Fall, 2003 CIS 610

Advanced geometric methods

Homework 3

November 11, 2003; Due November 25, beginning of class

You may work in groups of 2 or 3. Please, write up your solutions as clearly and concisely as possible. Be rigorous! You will have to present your solutions of the problems during a special problem session.

"A problems" are for practice only, and should not be turned in.

Problem A1. (1) Given a unit vector $(-\sin\theta, \cos\theta)$, prove that the Householder matrix determined by the vector $(-\sin\theta, \cos\theta)$ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Give a geometric interpretation (i.e., why the choice $(-\sin\theta, \cos\theta)$?).

(2) Given any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

prove that there is a Householder matrix H such that AH is lower triangular, i.e.,

$$AH = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}$$

for some $a', c', d' \in \mathbb{R}$.

Problem A2. Given a Euclidean space E of dimension n, if h is a reflection about some hyperplane orthogonal to a nonnull vector u and f is any isometry, prove that $f \circ h \circ f^{-1}$ is the reflection about the hyperplane orthogonal to $f(u)$.

Problem A3. Let E be a Euclidean space of dimension $n = 2$. Prove that given any two unit vectors $u_1, u_2 \in E$ (unit means that $\|u_1\| = \|u_2\| = 1$), there is a unique rotation r such that

$$r(u_1) = u_2.$$

Prove that there is a rotation mapping the pair $\langle u_1, u_2\rangle$ to the pair $\langle u_3, u_4\rangle$ iff there is a rotation mapping the pair $\langle u_1, u_3\rangle$ to the pair $\langle u_2, u_4\rangle$ (all vectors being unit vectors).

"B problems" must be turned in.

Problem B1 (30 pts). This problem is a warm-up for the next problem. Consider the set of matrices of the form

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix},$$

where $a \in \mathbb{R}$.

(a) Show that these matrices are invertible when $a \neq 0$ (give the inverse explicitly). Given any two such matrices A, B, show that $AB = BA$. Describe geometrically the action of such a matrix on points in the affine plane $\mathbb{A}^2$, with its usual Euclidean inner product. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}, +)$. This vector space is denoted by $\mathfrak{so}(2)$.

(b) Given an $n \times n$ matrix A, we define the *exponential* e^A as

$$e^A = I_n + \sum_{k \geq 1} \frac{A^k}{k!},$$

where I_n denotes the $n \times n$ identity matrix. It can be shown rigorously that this power series is indeed convergent for every A (over $\mathbb{R}$ or $\mathbb{C}$), so that e^A makes sense (and you do not have to prove it!).

Given any matrix

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

prove that

$$e^A = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Hint. Check that

$$\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^2 = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and use the power series for $\cos\theta$ and $\sin\theta$. Conclude that the exponential map provides a surjective map $\exp\colon \mathfrak{so}(2) \to \mathbf{SO}(2)$ from $\mathfrak{so}(2)$ onto the group $\mathbf{SO}(2)$ of plane rotations. Is this map injective? How do you need to restrict θ to get an injective map?

Remark: By the way, $\mathfrak{so}(2)$ is the *Lie algebra* of the (Lie) group $\mathbf{SO}(2)$.

(c) Consider the set $\mathbf{U}(1)$ of complex numbers of the form $\cos\theta + i\sin\theta$. Check that this is a group under multiplication. Assuming that we use the standard affine frame for the affine plane $\mathbb{A}^2$, every point (x, y) corresponds to the complex number $z = x + iy$, and this correspondence is a bijection. Then, every $\alpha = \cos\theta + i\sin\theta \in \mathbf{U}(1)$ induces the map $R_\alpha \colon \mathbb{A}^2 \to \mathbb{A}^2$ defined such that

$$R_\alpha(z) = \alpha z.$$

Prove that R_α is the rotation of matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Prove that the map $R \colon \mathbf{U}(1) \to \mathbf{SO}(2)$ defined such that $R(\alpha) = R_\alpha$ is an isomorphism. Deduce that topologically, $\mathbf{SO}(2)$ is a circle. Using the exponential map from $\mathbb{R}$ to $\mathbf{U}(1)$ defined such that $\theta \mapsto e^{i\theta} = \cos\theta + i\sin\theta$, prove that there is a surjective homomorphism from $(R, +)$ to $\mathbf{SO}(2)$. What is the connection with the exponential map from $\mathfrak{so}(2)$ to $\mathbf{SO}(2)$?

Problem B2 (60 pts).

(a) Recall that the coordinates of the cross product $u \times v$ of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ are

$$(u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

Letting U be the matrix

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

check that the coordinates of the cross product $u \times v$ are given by

$$\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

(b) Show that the set of matrices of the form

$$U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

is a vector space isomorphic to $(\mathbb{R}^3 +)$. This vector space is denoted by $\mathfrak{so}(3)$. Show that such matrices are never invertible. Find the kernel of the linear map associated with a matrix U. Describe geometrically the action of the linear map defined by a matrix U. Show that when restricted to the plane orthogonal to $u = (u_1, u_2, u_3)$ through the origin, it is a rotation by $\pi/2$.

(c) Consider the map $\psi\colon (\mathbb{R}^3, \times) \to \mathfrak{so}(3)$ defined by the formula

$$\psi(u_1, u_2, u_3) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

For any two matrices $A, B \in \mathfrak{so}(3)$, defining $[A,\, B]$ as

$$[A,\, B] = AB - BA,$$

verify that

$$\psi(u \times v) = [\psi(u),\, \psi(v)].$$

Show that $[-,\, -]$ is not associative. Show that $[A,\, A] = 0$, and that the so-called *Jacobi identity* holds:

$$[A,\, [B,\, C]] + [C,\, [A,\, B]] + [B,\, [C,\, A]] = 0.$$

Show that $[A\, B]$ is bilinear (linear in both A and B).

Remark: $[A,\, B]$ is called a *Lie bracket*, and under this operation, the vector space $\mathfrak{so}(3)$ is called a *Lie algebra*. In fact, it is the Lie algebra of the (Lie) group $\mathbf{SO}(3)$.

(d) For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

letting $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A,$$

and for any $k \geq 0$,

$$\begin{aligned} A^{4k+1} &= \theta^{4k} A, \\ A^{4k+2} &= \theta^{4k} A^2, \\ A^{4k+3} &= -\theta^{4k+2} A, \\ A^{4k+4} &= -\theta^{4k+2} A^2. \end{aligned}$$

Then prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2,$$

if $\theta \neq k2\pi$ $(k \in \mathbb{Z})$, with $\exp(0_3) = I_3$.

Remark: This formula is known as Rodrigues's formula (1840).

(e) Prove that $\exp A$ is a rotation of axis (a, b, c) and of angle $\theta = \sqrt{a^2 + b^2 + c^2}$.
Hint. Check that e^A is an orthogonal matrix of determinant $+1$, etc., or look up any textbook on kinematics or classical dynamics!

(f) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective. Prove that if R is a rotation matrix different from I_3, letting $\omega = (a, b, c)$ be a unit vector defining the axis of rotation, if $\mathrm{tr}(R) = -1$, then

$$(\exp(R))^{-1} = \left\{ \pm\pi \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \right\},$$

and if $\mathrm{tr}(R) \neq -1$, then

$$(\exp(R))^{-1} = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{11} + r_{22} + r_{33}$, the *trace* of the matrix R). Note that both θ and $2\pi - \theta$ yield the same matrix $\exp(R)$.

Problem B3 (30 pts). Given p vectors $(u_1, \ldots, u_p)$ in a Euclidean space E of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\mathrm{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2\rangle & \ldots & \langle u_1, u_p\rangle \\ \langle u_2, u_1\rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1\rangle & \langle u_p, u_2\rangle & \ldots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\mathrm{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$

Hint. By a previous problem, if $(e_1, \ldots, e_n)$ is an orthonormal basis of E and A is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A_i \cdot A_j),$$

where A_i denotes the ith column of the matrix A, and $(A_i \cdot A_j)$ denotes the $n \times n$ matrix with entries $A_i \cdot A_j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}).$$

Hint. Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\begin{aligned} \|w\|^4 &= \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}, w) \\ &= \mathrm{Gram}(u_1, \ldots, u_{n-1})\|w\|^2. \end{aligned}$$

Problem B4 (50 pts). Given a Euclidean space E, let U be a nonempty affine subspace of E, and let a be any point in E. We define the *distance* $d(a, U)$ of a to U as

$$d(a, U) = \inf\{\|\mathbf{ab}\| \mid b \in U\}.$$

(a) Prove that the affine subspace $U_a^\perp$ defined such that

$$U_a^\perp = a + \overrightarrow{U}^\perp$$

intersects U in a single point b such that $d(a, U) = \|\mathbf{ab}\|$.
Hint. Recall the discussion after Lemma 2.11.2.

(b) Let $(a_0, \ldots, a_p)$ be a frame for U (not necessarily orthonormal). Prove that

$$d(a, U)^2 = \frac{\mathrm{Gram}(\mathbf{a_0a}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p})}{\mathrm{Gram}(\mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p})}.$$

Hint. Gram is unchanged when a linear combination of other vectors is added to one of the vectors, and thus

$$\mathrm{Gram}(\mathbf{a_0a}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p}) = \mathrm{Gram}(\mathbf{ba}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p}),$$

where b is the unique point defined in question (a).

(c) If D and D' are two lines in E that are not coplanar, $a, b \in D$ are distinct points on D, and $a', b' \in D'$ are distinct points on D', prove that if $d(D, D')$ is the shortest distance between D and D' (why does it exist?), then

$$d(D, D')^2 = \frac{\mathrm{Gram}(\mathbf{aa'}, \mathbf{ab}, \mathbf{a'b'})}{\mathrm{Gram}(\mathbf{ab}, \mathbf{a'b'})}.$$

Problem B5 (30 pts). In $\mathbb{E}^3$, consider the closed convex set (cone), A, defined by the inequalities

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq xy,$$

and let D be the line given by $x = 0, z = 1$. Prove that $D \cap A = \emptyset$, both A and D are convex and closed, yet every plane containing D meets A. Therefore, A and D give another counter-example to the Hahn-Banach theorem where A is closed (one cannot relax the hypothesis that A is open).

Problem B6 (30 pts). In $\mathbb{E}^n$, consider the polar duality $A \mapsto A^*$, with respect to the origin, O.

(i) For any nonempty subsets, $A, B \subseteq \mathbb{E}^n$, prove the following properties:

(a) $A^* = A^{***}$.

(b) For any $\lambda \neq 0$, we have $(\lambda A)^* = (1/\lambda)A^*$.

(c) $(A \cup B)^* = A^* \cap B^*$.

(d) $A^{**} = \overline{\mathrm{conv}(A \cup \{O\})}$, the topological closure of the convex hull of A and the origin.

(e) If A and B are closed, convex and both contain O, then $(A \cap B)^* = \overline{\mathrm{conv}(A^* \cup B^*)}$, the topological closure of the convex hull of $A^* \cup B^*$.

(ii) If $A, B \subseteq \mathbb{E}^n$, for any λ, the set

$$(1 - \lambda)A + \lambda B = \{(1 - \lambda)a + \lambda b \mid a \in A, \ b \in B\}$$

is the *Minkowski sum* of A and B. Assume that $0 \leq \lambda \leq 1$. Check that $(1 - \lambda)A + \lambda B$ is convex if A and B are convex. Prove that $(1 - \lambda)A + \lambda B$ is a polytope if A and B are polytopes.

(iii) (**Extra Credit**) The Minkowski sum can be viewed as a way of interpolating between two polytopes and gives a way of *morphing* one polytope to another. The vertices of $(1-\lambda)A+\lambda B$ are convex combinations of the vertices of A and B. However, not all of these convex combinations are vertices. Investigate (in $\mathbb{E}^3$) practical algorithms for displaying the result of morphing polytopes using the Minkowski sum (You may want to begin with the case of polygons in $\mathbb{E}^2$.)

TOTAL: 230 points.