Spring, 2005 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 2

February 23, 2005; Due March 16, 2005
(Note the new due date!)

"A problems" are for practice only, and should not be turned in.

Problem A1. (a) Find two symmetric matrices, A and B, such that AB is not symmetric.

(b) Find two matrices, A and B, such that

$$e^A e^B \neq e^{A+B}.$$

Try

$$A = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Problem A2. (a) If $K = \mathbb{R}$ or $K = \mathbb{C}$, recall that the projective space, $\mathbf{P}(K^{n+1})$, is the set of equivalence classes of the equivalence relation, $\sim$, on $K^{n+1} - \{0\}$, defined so that, for all $u, v \in K^{n+1} - \{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \in K - \{0\}.$$

The map, $p\colon (K^{n+1} - \{0\}) \to \mathbf{P}(K^{n+1})$, is the projection mapping any nonzero vector in K^{n+1} to its equivalence class modulo $\sim$. We let $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$ and $\mathbb{CP}^n = \mathbf{P}(\mathbb{C}^{n+1})$.

Prove that for any $n \geq 0$, there is a bijection between $\mathbf{P}(K^{n+1})$ and $K^n \cup \mathbf{P}(K^n)$ (which allows us to identify them).

(b) Prove that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are connected and compact.

Hint. If

$$S^n = \{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

prove that $p(S^n) = \mathbf{P}(K^{n+1})$, and recall that S^n is compact for all $n \geq 0$ and connected for $n \geq 1$. For $n = 0$, $\mathbf{P}(K)$ consists of a single point.

Problem A3. Recall that $\mathbb{R}^2$ and $\mathbb{C}$ can be identified using the bijection $(x, y) \mapsto x + iy$. Also recall that the subset $U(1) \subseteq \mathbb{C}$ consisting of all complex numbers of the form $\cos\theta + i\sin\theta$

is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If $c\colon U(1) \to U(1)$ is the map defined such that

$$c(z) = z^2,$$

prove that $c(z_1) = c(z_2)$ iff either $z_2 = z_1$ or $z_2 = -z_1$, and thus that c induces a bijective map $\widehat{c}\colon \mathbb{RP}^1 \to S^1$. Prove that $\widehat{c}$ is a homeomorphism (remember that $\mathbb{RP}^1$ is compact).

"B problems" must be turned in.

Problem B1 (20 pts). Let $A = (a_{i\,j})$ be a real or complex $n \times n$ matrix.

(1) If λ is an eigenvalue of A, prove that there is some eigenvector $u = (u_1, \ldots, u_n)$ of A for λ such that

$$\max_{1 \leq i \leq n} |u_i| = 1.$$

(2) If $u = (u_1, \ldots, u_n)$ is an eigenvector of A for λ as in (1), assuming that i, $1 \leq i \leq n$, is an index such that $|u_i| = 1$, prove that

$$(\lambda - a_{i\,i})u_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{i\,j} u_j,$$

and thus that

$$|\lambda - a_{i\,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i\,j}|.$$

Conclude that the eigenvalues of A are inside the union of the closed disks D_i defined such that

$$D_i = \Big\{ z \in \mathbb{C} \mid |z - a_{i\,i}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i\,j}| \Big\}.$$

Remark: This result is known as *Gershgorin's theorem*.

Problem B2 (10). Recall that a real $n \times n$ symmetric matrix, A, is *positive semi-definite* iff its eigenvalues, $\lambda_1, \ldots, \lambda_n$ are non-negative (i.e., $\lambda_i \geq 0$ for $i = 1, \ldots, n$) and *positive definite* iff its eigenvalues are positive (i.e., $\lambda_i > 0$ for $i = 1, \ldots, n$).

(a) Prove that a symmetric matrix, A, is positive semi-definite iff $X^\top A X \geq 0$, for all $X \neq 0$ ($X \in \mathbb{R}^n$) and positive definite iff $X^\top A X > 0$, for all $X \neq 0$ ($X \in \mathbb{R}^n$).

(b) Prove that for any two positive definite matrices, A, B, for all $\lambda, \mu \in \mathbb{R}$, with $\lambda, \mu \geq 0$ and $\lambda + \mu > 0$, the matrix $\lambda A + \mu B$ is still symmetric, positive definite. Deduce that the set of $n \times n$ symmetric positive definite matrices is convex (in fact, a cone).

Problem B3 (40). (i) In $\mathbb{R}^3$, the sphere S^2 is the set of points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$. The point $N = (0, 0, 1)$ is called the *north pole*, and the point $S = (0, 0, -1)$ is called the *south pole*. The *stereographic projection map* $\sigma_N \colon (S^2 - \{N\}) \to \mathbb{R}^2$ is defined as follows: For every point $M \neq N$ on S^2, the point $\sigma_N(M)$ is the intersection of the line through N and M and the plane of equation $z = 0$. Show that if M has coordinates (x, y, z) (with $x^2 + y^2 + z^2 = 1$), then

$$\sigma_N(M) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big).$$

Prove that σ_N is bijective and that its inverse is given by the map $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$, with

$$(x, y) \mapsto \Big(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\Big).$$

Similarly, $\sigma_S \colon (S^2 - \{S\}) \to \mathbb{R}^2$ is defined as follows: For every point $M \neq S$ on S^2, the point $\sigma_S(M)$ is the intersection of the line through S and M and the plane of equation $z = 0$. Show that

$$\sigma_S(M) = \Big(\frac{x}{1+z}, \frac{y}{1+z}\Big).$$

Prove that σ_S is bijective and that its inverse is given by the map $\tau_S \colon \mathbb{R}^2 \to (S^2 - \{S\})$, with

$$(x, y) \mapsto \Big(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1}\Big).$$

Using the complex number $u = x + iy$ to represent the point (x, y), the maps $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$ and $\sigma_N \colon (S^2 - \{N\}) \to \mathbb{R}^2$ can be viewed as maps from $\mathbb{C}$ to $(S^2 - \{N\})$ and from $(S^2 - \{N\})$ to $\mathbb{C}$, defined such that

$$\tau_N(u) = \Big(\frac{2u}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1}\Big)$$

and

$$\sigma_N(u, z) = \frac{u}{1-z},$$

and similarly for τ_S and σ_S. Prove that if we pick two suitable orientations for the xy-plane, we have

$$\sigma_N(M)\sigma_S(M) = 1,$$

for every $M \in S^2 - \{N, S\}$.

(ii) Identifying $\mathbb{C}^2$ and $\mathbb{R}^4$, for $z = x + iy$ and $z' = x' + iy'$, we define

$$\|(z, z')\| = \sqrt{x^2 + y^2 + x'^2 + y'^2}.$$

The sphere S^3 is the subset of $\mathbb{C}^2$ (or $\mathbb{R}^4$) consisting of those points (z, z') such that $\|(z, z')\|^2 = 1$.

Prove that $\mathbf{P}(\mathbb{C}^2) = p(S^3)$, where $p\colon (\mathbb{C}^2 - \{(0,0)\}) \to \mathbf{P}(\mathbb{C}^2)$ is the projection map. If we let $u = z/z'$ (where $z, z' \in \mathbb{C}$) in the map

$$u \mapsto \Big(\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\Big)$$

and require that $\|(z,z')\|^2 = 1$, show that we get the map $HF\colon S^3 \to S^2$ defined such that

$$HF((z,z')) = (2z\overline{z'},\ |z|^2 - |z'|^2).$$

Prove that $HF\colon S^3 \to S^2$ induces a bijection $\widehat{HF}\colon \mathbf{P}(\mathbb{C}^2) \to S^2$, and thus that $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$ is homeomorphic to S^2.

(iii) Prove that the inverse image $HF^{-1}(s)$ of every point $s \in S^2$ is a circle. Thus S^3 can be viewed as a union of disjoint circles. The map HF is called the *Hopf fibration*.

Problem B4 (60). (a) Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x,y,z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x,y,z) = \mathcal{H}(x',y',z')$ iff $(x',y',z') = (x,y,z)$ or $(x',y',z') = (-x,-y,-z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\begin{aligned}
\psi_1(u,v) &= \left(\frac{uv}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1}\right),\\
\psi_2(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{u^2-1}{u^2+v^2+1}\right),\\
\psi_3(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1-u^2}{u^2+v^2+1}\right).
\end{aligned}$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}\right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian is never zero).

Prove that $\psi_i(\mathbb{R}^2)$ is an open subset, U_i, of $\mathcal{H}(S^2)$ for $i = 1, 2, 3$ and that the union of the U_i's covers $\mathcal{H}(S^2)$.

Prove that each $\psi_i^{-1} \colon U_i \to \mathbb{R}^2$ is continuous. This is a little tricky. For example, for ψ_1, first prove that if the coordinates in $\mathbb{R}^4$ are (x, y, z, t), then

$$yzt = x(y^2 - z^2).$$

Then, ψ_1^{-1} is defined as follows: If $y \neq 0$ and $z \neq 0$,

$$u = \frac{x}{z} = \frac{yt}{y^2 - z^2}, \quad v = \frac{x}{y} = \frac{zt}{y^2 - z^2}.$$

If $y = 0$ and $z \neq 0$, then

$$u = 0, \quad v = -\frac{t}{z},$$

if $y \neq 0$ and $z = 0$, then

$$u = \frac{t}{y} \quad v = 0,$$

and if $y = z = 0$, then

$$u = 0, \quad v = 0.$$

Finally, you have to show continuity of the above functions, and do a similar thing for ψ_2^{-1} and ψ_3^{-1}.

Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a manifold in $\mathbb{R}^4$.

(c) Investigate the surfaces in $\mathbb{R}^3$ obtained by dropping one of the four coordinates. Show that there are only two of them (the "Steiner Roman surface" and the "crosscap", up to a rigid motion).

Problem B5 (20). (a) Let A be any invertible (real) $n \times n$ matrix. Prove that for every SVD, $A = VDU^\top$, of A, the product $VU^\top$ is the same (i.e., if $V_1 D U_1^\top = V_2 D U_2^\top$, then $V_1 U_1^\top = V_2 U_2^\top$). What does $VU^\top$ have to do with the polar form of A?

(b) Given any invertible (real) $n \times n$ matrix, A, prove that there is a unique orthogonal matrix, $Q \in \mathbf{O}(n)$, such that $\|A - Q\|_F$ is minimal (under the Frobenius norm). In fact, prove that $Q = VU^\top$, where $A = VDU^\top$ is an SVD of A. Moreover, if $\det(A) > 0$, show that $Q \in \mathbf{SO}(n)$.

What can you say if A is singular (i.e., non-invertible)?

Problem B6 (40). (a) Consider the map, $f\colon \mathbf{GL}^+(n) \to \mathbf{S}(n)$, given by

$$f(A) = A^\top A - I.$$

Check that

$$df(A)(H) = A^\top H + H^\top A,$$

for any matrix, H.

(b) Consider the map, $f\colon \mathbf{GL}(n) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of B, for any matrix B (here, I is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n)$.

(c) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n)$ is a manifold of dimension $n^2 - 1$.

(d) Let J be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n, 1)$ the group of real $(n+1) \times (n+1)$ matrices

$$\mathbf{SO}(n, 1) = \{A \in \mathbf{GL}(n+1) \mid A^\top J A = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n, 1)$ is indeed a group with the inverse of A given by $A^{-1} = JA^\top J$ (this is the *special Lorentz group*.) Consider the function $f\colon \mathbf{GL}^+(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^\top J A - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1) \times (n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top J H + H^\top J A$$

for any matrix, H. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n, 1)$ and that $\mathbf{SO}(n, 1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

Problem B7 (40 pts). (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2\cosh\omega$.

Prove that the exponential map, $\exp\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2,\mathbb{C})$.

(b) Recall that a matrix, N, is *nilpotent* iff there is some $m \geq 0$ so that $N^m = 0$. Let A be any $n \times n$ matrix of the form $A = I - N$, where N is nilpotent. Why is A invertible? prove that there is some B so that $e^B = I - N$ as follows: Recall that for any $y \in \mathbb{R}$ so that $|y - 1|$ is small enough, we have

$$\log(y) = -(1-y) - \frac{(1-y)^2}{2} - \cdots - \frac{(1-y)^k}{k} - \cdots.$$

As N is nilpotent, we have $N^m = 0$, where m is the smallest integer with this propery. Then, the expression

$$B = \log(I - N) = -N - \frac{N^2}{2} - \cdots - \frac{N^{m-1}}{m-1}$$

is well defined. Use a formal power series argument to show that

$$e^B = A.$$

We denote B by $\log(A)$.

(c) Let $A \in \mathbf{GL}(n,\mathbb{C})$. Prove that there is some matrix, B, so that $e^B = A$. Thus, the exponential map, $\exp\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathbf{GL}(n,\mathbb{C})$, is surjective.

First, use the fact that A has a Jordan form, PJP^{-1}. Then, show that finding a log of A reduces to finding a log of every Jordan block of J. As every Jordan block, J, has a fixed nonzero constant, λ, on the diagonal, with 1's immediately above each diagonal entry and zero's everywhere else, we can write J as $(\lambda I)(I - N)$, where N is niplotent. Find B_1 and B_2 so that $\lambda I = e^{B_1}$, $I - N = e^{B_2}$, and $B_1 B_2 = B_2 B_1$. Conclude that $J = e^{B_1 + B_2}$.

Problem B8 (50 pts). Recall that for any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

then the exponential map, $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$, is given by

$$\exp A = e^A = \cos\theta \, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2,$$

if $\theta \neq k2\pi$ ($k \in \mathbb{Z}$), with $\exp(0_3) = I_3$ (Rodrigues's formula (1840))

(a) Let $R \in \mathbf{SO}(3)$ and assume that $R \neq I$ and $\mathrm{tr}(R) \neq -1$. Then, prove that a log of R (i.e., a skew symmetric matrix, S, so that $e^S = R$) is given by

$$\log(R) = \frac{\theta}{2\sin\theta}(R - R^T),$$

where $1 + 2\cos\theta = \mathrm{tr}(R)$ and $0 < \theta < \pi$.

(b) Now, assume that $\mathrm{tr}(R) = -1$. In this case, show that R is a rotation of angle π, that R is symmetric and has eigenvalues, $-1, -1, 1$. Assuming that $e^A = R$, Rodrigues formula becomes

$$R = I + \frac{2}{\pi^2} A^2,$$

so

$$A^2 = \frac{\pi^2}{2}(R - I).$$

If we let $S = A/\pi$, we see that we need to find a skew-symmetric matrix, S, so that

$$S^2 = \frac{1}{2}(R - I) = C.$$

Observe that C is also symmetric and has eigenvalues, $-1, -1, 0$. Thus, we can diagonalize C, as

$$C = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^\top,$$

and if we let

$$S = P \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^\top,$$

check that $S^2 = C$.

(c) From (a) and (b), we know that we can compute explicity a log of a rotation matrix, although when $\theta \approx 0$, we have to be careful in computing $\frac{\sin\theta}{\theta}$; in this case, we may want to use

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots.$$

Given two rotations, $R_1, R_2 \in \mathbf{SO}(3)$, there are three natural interpolation formulae:

$$e^{(1-t)\log R_1 + t\log R_2}; \quad R_1 e^{t\log(R_1^\top R_2)}; \quad e^{t\log(R_2 R_1^\top)} R_1,$$

with $0 \leq t \leq 1$.

Write a computer program to investigate the difference between these interpolation formulae. The position of a rigid body spinning around its center of gravity is determined by a rotation matrix, $R \in \mathbf{SO}(3)$. If R_1 denotes the initial position and R_2 the final position of this rigid body, by computing interpolants of R_1 and R_2, we get a motion of the rigid body and we can create an animation of this motion by displaying several interpolants. The rigid body can be a "funny" object, for example a banana, a bottle, etc.

TOTAL: 280 points.