$\boxed{\textbf{Fall, 2003} \quad \textbf{CIS 610}}$

Advanced geometric methods

Homework 2

October 27, 2003; Due November 11, beginning of class

You may work in groups of 2 or 3. Please, write up your solutions as clearly and concisely as possible. Be rigorous! You will have to present your solutions of the problems during a special problem session.

"A problems" are for practice only, and should not be turned in.

Problem A1. Let $(e_1, \ldots, e_n)$ be an orthonormal basis for E. If X and Y are arbitrary $n \times n$ matrices, denoting as usual the jth column of X by X_j, and similarly for Y, show that

$$X^\top Y = (X_i \cdot Y_j)_{1 \le i,j \le n}.$$

Use this to prove that

$$A^\top A = A\,A^\top = I_n$$

iff the column vectors $(A_1, \ldots, A_n)$ form an orthonormal basis. Show that the conditions $A\,A^\top = I_n$, $A^\top A = I_n$, and $A^{-1} = A^\top$ are equivalent.

Problem A2. Compute the real Fourier coefficients of the function $id(x) = x$ over $[-\pi, \pi]$ and prove that

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right).$$

What is the value of the Fourier series at $\pm\pi$? What is the value of the Fourier near $\pm\pi$? Do you find this surprising?

Problem A3. Prove Lemma 6.2.2 from my book.

"B problems" must be turned in.

Problem B1 (30 pts). (1) If an upper triangular $n \times n$ matrix R is invertible, prove that its inverse is also upper triangular.

(2) If an upper triangular matrix is orthogonal, prove that it must be a diagonal matrix.

If A is an invertible $n \times n$ matrix and if $A = Q_1 R_1 = Q_2 R_2$, where R_1 and R_2 are upper triangular with positive diagonal entries and Q_1, Q_2 are orthogonal, prove that $Q_1 = Q_2$ and $R_1 = R_2$.

Problem B2 (30 pts). Consider the Euclidean space $\mathbb{E}^n$, and let $O = (0, \ldots, 0)$. Given any $x \in \mathbb{E}^n$, $x \neq O$, let $H(x)$ be the affine hyperplane perpendicular to Ox and passing through the point x' on the line Ox and such that $\mathbf{Ox}\cdot\mathbf{Ox}' = 1$. Equivalently, $H(x)$ is the affine hyperplane defined by

$$H(x) = \{y \in \mathbb{E}^n \mid x \cdot y = 1\}.$$

We call $H(x)$ the *polar* or *dual* of x. Conversely, given any affine hyperplane H not passing through O, there is clearly a unique $x \in \mathbb{E}^n$ so that $H(x) = H$, and we call x the *pole* or *dual* of H.

Given a subset A of $\mathbb{E}^n$, let

$$A^* = \{y \in \mathbb{E}^n \mid x \cdot y \leq 1, \ \forall x \in A\}.$$

We call A^* the *polar* or *reciprocal* of A.

(a) Check that A^* is the intersection of all the closed half-spaces containing O determined by the polar hyperplanes of points of A. Thus, conclude that A^* is convex.

Let $B^n(r)$ be the ball of radius $r > 0$ and center O, i.e.,

$$B^n(r) = \{x \in \mathbb{E}^n \mid \|x\| \leq r\}.$$

Show that $B^n(r)^* = B^n(1/r)$.

Prove that the dual C^* of the cube $C = [-1, 1]^n$ is the convex hull of the $2n$ points $\{e_i, -e_i \mid 1 \leq i \leq n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the *ith* vector in the standard basis. The dual of a cube is called a *cross-polytope*. Check that the cube C has 2^n vertices and $2n$ faces, whereas its dual C^* has $2n$ vertices and 2^n faces. Draw C^* for $n = 3$.

(b) A *convex polyhedron* or *convex body* P is a bounded subset of $\mathbb{E}^n$ with nonempty interior obtained as the intersection of a finite number of closed half–spaces. We will prove in class that a convex polyhedron P is the convex hull of a finite set of points with nonempty interior and conversely. We will also prove that the dual of a convex polyhedron containing O is a convex polyhedron. Observe that the duality exchanges vertices of P and the faces of P^*.

What is the dual of an n-simplex?

(c) Consider in $\mathbb{E}^3$ the polyhedron I defined as follows. If $\tau = (\sqrt{5} + 1)/2$, then the vertices of I are the twelve points

$$(0,\ \pm\tau,\ \pm 1), \quad (\pm 1,\ 0,\ \pm\tau), \quad (\pm\tau,\ \pm 1,\ 0).$$

This polyhedron is called an *icosahedron*. Check that the icosahedron has 20 faces. Draw an icosahedron (or better, make a cardboard model).

Prove that the dual D of the icosahedron is a convex polyhedron whose twenty vertices are

$$(\pm 1,\ \pm 1,\ \pm 1), \quad (0,\ \pm 1/\tau,\ \pm\tau), \quad (\pm\tau,\ 0,\ \pm 1/\tau), \quad (\pm 1/\tau,\ \pm\tau,\ 0).$$

This polyhedron D is called a *dodecahedron*. Observe that it is "built up" on the cube $[-1, 1]^3$. Can you explain how? Check that the dodecahedron has 12 faces. Draw a dodecahedron (or better, make a cardboard model).

Problem B3 (50 pts). (1) Review the modified Gram–Schmidt method. Recall that to compute Q'_{k+1}, instead of projecting A_{k+1} onto $Q_1, \ldots, Q_k$ in a single step, it is better to perform k projections. We compute $Q^1_{k+1}, Q^2_{k+1}, \ldots, Q^k_{k+1}$ as follows:

$$\begin{aligned} Q^1_{k+1} &= A_{k+1} - (A_{k+1} \cdot Q_1)\, Q_1, \\ Q^{i+1}_{k+1} &= Q^i_{k+1} - (Q^i_{k+1} \cdot Q_{i+1})\, Q_{i+1}, \end{aligned}$$

where $1 \leq i \leq k-1$.

Prove that $Q'_{k+1} = Q^k_{k+1}$.

(2) Write two computer programs to compute the QR-decomposition of an invertible matrix. The first one should use the standard Gram–Schmidt method, and the second one the modified Gram–Schmidt method. Run both on a number of matrices, up to dimension at least 10. Do you observe any difference in their performance in terms of numerical stability?

Run your programs on the Hilbert matrix $H_n = (1/(i+j-1))_{1 \leq i,j \leq n}$. What happens?

Extra Credit. (20 points) Write a program to solve linear systems of equations $Ax = b$, using your version of the QR-decomposition program, where A is an $n \times n$ matrix.

Problem B4 (30 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a bilinear form on a real vector space E of finite dimension n. Given any basis $(e_1, \ldots, e_n)$ of E, let $A = (\alpha_{i\,j})$ be the matrix defined such that

$$\alpha_{i\,j} = \varphi(e_i, e_j),$$

$1 \leq i, j \leq n$. We call A *the matrix of* φ *w.r.t. the basis* $(e_1, \ldots, e_n)$.

(a) For any two vectors x and y, if X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^\top AY.$$

(b) Recall that A is a *symmetric* matrix if $A = A^\top$. Prove that φ is symmetric if A is a symmetric matrix.

(c) If $(f_1, \ldots, f_n)$ is another basis of E and P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of φ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^\top AP.$$

The common rank of all matrices representing φ is called the *rank* of φ.

Problem B5 (80 pts). Let $\varphi: E \times E \to \mathbb{R}$ be a symmetric bilinear form on a real vector space E of finite dimension n. Two vectors x and y are said to be *conjugate w.r.t.* φ if $\varphi(x, y) = 0$. The main purpose of this problem is to prove that there is a basis of vectors that are pairwise conjugate w.r.t. φ.

(a) Prove that if $\varphi(x, x) = 0$ for all $x \in E$, then φ is identically null on E.

Otherwise, we can assume that there is some vector $x \in E$ such that $\varphi(x, x) \neq 0$. Use induction to prove that there is a basis of vectors that are pairwise conjugate w.r.t. φ.

For the induction step, proceed as follows. Let $(e_1, e_2, \ldots, e_n)$ be a basis of E, with $\varphi(e_1, e_1) \neq 0$. Prove that there are scalars $\lambda_2, \ldots, \lambda_n$ such that each of the vectors

$$v_i = e_i + \lambda_i e_1$$

is conjugate to e_1 w.r.t. φ, where $2 \leq i \leq n$, and that $(e_1, v_2, \ldots, v_n)$ is a basis.

(b) Let $(e_1, \ldots, e_n)$ be a basis of vectors that are pairwise conjugate w.r.t. φ, and assume that they are ordered such that

$$\varphi(e_i, e_i) = \begin{cases} \theta_i \neq 0 & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r + 1 \leq i \leq n, \end{cases}$$

where r is the rank of φ. Show that the matrix of φ w.r.t. $(e_1, \ldots, e_n)$ is a diagonal matrix, and that

$$\varphi(x, y) = \sum_{i=1}^{r} \theta_i x_i y_i,$$

where $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$.

Prove that for every symmetric matrix A, there is an invertible matrix P such that

$$P^{\top} A P = D,$$

where D is a diagonal matrix.

(c) Prove that there is an integer p, $0 \leq p \leq r$ (where r is the rank of φ), such that $\varphi(u_i, u_i) > 0$ for exactly p vectors of every basis $(u_1, \ldots, u_n)$ of vectors that are pairwise conjugate w.r.t. φ (*Sylvester's inertia theorem*).

Proceed as follows. Assume that in the basis $(u_1, \ldots, u_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \alpha_1 x_1^2 + \cdots + \alpha_p x_p^2 - \alpha_{p+1} x_{p+1}^2 - \cdots - \alpha_r x_r^2,$$

where $x = \sum_{i=1}^{n} x_i u_i$, and that in the basis $(v_1, \ldots, v_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \beta_1 y_1^2 + \cdots + \beta_q y_q^2 - \beta_{q+1} y_{q+1}^2 - \cdots - \beta_r y_r^2,$$

where $x = \sum_{i=1}^{n} y_i v_i$, with $\alpha_i > 0$, $\beta_i > 0$, $1 \leq i \leq r$.

Assume that $p > q$ and derive a contradiction. First, consider x in the subspace F spanned by

$$(u_1, \ldots, u_p, u_{r+1}, \ldots, u_n),$$

and observe that $\varphi(x, x) \geq 0$ if $x \neq 0$. Next, consider x in the subspace G spanned by

$$(v_{q+1}, \ldots, v_r),$$

and observe that $\varphi(x, x) < 0$ if $x \neq 0$. Prove that $F \cap G$ is nontrivial (i.e., contains some nonnull vector), and derive a contradiction. This implies that $p \leq q$. Finish the proof.

The pair $(p, r - p)$ is called the *signature* of φ.

(d) A symmetric bilinear form φ is *definite* if for every $x \in E$, if $\varphi(x, x) = 0$, then $x = 0$.

Prove that a symmetric bilinear form is definite iff its signature is either $(n, 0)$ or $(0, n)$. In other words, a symmetric definite bilinear form has rank n and is either positive or negative.

(e) The *kernel* of a symmetric bilinear form φ is the subspace consisting of the vectors that are conjugate to all vectors in E. We say that a symmetric bilinear form φ is *nondegenerate* if its kernel is trivial (i.e., equal to $\{0\}$).

Prove that a symmetric bilinear form φ is nondegenerate iff its rank is n, the dimension of E. Is a definite symmetric bilinear form φ nondegenerate? What about the converse?

Prove that if φ is nondegenerate, then there is a basis of vectors that are pairwise conjugate w.r.t. φ and such that φ is represented by the matrix

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

where (p, q) is the signature of φ.

(f) Given a nondegenerate symmetric bilinear form φ on E, prove that for every linear map $f \colon E \to E$, there is a unique linear map $f^* \colon E \to E$ such that

$$\varphi(f(u),\, v) = \varphi(u,\, f^*(v)),$$

for all $u, v \in E$. The map f^* is called the *adjoint of f (w.r.t. to φ)*. Given any basis $(u_1, \ldots, u_n)$, if Ω is the matrix representing φ and A is the matrix representing f, prove that f^* is represented by $\Omega^{-1} A^\top \Omega$.

Prove that Lemma 6.2.4 of my book also holds, i.e., the map $\flat \colon E \to E^*$ is a canonical isomorphism.

A linear map $f \colon E \to E$ is an *isometry w.r.t. φ* if

$$\varphi(f(x),\, f(y)) = \varphi(x,\, y)$$

for all $x, y \in E$. Prove that a linear map f is an isometry w.r.t. φ iff

$$f^* \circ f = f \circ f^* = \mathrm{id}.$$

Prove that the set of isometries w.r.t. φ is a group. This group is denoted by $\mathbf{O}(\varphi)$, and its subgroup consisting of isometries having determinant $+1$ by $\mathbf{SO}(\varphi)$. Given any basis of E, if Ω is the matrix representing φ and A is the matrix representing f, prove that $f \in \mathbf{O}(\varphi)$ iff

$$A^\top \Omega A = \Omega.$$

Given another nondegenerate symmetric bilinear form ψ on E, we say that φ and ψ are *equivalent* if there is a bijective linear map $h\colon E \to E$ such that

$$\psi(x,\, y) = \varphi(h(x),\, h(y)),$$

for all $x, y \in E$. Prove that the groups of isometries $\mathbf{O}(\varphi)$ and $\mathbf{O}(\psi)$ are isomomorphic (use the map $f \mapsto h \circ f \circ h^{-1}$ from $\mathbf{O}(\psi)$ to $\mathbf{O}(\varphi)$).

If φ is a nondegenerate symmetric bilinear form of signature (p, q), prove that the group $\mathbf{O}(\varphi)$ is isomorphic to the group of $n \times n$ matrices A such that

$$A^\top \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} A = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Remark: In view of question (f), the groups $\mathbf{O}(\varphi)$ and $\mathbf{SO}(\varphi)$ are also denoted by $\mathbf{O}(p, q)$ and $\mathbf{SO}(p, q)$ when φ has signature (p, q). They are Lie groups. In particular, the group $\mathbf{SO}(3, 1)$, known as the *Lorentz group*, plays an important role in the theory of special relativity.

Problem B6 (50 pts). (a) Let C be a circle of radius R and center O, and let P be any point in the Euclidean plane $\mathbb{E}^2$. Consider the lines Δ through P that intersect the circle C, generally in two points A and B. Prove that for all such lines,

$$\mathbf{PA} \cdot \mathbf{PB} = \|\mathbf{PO}\|^2 - R^2.$$

Hint. If P is not on C, let B' be the antipodal of B (i.e., $\mathbf{OB'} = -\mathbf{OB}$). Then $\mathbf{AB} \cdot \mathbf{AB'} = 0$ and

$$\mathbf{PA} \cdot \mathbf{PB} = \mathbf{PB'} \cdot \mathbf{PB} = (\mathbf{PO} - \mathbf{OB}) \cdot (\mathbf{PO} + \mathbf{OB}) = \|\mathbf{PO}\|^2 - R^2.$$

The quantity $\|\mathbf{PO}\|^2 - R^2$ is called the *power of P w.r.t. C*, and it is denoted by $\mathcal{P}(P, C)$. Show that if Δ is tangent to C, then $A = B$ and

$$\|\mathbf{PA}\|^2 = \|\mathbf{PO}\|^2 - R^2.$$

Show that P is inside C iff $\mathcal{P}(P, C) < 0$, on C iff $\mathcal{P}(P, C) = 0$, outside C if $\mathcal{P}(P, C) > 0$.

If the equation of C is

$$x^2 + y^2 - 2ax - 2by + c = 0,$$

prove that the power of $P = (x, y)$ w.r.t. C is given by

$$\mathcal{P}(P, C) = x^2 + y^2 - 2ax - 2by + c.$$

(b) Given two nonconcentric circles C and C', show that the set of points having equal power w.r.t. C and C' is a line orthogonal to the line through the centers of C and C'. If the equations of C and C' are

$$x^2 + y^2 - 2ax - 2by + c = 0 \quad \text{and} \quad x^2 + y^2 - 2a'x - 2b'y + c' = 0,$$

show that the equation of this line is

$$2(a - a')x + 2(b - b')y + c' - c = 0.$$

This line is called the *radical axis* of C and C'.

(c) Given three distinct nonconcentric circles C, C', and C'', prove that either the three pairwise radical axes of these circles are parallel or that they intersect in a single point ω that has equal power w.r.t. C, C', and C''. In the first case, the centers of C, C', and C'' are collinear. In the second case, if the power of ω is positive, prove that ω is the center of a circle Γ orthogonal to C, C', and C'', and if the power of ω is negative, ω is inside C, C', and C''.

(d) Given any $k \in \mathbb{R}$ with $k \neq 0$ and any point a, recall that an *inversion of pole a and power k* is a map $h\colon (\mathbb{E}^n - \{a\}) \to \mathbb{E}^n$ defined such that for every $x \in \mathbb{E}^n - \{a\}$,

$$h(x) = a + k\frac{\mathbf{ax}}{\|\mathbf{ax}\|^2}.$$

For example, when $n = 2$, chosing any orthonormal frame with origin a, h is defined by the map

$$(x,\, y) \mapsto \left(\frac{kx}{x^2 + y^2}, \frac{ky}{x^2 + y^2}\right).$$

When the centers of C, C' and C'' are not collinear and the power of ω is positive, prove that by a suitable inversion, C, C' and C'' are mapped to three circles whose centers are collinear.

Prove that if three distinct nonconcentric circles C, C', and C'' have collinear centers, then there are at most eight circles simultaneously tangent to C, C', and C'', and at most two for those exterior to C, C', and C''.

(e) Prove that an inversion in $\mathbb{E}^3$ maps a sphere to a sphere or to a plane. Prove that inversions preserve tangency and orthogonality of planes and spheres.

TOTAL: 270 points.