Problem A1. Let $B_r = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r \}$ be the open ball of radius $r$ (centered at the origin) in $\mathbb{R}^n$ (where $r > 0$). Prove that the map $x \mapsto \frac{rx}{\sqrt{r^2 - (x_1^2 + \cdots + x_n^2)}}$ is a diffeomorphism of $B_r$ onto $\mathbb{R}^n$ (where $x = (x_1, \ldots, x_n)$).

*Hint.* Compute explicitly the inverse of this map.

Problem A2. A smooth bijective map of manifolds need not be a diffeomorphism. For example, show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is not a diffeomorphism.

Problem A3. (a) Let $X \subseteq \mathbb{R}^M$ and $Y \subseteq \mathbb{R}^N$ be two smooth manifolds of dimension $m$ and $n$ respectively. We can make $X \times Y \subseteq \mathbb{R}^{M+N}$ into a smooth manifold of dimension $m+n$ as follows: for any $(p,q) \in X \times Y$, if $\varphi: \Omega_1 \to U$ and $\psi: \Omega_2 \to V$ are parametrizations at $p \in U \subseteq X$ and $q \in V \subseteq Y$ respectively, then show that $\varphi \times \psi: \Omega_1 \times \Omega_2 \to U \times V$ is indeed a parametrization at $(p,q) \in X \times Y$. As the $U \times V$’s cover $X \times Y$, these parametrizations make $X \times Y$ into a manifold.

Check that $T_{(p,q)}(X \times Y) = T_pX \times T_qY$.

(b) Given a set, $X$, let $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$, called the diagonal of $X$. If $X$ is a manifold, then prove that $\Delta$ is a manifold diffeomorphic to $X$.

(c) The graph of a function, $f: X \to Y$, is the subset of $X \times Y$ given by $\text{graph}(f) = \{(x,f(x)) \mid x \in X\}$. Define $F: X \to \text{graph}(f)$ by $F(x) = (x, f(x))$. Prove that if $X$ and $Y$ are smooth manifolds and if $f$ is smooth, then $F$ is a diffeomorphism and thus, $\text{graph}(f)$ is a manifold diffeomorphic to $X$.

(d) Given any (smooth) map, $f: X \to X$, some $x \in X$ is a fixed point of $f$ iff $f(x) = x$. Prove that $f$ has a fixed point iff $\text{graph}(f) \cap \Delta \neq \emptyset$ (where $\Delta$ is the diagonal in $X \times X$).
“B problems” must be turned in.

**Problem B1 (60 pts).** Recall from Homework 1, Problem B6, the Cayley parametrization of rotation matrices in $\text{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where $B$ is any $n \times n$ skew symmetric matrix. In that problem, it was shown that $C(B)$ is a rotation matrix that does not admit $-1$ as an eigenvalue and that every such rotation matrix is of the form $C(B)$.

(a) If you have not already done so, prove that the map $B \mapsto C(B)$ is injective.

(b) Prove that

$$dC(B)(A) = D_A((I - B)(I + B)^{-1}) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1}.$$ 

*Hint.* First, show that $D_A(B^{-1}) = -B^{-1}AB^{-1}$ (where $B$ is invertible) and that $D_A(f(B))g(B)) = (D_Af(B))g(B) + f(B)(D_Ag(B))$, where $f$ and $g$ are differentiable matrix functions.

Deduce that $dC(B)$ is injective, for every skew-symmetric matrix, $B$. If we identify the space of $n \times n$ skew symmetric matrices with $\mathbb{R}^{n(n-1)/2}$, show that the Cayley map, $C: \mathbb{R}^{n(n-1)/2} \to \text{SO}(n)$, is a parametrization of $\text{SO}(n)$.

(c) Now, consider $n = 3$, i.e., $\text{SO}(3)$. Let $E_1$, $E_2$ and $E_3$ be the rotations about the $x$-axis, $y$-axis, and $z$-axis, respectively, by the angle $\pi$, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$B \mapsto C(B)$$
$$B \mapsto E_1C(B)$$
$$B \mapsto E_2C(B)$$
$$B \mapsto E_3C(B)$$

where $B$ is skew symmetric, are parametrizations of $\text{SO}(3)$ and that the union of the images of $C$, $E_1C$, $E_2C$ and $E_3C$ covers $\text{SO}(3)$, so that $\text{SO}(3)$ is a manifold.

(d) Let $A$ be any matrix (not necessarily invertible). Prove that there is some diagonal matrix, $E$, with entries $+1$ or $-1$, so that $EA + I$ is invertible.

(e) Prove that every rotation matrix, $A \in \text{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$
for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$, and where the number of $-1$ is even. Moreover, prove that every orthogonal matrix $A \in \text{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$. The above provide parametrizations for $\text{SO}(n)$ (resp. $\text{O}(n)$) that show that $\text{SO}(n)$ and $\text{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with $n$.

**Problem B2 (20 pts).**

1. For every symmetric, positive, definite matrix, $S$, and for every invertible matrix, $A$, prove that $ASA^\top$ is symmetric, positive, definite.

2. Prove that for any symmetric, positive, definite matrix, $S$, there is some symmetric, positive, definite matrix, $S_1$, so that $S = S_1^2 = S_1S_1^\top$.

3. Use (2) to prove that given any two symmetric, positive, definite matrices, $S$ and $S'$, there is some invertible matrix, $A$, so that

$$ASA^\top = S'.$$

Conclude that the action of $\text{GL}(n, \mathbb{R})$ on $\text{SPD}(n)$ given by $A \cdot S = ASA^\top$ is well-defined and transitive.

**Problem B3 (100 pts).** Consider the action of the group $\text{SL}(2, \mathbb{R})$ on the upper half-plane, $H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, given by

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \cdot z = \frac{az + b}{cz + d}.$$

(a) Check that for any $g \in \text{SL}(2, \mathbb{R})$,

$$\Im(g \cdot z) = \frac{\Im(z)}{|cz + d|^2},$$

and conclude that if $z \in H$, then $g \cdot z \in H$, so that the action of $\text{SL}(2, \mathbb{R})$ on $H$ is indeed well-defined (Recall, $\Re(z) = x$ and $\Im(z) = y$, where $z = x + iy$.)

(b) Check that if $c \neq 0$, then

$$\frac{az + b}{cz + d} = \frac{-1}{c^2z + cd} + \frac{a}{c}.$$

Prove that the group of Möbius transformations induced by $\text{SL}(2, \mathbb{R})$ is generated by Möbius transformations of the form

1. $z \mapsto z + b,$
2. \( z \mapsto kz \),
3. \( z \mapsto -1/z \),

where \( b \in \mathbb{R} \) and \( k \in \mathbb{R} \), with \( k > 0 \). Deduce from the above that the action of \( \text{SL}(2, \mathbb{R}) \) on \( H \) is transitive and that transformations of type (1) and (2) suffice for transitivity.

(c) Now, consider the action of the discrete group \( \text{SL}(2, \mathbb{Z}) \) on \( H \), where \( \text{SL}(2, \mathbb{Z}) \) consists of all matrices
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.
\]

Why is this action not transitive? Consider the two transformations
\[
S: z \mapsto -1/z
\]
asociated with \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and
\[
T: z \mapsto z + 1
\]
asociated with \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

Define the subset, \( D \), of \( H \), as the set of points, \( z = x + iy \), such that \( -1/2 \leq x \leq -1/2 \) and \( x^2 + y^2 \geq 1 \). Observe that \( D \) contains the three special points, \( i, \rho = e^{2\pi i/3} \) and \( -\rho = e^{\pi i/3} \).

Draw a picture of this set, known as a fundamental domain of the action of \( G = \text{SL}(2, \mathbb{Z}) \) on \( H \). You will now prove the following result first established by Gauss:

**Theorem I.** Let \( G' \) be the subgroup of \( G = \text{SL}(2, \mathbb{Z}) \) generated by \( S \) and \( T \).

1. For every point, \( z \in H \), there is some \( g \in G' \) so that \( g \cdot z \in D \).
2. If two distinct points \( z, z' \in D \) are in the same orbit under the action of \( G = \text{SL}(2, \mathbb{Z}) \), then either \( \Re(z) = \pm 1/2 \) and \( z = z' \pm 1 \) or \( |z| = 1 \) and \( z' = -1/z \).
3. Let \( z \in D \) and consider the stabilizer, \( G_z \) of \( z \) (under the action of \( G \)). Then, \( G_z = \{1\} \), unless
   1. \( z = i \), in which case, \( G_z \) is the group of order 2 generated by \( S \) (note, \( S^2 = 1 \))
   2. \( z = \rho = e^{2\pi i/3} \), in which case \( G_z \) is the group of order 3 generated by \( ST \) (composition is written from right to left, as usual, and note that \( (ST)^3 = 1 \))
   3. \( z = -\rho = e^{\pi i/3} \), in which case \( G_z \) is the group of order 3 generated by \( TS \) (note that \( (TS)^3 = 1 \)).
Deduce from Theorem I that the natural map $D \rightarrow H/G$ is surjective and that its restriction to the interior of $D$ is injective.

*Hints.* Observe that since $c,d$ are integers, for a fixed $z$, there are finitely many pairs $(c,d)$ so that $|cz + d| < K$, for any fixed $K > 0$. Thus, there is some $g \in G'$ so that $\Im(g \cdot z)$ is maximum. Next, show that there is some $n$ so that the real part of $T^n g z$ is between $-1/2$ and $1/2$. Show that this element, $z' = T^n g z$, is actually in $D$.

For (2) and (3), show that it may be assumed that $\Im(g \cdot z) \geq \Im(z)$, i.e., $|cz + d| \leq 1$.

(d) Use Theorem I to prove

**Theorem II.** The group $G = \text{SL}(2, \mathbb{Z})$ is generated by $S$ and $T$, i.e., $G' = G$.

(e) In view of Theorem I, as every point in the interior of $D$ corresponds to a unique orbit and every orbit has some representative in $D$, by applying all elements of $G = \text{SL}(2, \mathbb{Z})$ to $D$, we get a *tesselation* of $H$, i.e., we get

$$H = \bigcup_{g \in G} g \cdot D,$$

where the interiors of $g \cdot D$ and $g' \cdot D$ are disjoint whenever $g \cdot D$ and $g' \cdot D$ are distinct. By Theorem II, we get all $g \cdot D$’s by applying $S$ and $T$ to $D$. Draw the picture obtained by applying

$$1, T, TS, ST^{-1} S, ST, S, ST, STS, T^{-1} S, T^{-1}.$$

**Problem B4 (30 pts).** Let $J$ be the $2 \times 2$ matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let $\text{SU}(1, 1)$ be the set of $2 \times 2$ complex matrices

$$\text{SU}(1, 1) = \{ A \mid A^* J A = J, \quad \det(A) = 1 \},$$

where $A^*$ is the conjugate transpose of $A$.

(a) Prove that $\text{SU}(1, 1)$ is the group of matrices of the form

$$A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{with} \quad a \overline{a} - b \overline{b} = 1.$$

If

$$g = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

prove that the map from $\text{SL}(2, \mathbb{R})$ to $\text{SU}(1, 1)$ given by

$$A \mapsto gAg^{-1}$$

is...
is a group isomorphism.

(b) Prove that the Möbius transformation associated with \( g \),

\[
    z \mapsto \frac{z - i}{z + i}
\]
is a bijection between the upper half-plane, \( H \), and the unit open disk, \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Prove that the map from \( SU(1,1) \) to \( S^1 \times D \) given by

\[
    \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a/|a|, b/a)
\]
is a continuous bijection (in fact, a homeomorphism). Conclude that \( SU(1,1) \) is topologically an open solid torus.

(c) Check that \( SU(1,1) \) acts transitively on \( D \) by

\[
    \begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}.
\]

Find the stabilizer of \( z = 0 \) and conclude that

\[
    SU(1,1)/SO(2) \cong D.
\]

Problem B5 (100 pts). (a) Let \( U \subseteq \mathbb{R}^m \) be an open subset of \( \mathbb{R}^m \) and pick some \( a \in U \). If \( f: U \to \mathbb{R}^n \) is a submersion at \( a \), i.e., \( df_a \) is surjective (so, \( m \geq n \)), prove that there is an open set, \( W \subseteq U \subseteq \mathbb{R}^m \), with \( a \in W \) and a diffeomorphism, \( \psi \), with domain \( V \subseteq \mathbb{R}^m \), so that

\[
    \psi(V) = W \quad \text{and} \quad f(\psi(x_1, \ldots, x_m)) = (x_1, \ldots, x_n),
\]

for all \((x_1, \ldots, x_m) \in V\).

*Hint.* Since \( df_a \) is surjective, the rank of the Jacobian matrix, \((\partial f_i/\partial x_j)(a) \) (1 \( \leq i \leq n \), 1 \( \leq j \leq m \)), is \( n \) and after some permutation of \( \mathbb{R}^m \), we may assume that the square matrix, \( B = (\partial f_i/\partial x_j)(a) \) (1 \( \leq i, j \leq n \)), is invertible. Define the map, \( h: U \to \mathbb{R}^m \), by

\[
    h(x) = (f_1(x), \ldots, f_n(x), x_{n+1}, \ldots, x_m),
\]

where \( x = (x_1, \ldots, x_m) \). Check that the Jacobian matrix of \( h \) at \( a \) is invertible. Then, apply the inverse function theorem and finish up.

(b) Let \( f: M \to N \) be a map of smooth manifolds. A point, \( p \in M \), is called a **critical point (of \( f \))** iff \( df_p \) is not surjective and a point \( q \in N \) is called a **critical value (of \( f \))** iff \( q = f(p) \), for some critical point, \( p \in M \). A point \( p \in M \) is a **regular point (of \( f \))** iff \( p \) is not critical, i.e., \( df_p \) is surjective, and a point \( q \in N \) is a **regular value (of \( f \))** iff it is not a critical value. In particular, any \( q \in N - f(M) \) is a regular value and \( q \in f(M) \) is a regular
value iff every \( p \in f^{-1}(q) \) is a regular point (but, in contrast, \( q \) is a critical value iff some \( p \in f^{-1}(q) \) is critical).

Prove that for every regular value, \( q \in f(M) \), the preimage \( Z = f^{-1}(q) \) is a manifold of dimension \( \dim(M) - \dim(N) \).

*Hint.* Pick any \( p \in f^{-1}(q) \) and some parametrizations, \( \varphi \) at \( p \) and \( \psi \) at \( q \), with \( \varphi(0) = p \) and \( \psi(0) = q \) and consider \( h = \psi^{-1} \circ f \circ \varphi \). Prove that \( dh_0 \) is surjective and then apply (a).

(c) Under the same assumptions as (b), prove that for every point \( p \in Z = f^{-1}(q) \), the tangent space, \( T_pZ \), is the kernel of \( df_p: T_pM \to T_qN \).

(d) If \( X, Z \subseteq \mathbb{R}^N \) are manifolds and \( Z \subseteq X \), we say that \( Z \) is a *submanifold of \( X \). Assume there is a smooth function, \( g: X \to \mathbb{R}^k \), and that \( 0 \in \mathbb{R}^k \) is a regular value of \( g \). Then, by (b), \( Z = g^{-1}(0) \) is a submanifold of \( X \) of dimension \( \dim(X) - k \). Let \( g = (g_1, \ldots, g_k) \), with each \( g_i \) a function, \( g_i: X \to \mathbb{R} \). Prove that for any \( p \in X \), \( dg_p \) is surjective iff the linear forms, \( (dg_i)_p: T_pX \to \mathbb{R} \), are linearly independent. In this case, we say that \( g_1, \ldots, g_k \) are independent at \( p \). We also say that \( Z \) is cut out by \( g_1, \ldots, g_k \) when

\[
Z = \{ p \in X \mid g_1(p) = 0, \ldots, g_k(p) = 0 \}
\]

with \( g_1, \ldots, g_k \) independent for all \( p \in Z \).

Let \( f: X \to Y \) be a smooth maps of manifolds and let \( q \in f(X) \) be a regular value. Prove that \( Z = f^{-1}(q) \) is a submanifold of \( X \) cut out by \( k = \dim(X) - \dim(Y) \) independent functions.

*Hint.* Pick some parametrization, \( \psi \), at \( q \), so that \( \psi(0) = q \) and check that \( 0 \) is a regular value of \( g = \psi^{-1} \circ f \), so that \( g_1, \ldots, g_k \) work.

(e) Let \( U \subseteq \mathbb{R}^m \) be an open subset of \( \mathbb{R}^m \) and pick some \( a \in U \). If \( f: U \to \mathbb{R}^n \) is an immersion at \( a \), i.e., \( df_a \) is injective (so, \( m \leq n \)), prove that there is an open set, \( V \subseteq \mathbb{R}^n \), with \( f(a) \in V \), an open subset, \( U' \subseteq U \), with \( a \in U' \) and \( f(U') \subseteq V \) and a diffeomorphism, \( \varphi \), with domain \( V \), so that

\[
\varphi(f(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0),
\]

for all \( (x_1, \ldots, x_m) \in U' \).

*Hint.* Since \( df_a \) is injective, the rank of the Jacobian matrix, \( (\partial f_i/\partial x_j(a)) (1 \leq i \leq n, 1 \leq j \leq m) \), is \( m \) and after some permutation of \( \mathbb{R}^n \), we may assume that the square matrix, \( B = (\partial f_i/\partial x_j(a)) (1 \leq i, j \leq m) \), is invertible. Define the map, \( g: U \times \mathbb{R}^{n-m} \to \mathbb{R}^n \), by

\[
g(x, y) = (f_1(x), \ldots, f_m(x), y_1 + f_{m+1}(x), \ldots, y_{n-m} + f_n(x)),
\]

where \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_{n-m}) \). Check that the Jacobian matrix of \( g \) at \( (a, 0) \) is invertible. Then, apply the inverse function theorem and finish up.

Now, assume \( Z \) is a submanifold of \( X \). Prove that locally, \( Z \) is cut out by independent functions. This means that if \( k = \dim(X) - \dim(Z) \), the *codimension* of \( Z \) in \( X \), then for
every \( z \in Z \), there are \( k \) independent functions, \( g_1, \ldots, g_k \), defined on some open subset, \( W \subseteq X \), with \( z \in W \), so that \( Z \cap W \) is the common zero set of the \( g_i \)'s.

(f) We would like to generalize our result in (b) to the more general situation where we have a smooth map, \( f: X \to Y \), but this time, we have a submanifold, \( Z \subseteq Y \) and we are investigating whether \( f^{-1}(Z) \) is a submanifold of \( X \). In particular, if \( X \) is also a submanifold of \( Y \) and \( f \) is the inclusion of \( X \) into \( Y \), then \( f^{-1}(Z) = X \cap Z \).

Convince yourself that, in general, the intersection of two submanifolds is not a submanifold. Try examples involving curves and surfaces and you will see how bad the situation can be. What is needed is a notion generalizing that of a regular value, and this turns out to be the notion of transversality.

We say that \( f \) is transversal to \( Z \) iff
\[
df_p(T_pX) + T_f(p)Z = T_f(p)Y,
\]
for all \( p \in f^{-1}(Z) \). (Recall, if \( U \) and \( V \) are subspaces of a vector space, \( E \), then \( U + V \) is the subspace \( U + V = \{u + v \in E \mid u \in U, v \in V\} \).) In particular, if \( f \) is the inclusion of \( X \) into \( Y \), the transversality condition is
\[
T_pX + T_pZ = T_pY,
\]
for all \( p \in X \cap Z \).

Draw several examples of transversal intersections to understand better this concept. Prove that if \( f \) is transversal to \( Z \), then \( f^{-1}(Z) \) is a submanifold of \( X \) of codimension equal to \( \dim(Y) - \dim(Z) \).

Hint. The set \( f^{-1}(Z) \) is a manifold iff for every \( p \in f^{-1}(Z) \), there is some open subset, \( U \subseteq X \), with \( p \in U \), and \( f^{-1}(Z) \cap U \) is a manifold. First, use (e) to assert that locally near \( q = f(p) \), \( Z \) is cut out by \( k = \dim(Y) - \dim(Z) \) independent functions, \( g_1, \ldots, g_k \), so that locally near \( p \), the preimage \( f^{-1}(Z) \) is cut out by \( g_1 \circ f, \ldots, g_k \circ f \). If we let \( g = (g_1, \ldots, g_k) \), it is an immersion and the issue is to prove that \( 0 \) is a regular value of \( g \circ f \) in order to apply (b). Show that transversality is just what’s needed to show that \( 0 \) is a regular value of \( g \circ f \).

(g) With the same assumptions as in (f) (\( f \) is transversal to \( Z \)), if \( W = f^{-1}(Z) \), prove that for every \( p \in W \),
\[
T_pW = (df_p)^{-1}(T_{f(p)}Z),
\]
the preimage of \( T_{f(p)}Z \) by \( df_p: T_pX \to T_{f(p)}Y \). In particular, if \( f \) is the inclusion of \( X \) into \( Y \), then
\[
T_p(X \cap Z) = T_pX \cap T_pZ.
\]

(h) Let \( X, Z \subseteq Y \) be two submanifolds of \( Y \), with \( X \) compact, \( Z \) closed, \( \dim(X) + \dim(Z) = \dim(Y) \) and \( X \) transversal to \( Z \). Prove that \( X \cap Z \) consists of a finite set of points.

TOTAL: 310 points.