Fall, 2003 CIS 610

Advanced geometric methods

Homework 4

November 25, 2003; Due December 11, beginning of class

You may work in groups of 2 or 3. Please, write up your solutions as clearly and concisely as possible. Be rigorous! You will have to present your solutions of the problems during a special problem session.

"B problems" must be turned in.

Problem B1 (50 pts). The purpose of this problem is to prove that given any self-adjoint linear map $f\colon E \to E$ (i.e., such that $f^* = f$), where E is a Euclidean space of dimension $n \geq 3$, given an orthonormal basis $(e_1, \ldots, e_n)$, there are $n-2$ isometries h_i, hyperplane reflections or the identity, such that the matrix of

$$h_{n-2} \circ \cdots \circ h_1 \circ f \circ h_1 \circ \cdots \circ h_{n-2}$$

is a symmetric tridiagonal matrix.

(1) Prove that for any isometry $f\colon E \to E$, we have $f = f^* = f^{-1}$ iff $f \circ f = \mathrm{id}$.

Prove that if f and h are self-adjoint linear maps ($f^* = f$ and $h^* = h$), then $h \circ f \circ h$ is a self-adjoint linear map.

(2) Proceed by induction, taking inspiration from the proof of the triangular decomposition given in the class notes. Let V_k be the subspace spanned by $(e_{k+1}, \ldots, e_n)$. For the base case, proceed as follows.

Let

$$f(e_1) = a_1^0 e_1 + \cdots + a_n^0 e_n,$$

and let

$$r_{1,2} = \left\| a_2^0 e_2 + \cdots + a_n^0 e_n \right\|.$$

Find an isometry h_1 (reflection or id) such that

$$h_1(f(e_1) - a_1^0 e_1) = r_{1,2}\, e_2.$$

Observe that

$$w_1 = r_{1,2}\, e_2 + a_1^0 e_1 - f(e_1) \in V_1,$$

and prove that $h_1(e_1) = e_1$, so that

$$h_1 \circ f \circ h_1(e_1) = a_1^0 e_1 + r_{1,2}\, e_2.$$

Let $f_1 = h_1 \circ f \circ h_1$.

Assuming by induction that

$$f_k = h_k \circ \cdots \circ h_1 \circ f \circ h_1 \circ \cdots \circ h_k$$

has a tridiagonal matrix up to the kth row and column, $1 \leq k \leq n-3$, let

$$f_k(e_{k+1}) = a_k^k e_k + a_{k+1}^k e_{k+1} + \cdots + a_n^k e_n,$$

and let

$$r_{k+1,\,k+2} = \left\| a_{k+2}^k e_{k+2} + \cdots + a_n^k e_n \right\|.$$

Find an isometry h_{k+1} (reflection or id) such that

$$h_{k+1}(f_k(e_{k+1}) - a_k^k e_k - a_{k+1}^k e_{k+1}) = r_{k+1,\,k+2}\, e_{k+2}.$$

Observe that

$$w_{k+1} = r_{k+1,\,k+2}\, e_{k+2} + a_k^k e_k + a_{k+1}^k e_{k+1} - f_k(e_{k+1}) \in V_{k+1},$$

and prove that $h_{k+1}(e_k) = e_k$ and $h_{k+1}(e_{k+1}) = e_{k+1}$, so that

$$h_{k+1} \circ f_k \circ h_{k+1}(e_{k+1}) = a_k^k e_k + a_{k+1}^k e_{k+1} + r_{k+1,\,k+2}\, e_{k+2}.$$

Let $f_{k+1} = h_{k+1} \circ f_k \circ h_{k+1}$, and finish the proof.

Do f and f_{n-2} have the same eigenvalues? If so, explain why.

(3) Prove that given any symmetric $n \times n$-matrix A, there are $n-2$ matrices $H_1, \ldots, H_{n-2}$, Householder matrices or the identity, such that

$$B = H_{n-2} \cdots H_1 A H_1 \cdots H_{n-2}$$

is a symmetric tridiagonal matrix.

Problem B2 (40 pts).

Write a computer program implementing the method of problem 1(3).

Problem B3 (40 pts).

Let A be a symmetric tridiagonal $n \times n$-matrix

$$A = \begin{pmatrix} b_1 & c_1 & & & & \\ c_1 & b_2 & c_2 & & & \\ & c_2 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-2} & b_{n-1} & c_{n-1} \\ & & & & c_{n-1} & b_n \end{pmatrix}$$

where it is assumed that $c_i \neq 0$ for all i, $1 \leq i \leq n-1$, and let A_k be the $k \times k$-submatrix consisting of the first k rows and columns of A, $1 \leq k \leq n$. We define the polynomials $P_k(x)$ as follows ($0 \leq k \leq n$).

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= b_1 - x, \\
P_k(x) &= (b_k - x)P_{k-1}(x) - c_{k-1}^2 P_{k-2}(x),
\end{aligned}$$

where $2 \leq k \leq n$.

(1) Prove the following properties:

(i) $P_k(x)$ is the characteristic polynomial of A_k, where $1 \leq k \leq n$.

(ii) $\lim_{x \to -\infty} P_k(x) = +\infty$, where $1 \leq k \leq n$.

(iii) If $P_k(x) = 0$, then $P_{k-1}(x)P_{k+1}(x) < 0$, where $1 \leq k \leq n-1$.

(iv) $P_k(x)$ has k distinct real roots that separate the $k+1$ roots of P_{k+1}, where $1 \leq k \leq n-1$.

(2) (**Extra Credit 20 pts**) Given any real number $\mu > 0$, for every k, $1 \leq k \leq n$, define the function $sg_k(\mu)$ as follows:

$$sg_k(\mu) = \begin{cases} \text{sign of } P_k(\mu) & \text{if } P_k(\mu) \neq 0, \\ \text{sign of } P_{k-1}(\mu) & \text{if } P_k(\mu) = 0. \end{cases}$$

We encode the sign of a positive number as $+$, and the sign of a negative number as $-$. Then, let $E(k, \mu)$ be the ordered list

$$E(k, \mu) = \langle +,\ sg_1(\mu),\ sg_2(\mu),\ \ldots,\ sg_k(\mu) \rangle,$$

and let $N(k, \mu)$ be the number changes of sign between consecutive signs in $E(k, \mu)$.

Prove that $sg_k(\mu)$ is well defined, and that $N(k, \mu)$ is the number of roots λ of $P_k(x)$ such that $\lambda < \mu$.

Remark: The above can be used to compute the eigenvalues of a (tridiagonal) symmetric matrix (the method of Givens-Householder).

Problem B4 (50 pts). Let A and B be the following 4×4-matrices:

$$A = \begin{pmatrix} 0 & -\theta_1 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ 0 & 0 & \theta_2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

where $\theta_1, \theta_2 \geq 0$.

(i) Compute A^2, and prove that

$$B = e^A,$$

where

$$e^A = I_n + \sum_{p\geq 1} \frac{A^p}{p!} = \sum_{p\geq 0} \frac{A^p}{p!},$$

letting $A^0 = I_n$. Use this to prove that for every orthogonal 4×4-matrix B, there is a skew symmetric matrix A such that

$$B = e^A.$$

(ii) Given a skew-symmetric 4×4-matrix A, prove that there are two skew symmetric matrices A_1 and A_2 and some $\theta_1, \theta_2 \geq 0$, such that

$$\begin{aligned} A &= A_1 + A_2,\\ A_1^3 &= -\theta_1^2 A_1,\\ A_2^3 &= -\theta_2^2 A_2,\\ A_1A_2 &= A_2A_1 = 0,\\ tr(A_1^2) &= -2\theta_1^2,\\ tr(A_2^2) &= -2\theta_2^2, \end{aligned}$$

and where $A_i = 0$ if $\theta_i = 0$ and $A_1^2 + A_2^2 = -\theta_1^2 I_4$ if $\theta_2 = \theta_1$.

Using the above, prove that

$$e^A = I_4 + \frac{\sin\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2}{\theta_2}A_2 + \frac{(1-\cos\theta_1)}{\theta_1^2}A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2}A_2^2.$$

(iii) Given an orthogonal 4×4-matrix B, prove that there are two skew symmetric matrices A_1 and A_2 and some $\theta_1, \theta_2 \geq 0$, such that

$$B = I_4 + \frac{\sin\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2}{\theta_2}A_2 + \frac{(1-\cos\theta_1)}{\theta_1^2}A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2}A_2^2.$$

where

$$\begin{aligned} A_1^3 &= -\theta_1^2 A_1,\\ A_2^3 &= -\theta_2^2 A_2,\\ A_1A_2 &= A_2A_1 = 0,\\ tr(A_1^2) &= -2\theta_1^2,\\ tr(A_2^2) &= -2\theta_2^2, \end{aligned}$$

and where $A_i = 0$ if $\theta_i = 0$ and $A_1^2 + A_2^2 = -\theta_1^2 I_4$ if $\theta_2 = \theta_1$. Prove that

$$\begin{aligned}
1/2(B - B^\top) &= \frac{\sin\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2}{\theta_2}A_2, \\
1/2(B + B^\top) &= I_4 + \frac{(1-\cos\theta_1)}{\theta_1^2}A_1^2 + \frac{(1-\cos\theta_2)}{\theta_2^2}A_2^2, \\
tr(B) &= 2\cos\theta_1 + 2\cos\theta_2.
\end{aligned}$$

(iv) Prove that if $\sin\theta_1 = 0$ or $\sin\theta_2 = 0$, then A_1, A_2 and the $\cos\theta_i$ can be computed from B. Prove that if $\theta_2 = \theta_1$, then

$$B = \cos\theta_1 I_4 + \frac{\sin\theta_1}{\theta_1}(A_1 + A_2),$$

and $\cos\theta_1$ and $A_1 + A_2$ can be computed from B.

(v) Prove that

$$\frac{1}{4}tr((B - B^\top)^2) = 2\cos^2\theta_1 + 2\cos^2\theta_2 - 4.$$

Prove that $\cos\theta_1$ and $\cos\theta_2$ are solutions of the equation

$$x^2 - sx + p = 0,$$

where

$$s = \frac{1}{2}tr(B), \quad p = \frac{1}{8}\left(tr(B)\right)^2 - \frac{1}{16}tr((B - B^\top)^2) - 1.$$

Prove that we also have

$$\cos^2\theta_1\cos^2\theta_2 = \det\left(1/2(B + B^\top)\right).$$

If $\sin\theta_i \neq 0$ for $i = 1, 2$ and $\cos\theta_2 \neq \cos\theta_1$, prove that the system

$$\begin{aligned}
1/2(B - B^\top) &= \frac{\sin\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2}{\theta_2}A_2, \\
1/4(B + B^\top)(B - B^\top) &= \frac{\sin\theta_1\cos\theta_1}{\theta_1}A_1 + \frac{\sin\theta_2\cos\theta_2}{\theta_2}A_2,
\end{aligned}$$

has a unique solution for A_1 and A_2.

(vi) Prove that $A = A_1 + A_2$ has an orthonormal basis of eigenvectors such that the first two are a basis of the plane w.r.t. which B is a rotation of angle θ_1, and the last two are a basis of the plane w.r.t. which B is a rotation of angle θ_2.

Remark: I don't know a simple way to compute such an orthonormal basis of eigenvectors of $A = A_1 + A_2$, but it should be possible!

Problem B5 (50 pts). The motion of a rigid body in space can be described using rigid motions. Given a fixed Euclidean frame $(O, (e_1, e_2, e_3))$, we can assume that some moving frame $(C, (u_1, u_2, u_3))$ is attached (say glued) to a rigid body B (for example, at the center of gravity of B) so that the position and orientation of B in space is completely (and uniquely) determined by some rigid motion (R, U), where U specifies the position of C w.r.t. O, and R is a rotation matrix specifying the orientation of B w.r.t. the fixed frame $(O, (e_1, e_2, e_3))$. For simplicity, we can separate the motion of the center of gravity C of B from the rotation of B around its center of gravity. Then, a motion of B in space corresponds to two curves, the trajectory of the center of gravity, and a curve in $\mathbf{SO}(3)$ representing the various orientations of B. Given a sequence of "snapshots" of B, say $B_0, B_1, \ldots, B_m$, we may want to find an interpolating motion passing through the given snapshots.

Assuming that a rigid body B (say, a square box) spins around its center of gravity, which remains **fixed**, write a computer program to display an interpolated motion of B, given a sequence $B_0, B_1, \ldots, B_m$ of rotations specifying the orientation of B.

The problem is to ensure that the motion is smooth enough. You may use a cubic spline curve in the appropriate space, and either use quaternion interpolation, or the exponential map and Rodrigues' formula.

Extra credit (40 points): Also assume that the center of gravity is moving, and write a program performing motion interpolation.

TOTAL: 230 points.