$\boxed{\textbf{Spring, 2005} \quad \textbf{CIS 610}}$

# Advanced Geometric Methods in Computer Science
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## Homework 3

March 16, 2005; Due April 6, 2005
Note: New due date!

"A problems" are for practice only, and should not be turned in.

**Problem A1.** Let $B_r = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\}$ be the open ball of radius $r$ (centered at the origin) in $\mathbb{R}^n$ (where $r > 0$). Prove that the map

$$x \mapsto \frac{rx}{\sqrt{r^2 - (x_1^2 + \cdots + x_n^2)}}$$

is a diffeomorphism of $B_r$ onto $\mathbb{R}^n$ (where $x = (x_1, \ldots, x_n)$).

*Hint*. Compute explicity the inverse of this map.

**Problem A2.** A smooth bijective map of manifolds need not be a diffeomorphism. For example, show that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is not a diffeomorphism.

**Problem A3.** (a) Let $X \subseteq \mathbb{R}^M$ and $Y \subseteq \mathbb{R}^N$ be two smooth manifolds of dimension $m$ and $n$ respectively. We can make $X \times Y \subseteq \mathbb{R}^{M+N}$ into a smooth manifold of dimension $m + n$ as follows: for any $(p, q) \in X \times Y$, if $\varphi\colon \Omega_1 \to U$ and $\psi\colon \Omega_2 \to V$ are parametrizations at $p \in U \subseteq X$ and $q \in V \subseteq Y$ respectively, then show that $\varphi \times \psi\colon \Omega_1 \times \Omega_2 \to U \times V$ is indeed a parametrization at $(p, q) \in X \times Y$. As the $U \times V$'s cover $X \times Y$, these parametrizations make $X \times Y$ into a manifold.

Check that $T_{(p,q)}(X \times Y) = T_pX \times T_qY$.

(b) Given a set, $X$, let $\Delta = \{(x, x) \mid x \in X\} \subseteq X \times X$, called the *diagonal of* $X$. If $X$ is a manifold, then prove that $\Delta$ is a manifold diffeomorphic to $X$.

(c) The *graph* of a function, $f\colon X \to Y$, is the subset of $X \times Y$ given by

$$\mathrm{graph}(f) = \{(x, f(x)) \mid x \in X\}.$$

Define $F\colon X \to \mathrm{graph}(f)$ by $F(x) = (x, f(x))$. Prove that if $X$ and $Y$ are smooth manifolds and if $f$ is smooth, then $F$ is a diffeomorphism and thus, $\mathrm{graph}(f)$ is a manifold diffeomorphic to $X$.

(d) Given any (smooth) map, $f\colon X \to X$, some $x \in X$ is a *fixed point* of $f$ iff $f(x) = x$. Prove that $f$ has a fixed point iff $\mathrm{graph}(f) \cap \Delta \neq \emptyset$ (where $\Delta$ is the diagonal in $X \times X$).

"B problems" must be turned in.

**Problem B1 (60 pts).** Recall from Homework 1, Problem B6, the Cayley parametrization of rotation matrices in $\mathbf{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where $B$ is any $n \times n$ skew symmetric matrix. In that problem, it was shown that $C(B)$ is a rotation matrix that does not admit $-1$ as an eigenvalue and that every such rotation matrix is of the form $C(B)$.

(a) If you have not already done so, prove that the map $B \mapsto C(B)$ is injective.

(b) Prove that

$$dC(B)(A) = D_A((I - B)(I + B)^{-1}) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1}.$$

*Hint*. First, show that $D_A(B^{-1}) = -B^{-1}AB^{-1}$ (where $B$ is invertible) and that $D_A(f(B)g(B)) = (D_Af(B))g(B) + f(B)(D_Ag(B))$, where $f$ and $g$ are differentiable matrix functions.

Deduce that $dC(B)$ is injective, for every skew-symmetric matrix, $B$. If we identify the space of $n \times n$ skew symmetric matrices with $\mathbb{R}^{n(n-1)/2}$, show that the Cayley map, $C\colon \mathbb{R}^{n(n-1)/2} \to \mathbf{SO}(n)$, is a parametrization of $\mathbf{SO}(n)$.

(c) Now, consider $n = 3$, i.e., $\mathbf{SO}(3)$. Let $E_1$, $E_2$ and $E_3$ be the rotations about the $x$-axis, $y$-axis, and $z$-axis, respectively, by the angle $\pi$, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$\begin{aligned} B &\mapsto C(B) \\ B &\mapsto E_1C(B) \\ B &\mapsto E_2C(B) \\ B &\mapsto E_3C(B) \end{aligned}$$

where $B$ is skew symmetric, are parametrizations of $\mathbf{SO}(3)$ and that the union of the images of $C$, $E_1C$, $E_2C$ and $E_3C$ covers $\mathbf{SO}(3)$, so that $\mathbf{SO}(3)$ is a manifold.

(d) Let $A$ be *any* matrix (not necessarily invertible). Prove that there is some diagonal matrix, $E$, with entries $+1$ or $-1$, so that $EA + I$ is invertible.

(e) Prove that every rotation matrix, $A \in \mathbf{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$, and where the number of $-1$ is even. Moreover, prove that every orthogonal matrix $A \in \mathbf{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$. The above provide parametrizations for $\mathbf{SO}(n)$ (resp. $\mathbf{O}(n)$) that show that $\mathbf{SO}(n)$ and $\mathbf{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with $n$.

**Problem B2 (20 pts).** (1) For every symmetric, positive, definite matrix, $S$, and for every invertible matrix, $A$, prove that $ASA^\top$ is symmetric, positive, definite.

(2) Prove that for any symmetric, positive, definite matrix, $S$, there is some symmetric, positive, definite matrix, $S_1$, so that $S = S_1^2 = S_1 S_1^\top$.

(3) Use (2) to prove that given any two symmetric, positive, definite matrices, $S$ and $S'$, there is some invertible matrix, $A$, so that

$$ASA^\top = S'.$$

Conclude that the action of $\mathbf{GL}(n, \mathbb{R})$ on $\mathbf{SPD}(n)$ given by $A \cdot S = ASA^\top$ is well-defined and transitive.

**Problem B3 (100 pts).** Consider the action of the group $\mathbf{SL}(2, \mathbb{R})$ on the upper half-plane, $H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

(a) Check that for any $g \in \mathbf{SL}(2, \mathbb{R})$,

$$\Im(g \cdot z) = \frac{\Im(z)}{|cz + d|^2},$$

and conclude that if $z \in H$, then $g \cdot z \in H$, so that the action of $\mathbf{SL}(2, \mathbb{R})$ on $H$ is indeed well-defined (Recall, $\Re(z) = x$ and $\Im(z) = y$, where $z = x + iy$.)

(b) Check that if $c \neq 0$, then

$$\frac{az + b}{cz + d} = \frac{-1}{c^2 z + cd} + \frac{a}{c}.$$

Prove that the group of Möbius transformations induced by $\mathbf{SL}(2, \mathbb{R})$ is generated by Möbius transformations of the form

1. $z \mapsto z + b$,

2. $z \mapsto kz$,

3. $z \mapsto -1/z$,

where $b \in \mathbb{R}$ and $k \in \mathbb{R}$, with $k > 0$. Deduce from the above that the action of $\mathbf{SL}(2, \mathbb{R})$ on $H$ is transitive and that transformations of type (1) and (2) suffice for transitivity.

(c) Now, consider the action of the discrete group $\mathbf{SL}(2, \mathbb{Z})$ on $H$, where $\mathbf{SL}(2, \mathbb{Z})$ consists of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.$$

Why is this action not transitive? Consider the two transformations

$$S\colon z \mapsto -1/z$$

associated with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$T\colon z \mapsto z + 1$$

associated with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Define the subset, $D$, of $H$, as the set of points, $z = x + iy$, such that $-1/2 \leq x \leq -1/2$ and $x^2 + y^2 \geq 1$. Observe that $D$ contains the three special points, $i$, $\rho = e^{2\pi i/3}$ and $-\overline{\rho} = e^{\pi i/3}$.

Draw a picture of this set, known as a *fundamental domain* of the action of $G = \mathbf{SL}(2, \mathbb{Z})$ on $H$. You will now prove the following result first established by Gauss:

**Theorem I**. Let $G'$ be the subgroup of $G = \mathbf{SL}(2, \mathbb{Z})$ generated by $S$ and $T$.

(1) For every point, $z \in H$, there is some $g \in G'$ so that $g \cdot z \in D$.

(2) If two distinct points $z, z' \in D$ are in the same orbit under the action of $G = \mathbf{SL}(2, \mathbb{Z})$, then either $\Re(z) = \pm 1/2$ and $z = z' \pm 1$ or $|z| = 1$ and $z' = -1/z$.

(3) Let $z \in D$ and consider the stabilizer, $G_z$ of $z$ (under the action of $G$). Then, $G_z = \{1\}$, unless

   (i) $z = i$, in which case, $G_z$ is the group of order 2 generated by $S$ (note, $S^2 = 1$)

   (ii) $z = \rho = e^{2\pi i/3}$, in which case $G_z$ is the group of order 3 generated by $ST$ (composition is written from right to left, as usual, and note that $(ST)^3 = 1$)

   (iii) $z = -\overline{\rho} = e^{\pi i/3}$, in which case $G_z$ is the group of order 3 generated by $TS$ (note that $(TS)^3 = 1$).

Deduce from Theorem I that the natural map $D \longrightarrow H/G$ is surjective and that its restriction to the interior of $D$ is injective.

*Hints*. Observe that since $c, d$ are integers, for a fixed $z$, there are finitely many pairs $(c, d)$ so that $|cz + d| < K$, for any fixed $K > 0$. Thus, there is some $g \in G'$ so that $\Im(g \cdot z)$ is maximum. Next, show that there is some $n$ so that the real part of $T^n gz$ is between $-1/2$ and $1/2$. Show that this element, $z' = T^n gz$, is actually in $D$.

For (2) and (3), show that it may be assumed that $\Im(g \cdot z) \geq \Im(z)$, i.e., $|cz + d| \leq 1$.

(d) Use Theorem I to prove

**Theorem II**. The group $G = \mathbf{SL}(2, \mathbb{Z})$ is generated by $S$ and $T$, i.e., $G' = G$.

(e) In view of Theorem I, as every point in the interior of $D$ corresponds to a unique orbit and every orbit has some representative in $D$, by applying all elements of $G = \mathbf{SL}(2, \mathbb{Z})$ to $D$, we get a *tesselation* of $H$, i.e., we get

$$H = \bigcup_{g \in G} g \cdot D,$$

where the interiors of $g \cdot D$ and $g' \cdot D$ are disjoint whenever $g \cdot D$ and $g' \cdot D$ are distinct. By Theorem II, we get all $g \cdot D$'s by applying $S$ and $T$ to $D$. Draw the picture obtained by applying

$$1, T, TS, ST^{-1}S, ST^{-1}, S, ST, STS, T^{-1}S, T^{-1}.$$

**Problem B4 (30 pts).** Let $J$ be the $2 \times 2$ matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let $\mathbf{SU}(1, 1)$ be the set of $2 \times 2$ complex matrices

$$\mathbf{SU}(1, 1) = \{A \mid A^* J A = J, \quad \det(A) = 1\},$$

where $A^*$ is the conjugate transpose of $A$.

(a) Prove that $\mathbf{SU}(1, 1)$ is the group of matrices of the form

$$A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{with} \quad a\overline{a} - b\overline{b} = 1.$$

If

$$g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

prove that the map from $\mathbf{SL}(2, \mathbb{R})$ to $\mathbf{SU}(1, 1)$ given by

$$A \mapsto gAg^{-1}$$

is a group isomorphism.

(b) Prove that the Möbius transformation associated with $g$,

$$z \mapsto \frac{z-i}{z+i}$$

is a bijection between the upper half-plane, $H$, and the unit open disk, $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Prove that the map from $\mathbf{SU}(1,1)$ to $S^1 \times D$ given by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto (a/|a|, b/a)$$

is a continuous bijection (in fact, a homeomorphism). Conclude that $\mathbf{SU}(1,1)$ is topologically an open solid torus.

(c) Check that $\mathbf{SU}(1,1)$ acts transitively on $D$ by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot z = \frac{az+b}{\overline{b}z+\overline{a}}.$$

Find the stabilizer of $z = 0$ and conclude that

$$\mathbf{SU}(1,1)/\mathbf{SO}(2) \cong D.$$

**Problem B5 (100 pts).** (a) Let $U \subseteq \mathbb{R}^m$ be an open subset of $\mathbb{R}^m$ and pick some $a \in U$. If $f \colon U \to \mathbb{R}^n$ is a submersion at $a$, i.e., $df_a$ is surjective (so, $m \geq n$), prove that there is an open set, $W \subseteq U \subseteq \mathbb{R}^m$, with $a \in W$ and a diffeomorphism, $\psi$, with domain $V \subseteq \mathbb{R}^m$, so that $\psi(V) = W$ and

$$f(\psi(x_1, \ldots, x_m)) = (x_1, \ldots, x_n),$$

for all $(x_1, \ldots, x_m) \in V$.

*Hint*. Since $df_a$ is surjective, the rank of the Jacobian matrix, $(\partial f_i/\partial x_j(a))$ $(1 \leq i \leq n, 1 \leq j \leq m)$, is $n$ and after some permutation of $\mathbb{R}^m$, we may assume that the square matrix, $B = (\partial f_i/\partial x_j(a))$ $(1 \leq i, j \leq n)$, is invertible. Define the map, $h \colon U \to \mathbb{R}^m$, by

$$h(x) = (f_1(x), \ldots, f_n(x), x_{n+1}, \ldots, x_m),$$

where $x = (x_1, \ldots, x_m)$. Check that the Jacobian matrix of $h$ at $a$ is invertible. Then, apply the inverse function theorem and finish up.

(b) Let $f \colon M \to N$ be a map of smooth manifolds. A point, $p \in M$, is called a *critical point (of $f$)* iff $df_p$ is *not* surjective and a point $q \in N$ is called a *critical value (of $f$)* iff $q = f(p)$, for some critical point, $p \in M$. A point $p \in M$ is a *regular point (of $f$)* iff $p$ is not critical, i.e., $df_p$ is surjective, and a point $q \in N$ is a *regular value (of $f$)* iff it is not a critical value. In particular, any $q \in N - f(M)$ is a regular value and $q \in f(M)$ is a regular value iff *every* $p \in f^{-1}(q)$ is a regular point (but, in contrast, $q$ is a critical value iff *some* $p \in f^{-1}(q)$ is critical).

Prove that for every regular value, $q \in f(M)$, the preimage $Z = f^{-1}(q)$ is a manifold of dimension $\dim(M) - \dim(N)$.

*Hint*. Pick any $p \in f^{-1}(q)$ and some parametrizations, $\varphi$ at $p$ and $\psi$ at $q$, with $\varphi(0) = p$ and $\psi(0) = q$ and consider $h = \psi^{-1} \circ f \circ \varphi$. Prove that $dh_0$ is surjective and then apply (a).

(c) Under the same assumptions as (b), prove that for every point $p \in Z = f^{-1}(q)$, the tangent space, $T_pZ$, is the kernel of $df_p \colon T_pM \to T_qN$.

(d) If $X, Z \subseteq \mathbb{R}^N$ are manifolds and $Z \subseteq X$, we say that $Z$ *is a submanifold of* $X$. Assume there is a smooth function, $g \colon X \to \mathbb{R}^k$, and that $0 \in \mathbb{R}^k$ is a regular value of $g$. Then, by (b), $Z = g^{-1}(0)$ is a submanifold of $X$ of dimension $\dim(X) - k$. Let $g = (g_1, \ldots, g_k)$, with each $g_i$ a function, $g_i \colon X \to \mathbb{R}$. Prove that for any $p \in X$, $dg_p$ is surjective iff the linear forms, $(dg_i)_p \colon T_pX \to \mathbb{R}$, are linearly independent. In this case, we say that $g_1, \ldots, g_k$ are *independent at* $p$. We also say that $Z$ is *cut out* by $g_1, \ldots, g_k$ when

$$Z = \{p \in X \mid g_1(p) = 0, \ldots, g_k(p) = 0\}$$

with $g_1, \ldots, g_k$ independent for all $p \in Z$.

Let $f \colon X \to Y$ be a smooth maps of manifolds and let $q \in f(X)$ be a regular value. Prove that $Z = f^{-1}(q)$ is a submanifold of $X$ cut out by $k = \dim(X) - \dim(Y)$ independent functions.

*Hint*. Pick some parametrization, $\psi$, at $q$, so that $\psi(0) = q$ and check that 0 is a regular value of $g = \psi^{-1} \circ f$, so that $g_1, \ldots, g_k$ work.

(e) Let $U \subseteq \mathbb{R}^m$ be an open subset of $\mathbb{R}^m$ and pick some $a \in U$. If $f \colon U \to \mathbb{R}^n$ is an immersion at $a$, i.e., $df_a$ is injective (so, $m \leq n$), prove that there is an open set, $V \subseteq \mathbb{R}^n$, with $f(a) \in V$, an open subset, $U' \subseteq U$, with $a \in U'$ and $f(U') \subseteq V$ and a diffeomorphism, $\varphi$, with domain $V$, so that

$$\varphi(f(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0),$$

for all $(x_1, \ldots, x_m) \in U'$.

*Hint*. Since $df_a$ is injective, the rank of the Jacobian matrix, $(\partial f_i / \partial x_j(a))$ $(1 \leq i \leq n,$ $1 \leq j \leq m)$, is $m$ and after some permutation of $\mathbb{R}^n$, we may assume that the square matrix, $B = (\partial f_i / \partial x_j(a))$ $(1 \leq i, j \leq m)$, is invertible. Define the map, $g \colon U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$, by

$$g(x, y) = (f_1(x), \ldots, f_m(x), y_1 + f_{m+1}(x), \ldots, y_{n-m} + f_n(x)),$$

where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_{n-m})$. Check that the Jacobian matrix of $g$ at $(a, 0)$ is invertible. Then, apply the inverse function theorem and finish up.

Now, assume $Z$ is a submanifold of $X$. Prove that locally, $Z$ is cut out by independent functions. This means that if $k = \dim(X) - \dim(Z)$, the *codimension* of $Z$ in $X$, then for every $z \in Z$, there are $k$ independent functions, $g_1, \ldots, g_k$, defined on some open subset, $W \subseteq X$, with $z \in W$, so that $Z \cap W$ is the common zero set of the $g_i$'s.

(f) We would like to generalize our result in (b) to the more general situation where we have a smooth map, $f\colon X \to Y$, but this time, we have a submanifold, $Z \subseteq Y$ and we are investigating whether $f^{-1}(Z)$ is a submanifold of $X$. In particular, if $X$ is also a submanifold of $Y$ and $f$ is the inclusion of $X$ into $Y$, then $f^{-1}(Z) = X \cap Z$.

Convince yourself that, in general, the intersection of two submanifolds is *not* a submanifold. Try examples involving curves and surfaces and you will see how bad the situation can be. What is needed is a notion generalizing that of a regular value, and this turns out to be the notion of transversality.

We say that $f$ *is transveral to* $Z$ iff

$$df_p(T_pX) + T_{f(p)}Z = T_{f(p)}Y,$$

for all $p \in f^{-1}(Z)$. (Recall, if $U$ and $V$ are subspaces of a vector space, $E$, then $U + V$ is the subspace $U + V = \{u + v \in E \mid u \in U,\, v \in V\}$). In particular, if $f$ is the inclusion of $X$ into $Y$, the transversality condition is

$$T_pX + T_pZ = T_pY,$$

for all $p \in X \cap Z$.

Draw several examples of transversal intersections to understand better this concept. Prove that if $f$ is transversal to $Z$, then $f^{-1}(Z)$ is a submanifold of $X$ of codimension equal to $\dim(Y) - \dim(Z)$.

*Hint*. The set $f^{-1}(Z)$ is a manifold iff for every $p \in f^{-1}(Z)$, there is some open subset, $U \subseteq X$, with $p \in U$, and $f^{-1}(Z) \cap U$ is a manifold. First, use (e) to assert that locally near $q = f(p)$, $Z$ is cut out by $k = \dim(Y) - \dim(Z)$ independent functions, $g_1, \ldots, g_k$, so that locally near $p$, the preimage $f^{-1}(Z)$ is cut out by $g_1 \circ f, \ldots, g_k \circ f$. If we let $g = (g_1, \ldots, g_k)$, it is an immersion and the issue is to prove that 0 is a regular value of $g \circ f$ in order to apply (b). Show that transversality is just what's needed to show that 0 is a regular value of $g \circ f$.

(g) With the same assumptions as in (f) ($f$ is transversal to $Z$), if $W = f^{-1}(Z)$, prove that for every $p \in W$,

$$T_pW = (df_p)^{-1}(T_{f(p)}Z),$$

the preimage of $T_{f(p)}Z$ by $df_p\colon T_pX \to T_{f(p)}Y$. In particular, if $f$ is the inclusion of $X$ into $Y$, then

$$T_p(X \cap Z) = T_pX \cap T_pZ.$$

(h) Let $X, Z \subseteq Y$ be two submanifolds of $Y$, with $X$ compact, $Z$ closed, $\dim(X) + \dim(Z) = \dim(Y)$ and $X$ transversal to $Z$. Prove that $X \cap Z$ consists of a finite set of points.

**TOTAL: 310 points.**