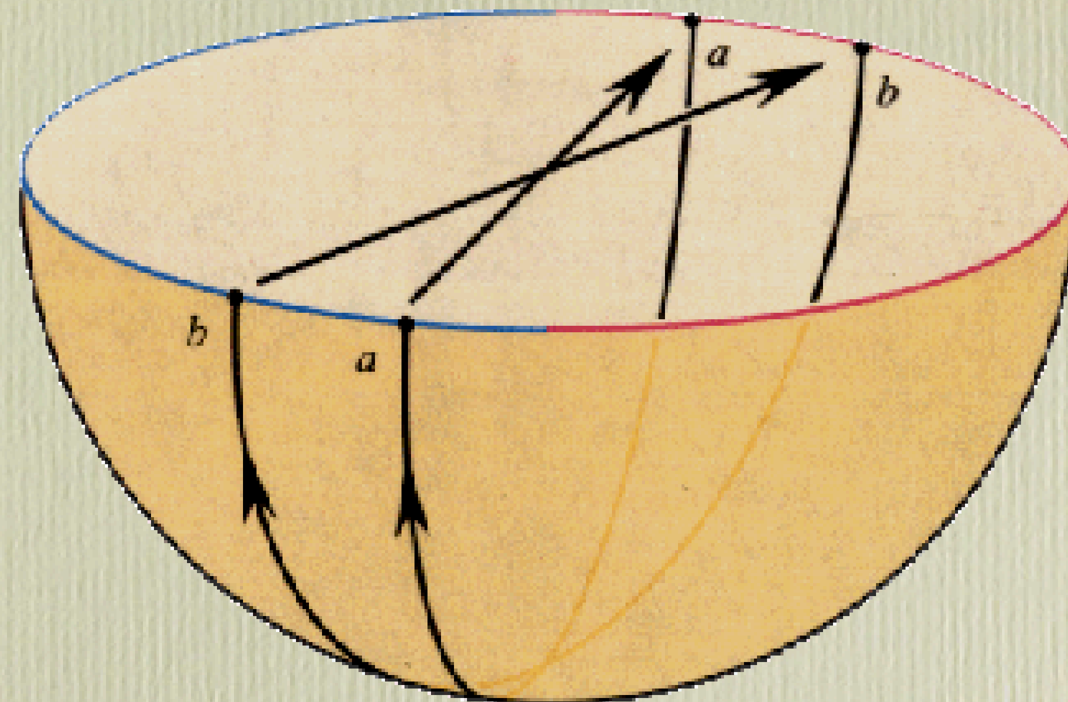
We obtain a space called the (real) projective line, denoted $\mathbb{RP}^1$.

2. The set of all lines in 3-space ($\mathbb{R}^3$) through the origin.

This time, we draw a sphere around the origin, and, again, we observe that every line intersects the sphere in two antipodal points.



Better, consider the upper half sphere. It has a great circle as boundary. Then, every line through the origin not in the plane containing this great circle intersects the sphere in exactly one point, but lines in that plane intersect this circle in two antipodal points. In order to obtain a bijection, we can form the surface obtained by gluing pairs of antipodal points together. The resulting surface is hard to visualize!



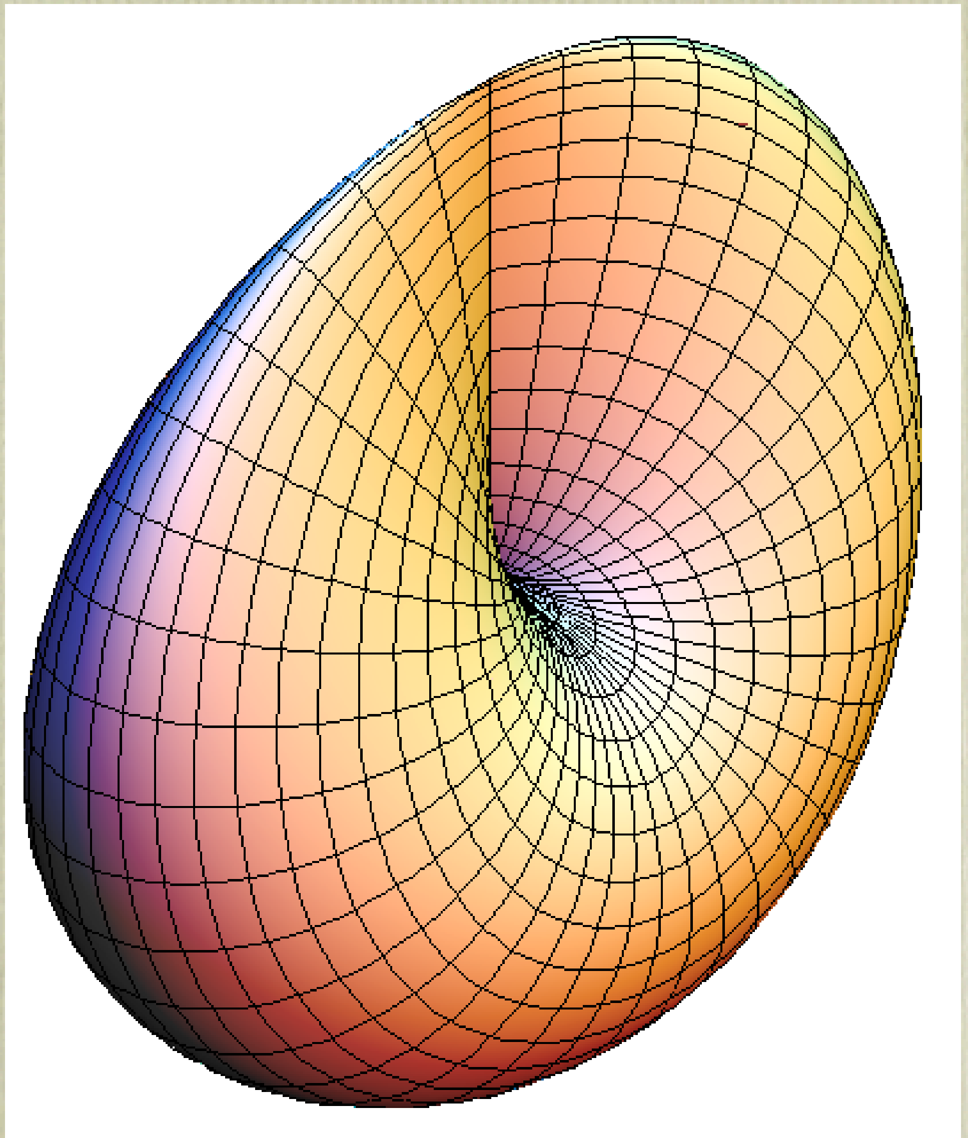
Antipodal points, denoted both by *a* (or *b*) are identified. If we perform these identifications carefuly, we get various models of the projective plane.

We obtain a surface called the (real) projective plane and denoted by $\mathbb{RP}^2$.

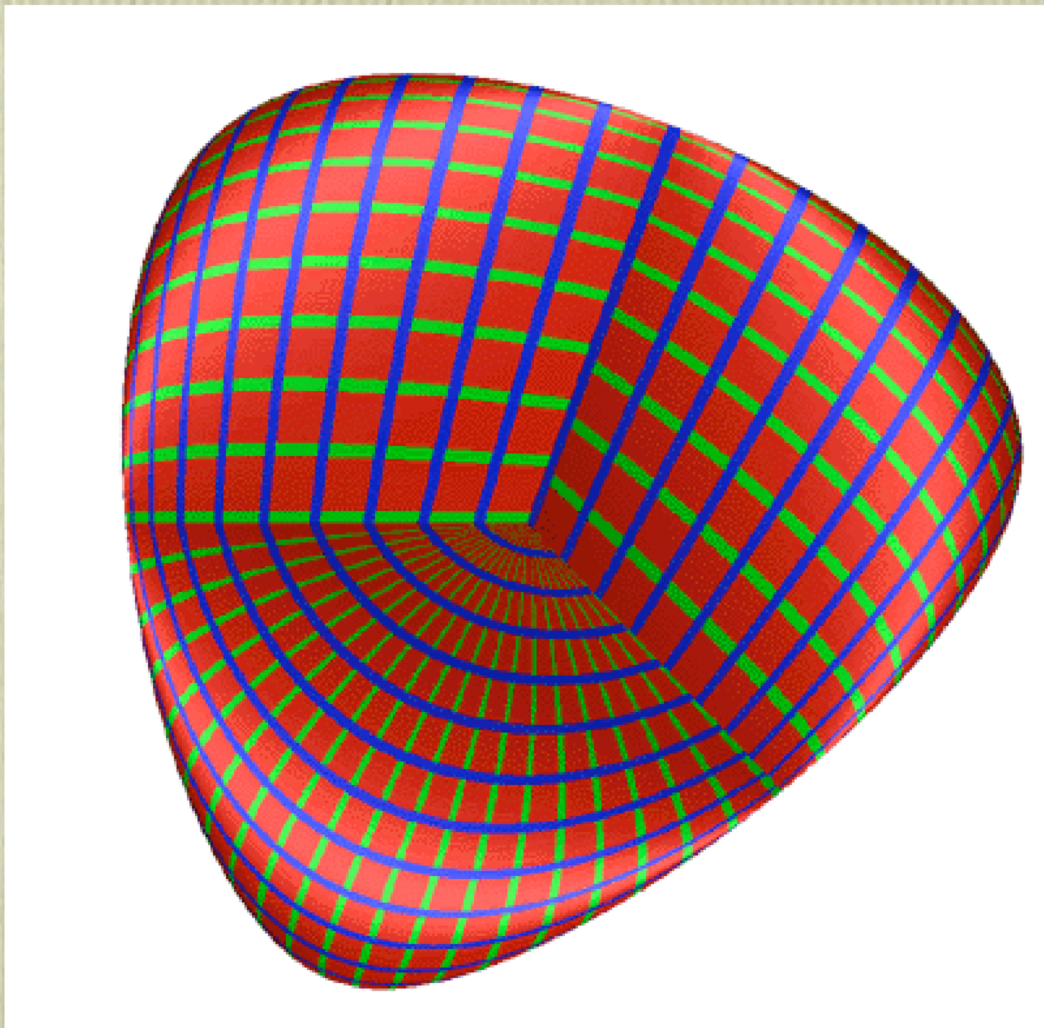
There are clever ways of getting a nice looking surface representing the real projective plane (see Hilbert and Cohn-Vossen's classic book). For example, there are the cross-cap, the Steiner roman surface and the Boy surface.

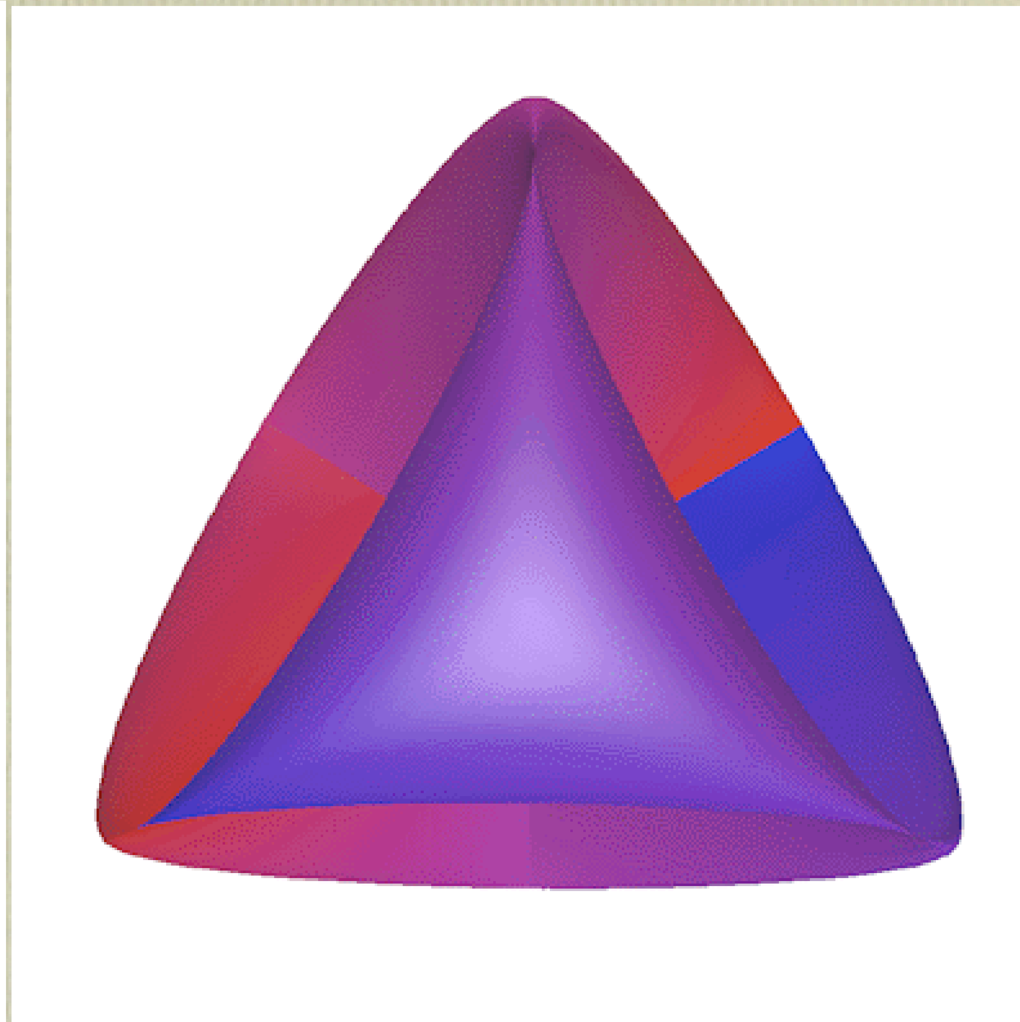
It turns out that in 3D, every surface representing the projective plane must self intersect (but not in 4D). Moreover, the projective plane is not orientable.

A cross-cap



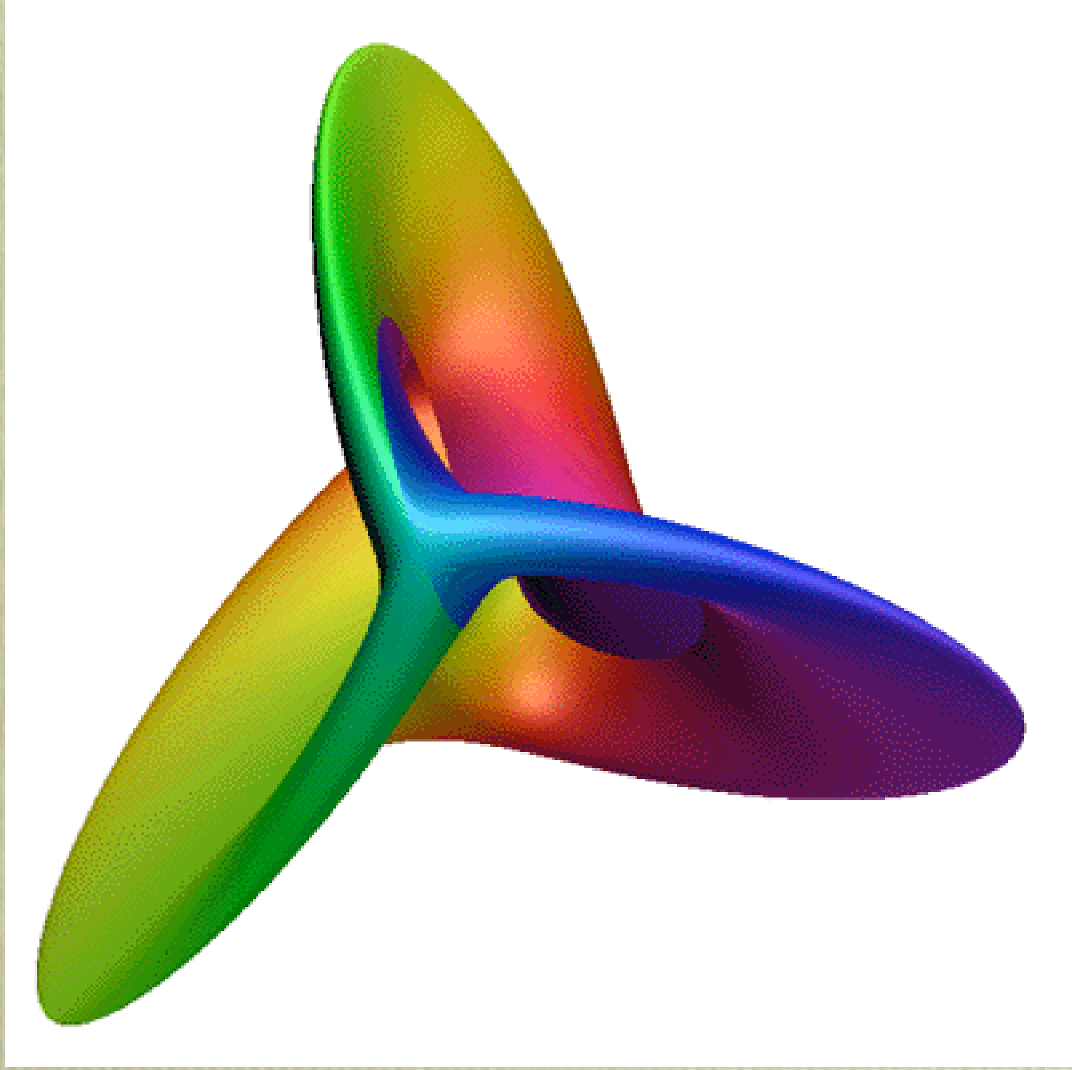
Steiner Roman Surfaces

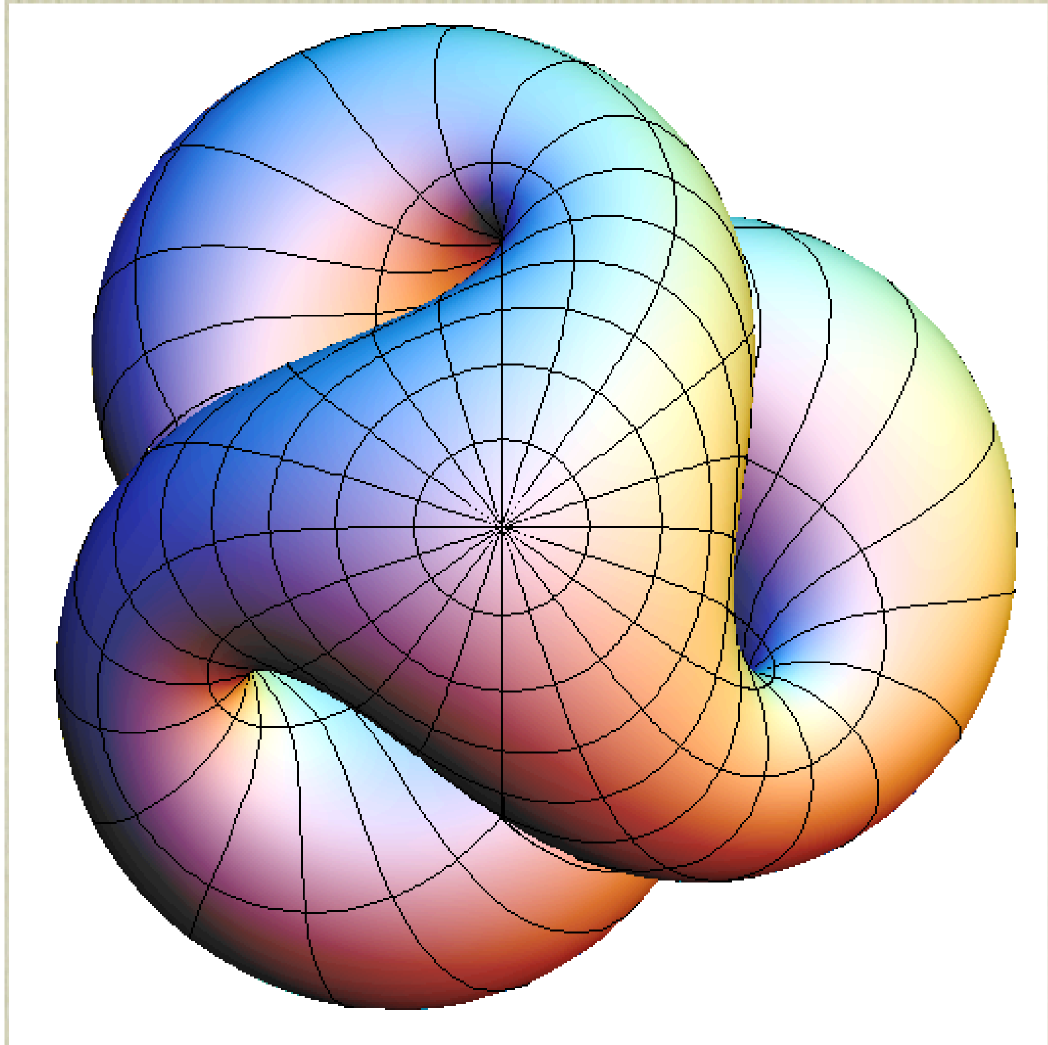




Knitted Versions of the cross-cap and Steiner surface

Boy Surfaces





3. Symmetric Positive Definite Matrices

Diffusion tensor magnetic resonance imaging produces a 3D symmetric, positive definite matrix, at each voxel of an imaging volume. In brain imaging, this method is used to track the white matter fibres, which demonstrate higher diffusivity of water in the direction of the fibre.

Diffusion tensor imaging has shown promise in clinical studies of brain pathologies and in the study of brain connectivity. One would hope to produce statistical atlases from diffusion tensor images and to understand the anatomical variability caused by a disease.

Unfortunately, the space of symmetric, positive definite matrices, $\mathbf{SPD}(n)$, is not a vector space. Consequently, standard linear statistical methods do not apply.

Recall that a matrix, A, is in $\mathbf{SPD}(n)$ iff it is symmetric and if its eigenvalues are all strictly positive.

For example, it is easy to show that a matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite iff $ac - b^2 > 0$ and $a > 0$.

To better understand $\mathbf{SPD}(n)$, we observe that the group, $\mathbf{GL}(n)$, of invertible $n \times n$ matrices acts on $\mathbf{SPD}(n)$ *via*: $\cdot : \mathbf{GL}(n) \times \mathbf{SPD}(n) \longrightarrow \mathbf{SPD}(n)$, where

$$A \cdot S = ASA^\top.$$

Furthermore, it turns out that this action is transitive and that the stabilizer of the identity is the orthogonal group, $\mathbf{O}(n)$. It follows that there is a bijection

$$\mathbf{SPD}(n) \cong \mathbf{GL}(n)/\mathbf{O}(n)$$

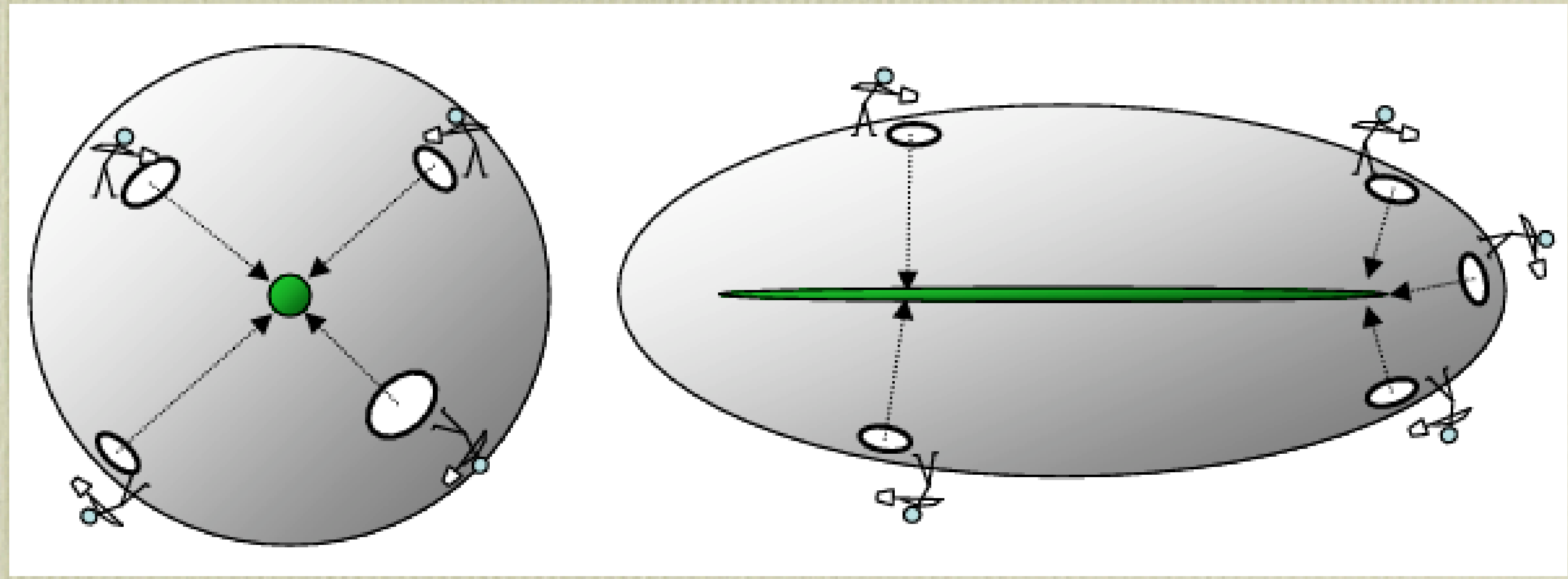
In fact, this even makes $\mathbf{SPD}(n)$ into a manifold, a homogeneous space, as we will see later.

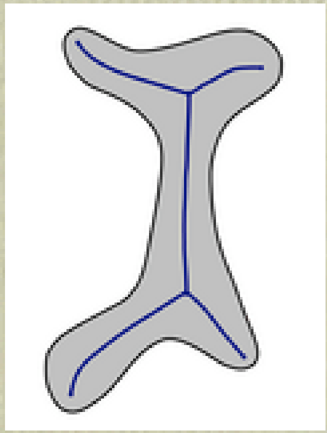
Some papers use this representation to define statistics of diffusion tensors; in particular, some paper of Fletcher and Joshi that will be considered later in this course.

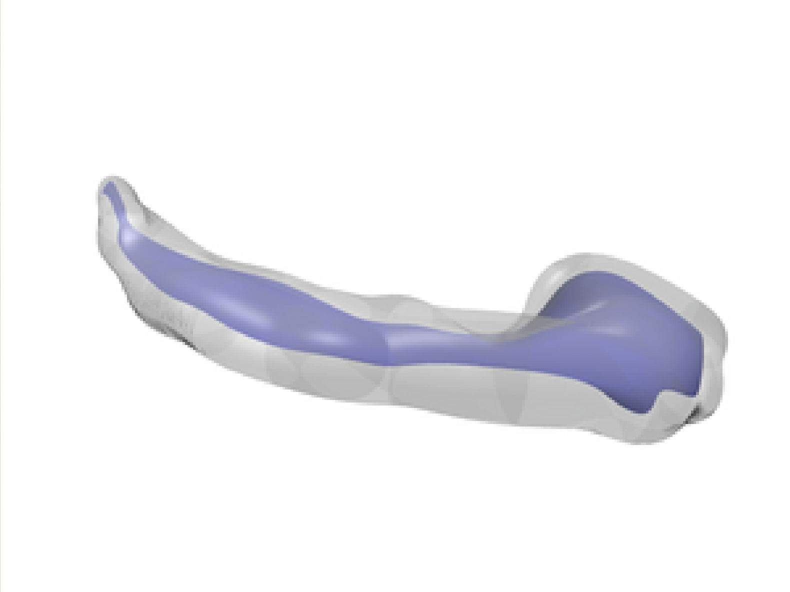
4. Medial axis representation of shapes

The medial axis representation (originally due to Blum) represents an object (2D or 3D) using the notion of medial axis. This is either a curve (2D) or a surface (3D). In the 3D case, this is the locus of centers of all spheres interior to the object and tangent to the boundary of the object in at least 2 points.

Here are some illustration taken from Paul Yushkevich (Radiology, Upenn):







Fletcher, Lu and Joshi show that medial descriptions are in fact elements of a Lie group. They develop a methodology based on Lie groups for the statistical analysis of medially-defined anatomical objects.

5. 2D-Shapes (Mumford and Sharon)

Mumford and Sharon propose to define a 2D shape as a simple closed smooth curve in the plane. They also postulate that two shapes are to be identified if one if obtained from the other by translation and scaling (but not rotation). Using complex analysis (namely, Riemann's conformal mapping theorem) they show that a simple closed curve, Γ , corresponds to a diffeomorphism of the unit circle, $\Psi : S^1 \longrightarrow S^1$, unique up to Mobius transformations of the form

$$z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} \ .$$

The group of diffeomorphisms is denoted $\mathbf{Diff}(S^1)$ and the above group of Mobius transformations is $\mathbf{SU}(1,1)/\{I,-I\}$, which is isomorphic to the group $\mathbf{PSL}(2,\mathbb{R})$. It follows that the space of closed simple curves, up to translation and scaling, is a space homeomorphic to the space $\mathbf{Diff}(S^1)/\mathbf{PSL}(2,\mathbb{R})$. This is also a homogeneous space, but it has infinite dimension!

Mumford and Sharon also define a Riemannian metric on this space (using the Weil-Petersson metric). This gives a precise way of telling how two shapes differ.

One of our goals is to come back and study this paper of Mumford and Sharon.

II. **Goals**

In discussing the previous examples, we ran into terms such as

- manifold
- group action
- homogeneous space
- Lie group (and Lie algebra)
- Riemannian metric
- curvature, diffeomorphism, etc.

In this course, we will explain what all these terms mean (and more!). We will also show how these concepts are used in various papers on shape analysis, and medical imaging, more specifically diffusion tensors and shape statistics.

I. Adjoint of a Linear Map in a Euclidean Space

Let E be a Euclidean space of dimension n .

This means E has an inner product $\langle u, v \rangle$.

Recall that a linear map, $f : E \longrightarrow E$, has an adjoint, $f^* : E \longrightarrow E$, so that

$$\langle f(u), v \rangle = \langle u, f^*(v) \rangle, \qquad \text{for all } u, v \in E .$$