Fall, 2003 CIS 610

Advanced geometric methods

Homework 1

September 30, 2003; Due October 21, beginning of class

You may work in groups of 2 or 3. Please, write up your solutions as clearly and concisely as possible. Be rigorous! You will have to present your solutions of the problems during a special problem session.

Instead of doing Problem B7, search the literature for algorithms for computing the convex hull of a finite set of points in the plane (or in space, but this is harder!). Write a short critical paper presenting two such algorithms. You will also have to give a presentation during the problem session. Some relevant sources are,

Computational Geometry in C, by O'Rourke, Joseph, Cambridge University Press, 1998, second edition, and

Computational Geometry. Algorithms and Applications, by Berg, M., Van Kreveld, M., Overmars, M., and Schwarzkopf, O., Springer, 1997.

Computational Geometry: An Introduction, by F.P. Preparata and M.I. Shamos, Springer-Verlag, 1985.

See also

"Convex Hull Computations", by Raimund Seidel, in *Discrete and Computational Geometry*, J. Goodman and J. O'Rourke, eds., CRC Press, pp. 361-375. 1997.

"A problems" are for practice only, and should not be turned in.

Problem A1. (a) Given a tetrahedron (a, b, c, d), given any two distinct points $x, y \in \{a, b, c, d\}$, let let $m_{x,y}$ be the middle of the edge (x, y). Prove that the barycenter g of the weighted points $(a, 1/4)$, $(b, 1/4)$, $(c, 1/4)$, and $(d, 1/4)$, is the common intersection of the line segments $(m_{a,b}, m_{c,d})$, $(m_{a,c}, m_{b,d})$, and $(m_{a,d}, m_{b,c})$. Show that if g_d is the barycenter of the weighted points $(a, 1/3), (b, 1/3), (c, 1/3)$ then g is the barycenter of $(d, 1/4)$ and $(g_d, 3/4)$.

Problem A2. Given any two affine spaces E and F, for any affine map $f\colon E \to F$, for any convex set U in E and any convex set V in F, prove that $f(U)$ is convex and that $f^{-1}(V)$ is convex. Recall that

$$f(U) = \{b \in F \mid \exists a \in U,\, b = f(a)\}$$

is the *direct image of* U *under* f, and that

$$f^{-1}(V) = \{a \in E \mid \exists b \in V,\, b = f(a)\}$$

is the *inverse image of* V *under* f.

Problem A3. Let E be a nonempty set and $\overrightarrow{E}$ be a vector space and assume that there is a function $\Phi\colon E \times E \to \overrightarrow{E}$, such that if we denote $\Phi(a, b)$ by $\mathbf{ab}$, the following properties hold:

(1) $\mathbf{ab} + \mathbf{bc} = \mathbf{ac}$, for all $a, b, c \in E$;

(2) For every $a \in E$, the map $\Phi_a\colon E \to \overrightarrow{E}$ defined such that for every $b \in E$, $\Phi_a(b) = \mathbf{ab}$, is a bijection.

Let $\Psi_a\colon \overrightarrow{E} \to E$ be the inverse of $\Phi_a\colon E \to \overrightarrow{E}$.

Prove that the function $+\colon E \times \overrightarrow{E} \to E$ defined such that

$$a + u = \Psi_a(u)$$

for all $a \in E$ and all $u \in \overrightarrow{E}$ makes $(E, \overrightarrow{E}, +)$ into an affine space.

Note: We showed in class that an affine space $(E, \overrightarrow{E}, +)$ satisfies the properties stated above. Thus, we obtain an equivalent characterization of affine spaces.

"B problems" must be turned in.

Problem B1 (30 pts). Given any two distinct points a, b in $\mathbb{A}^2$ of barycentric coordinates (a_0, a_1, a_2) and (b_0, b_1, b_2) with respect to any given affine frame, show that the equation of the line $\langle a, b\rangle$ determined by a and b is

$$\begin{vmatrix} a_0 & b_0 & x \\ a_1 & b_1 & y \\ a_2 & b_2 & z \end{vmatrix} = 0,$$

or equivalently

$$(a_1b_2 - a_2b_1)x + (a_2b_0 - a_0b_2)y + (a_0b_1 - a_1b_0)z = 0,$$

where (x, y, z) are the barycentric coordinates of the generic point on the line $\langle a, b\rangle$.

Prove that the equation of a line in barycentric coordinates is of the form

$$ux + vy + wz = 0,$$

where $u \neq v$, or $v \neq w$, or $u \neq w$. Show that two equations

$$ux + vy + wz = 0 \quad \text{and} \quad u'x + v'y + w'z = 0$$

represent the same line in barycentric coordinates iff $(u', v', w') = \lambda(u, v, w)$ for some $\lambda \in \mathbb{R}$ (with $\lambda \neq 0$).

A triple (u, v, w) where $u \neq v$, or $v \neq w$, or $u \neq w$, is called a system of *tangential coordinates* of the line defined by the equation

$$ux + vy + wz = 0.$$

Problem B2 (30 pts). Given two lines D and D' in $\mathbb{A}^2$ defined by tangential coordinates (u, v, w) and (u', v', w') (as defined in problem B1), let

$$d = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ 1 & 1 & 1 \end{vmatrix} = vw' - wv' + wu' - uw' + uv' - vu'.$$

(a) Prove that D and D' have a unique intersection point iff $d \neq 0$, and that when it exists, the barycentric coordinates of this intersection point are

$$\frac{1}{d}(vw' - wv',\, wu' - uw',\, uv' - vu').$$

(b) Letting (O, i, j) be any affine frame for $\mathbb{A}^2$, recall that when $x + y + z = 0$, for any point a, the vector

$$x\overrightarrow{aO} + y\overrightarrow{ai} + z\overrightarrow{aj}$$

is independent of a and equal to

$$y\overrightarrow{Oi} + z\overrightarrow{Oj} = (y, z).$$

The triple (x, y, z) such that $x + y + z = 0$ is called the *barycentric coordinates* of the vector $y\overrightarrow{Oi} + z\overrightarrow{Oj}$ w.r.t. the affine frame (O, i, j).

Given any affine frame (O, i, j), prove that for $u \neq v$, or $v \neq w$, or $u \neq w$, the line of equation

$$ux + vy + wz = 0$$

in barycentric coordinates (x, y, z) (where $x + y + z = 1$) has for direction the set of vectors of barycentric coordinates (x, y, z) such that

$$ux + vy + wz = 0$$

(where $x + y + z = 0$).

Prove that D and D' are parallel iff $d = 0$. In this case, if $D \neq D'$, show that the common direction of D and D' is defined by the vector of barycentric coordinates

$$(vw' - wv',\, wu' - uw',\, uv' - vu').$$

(c) Given three lines D, D', and D'', at least two of which are distinct, and defined by tangential coordinates (u, v, w), (u', v', w'), and (u'', v'', w''), prove that D, D', and D'' are parallel or have a unique intersection point iff

$$\begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix} = 0.$$

Problem B3 (20 pts). Given an affine space E of dimension n and an affine frame $(a_0, \ldots, a_n)$ for E, let $f: E \to E$ and $g: E \to E$ be two affine maps represented by the two $(n+1) \times (n+1)$-matrices

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & c \\ 0 & 1 \end{pmatrix},$$

w.r.t. the frame $(a_0, \ldots, a_n)$. We also say that f and g are represented by (A, b) and (B, c).

(1) Prove that the composition $f \circ g$ is represented by the matrix

$$\begin{pmatrix} AB & Ac + b \\ 0 & 1 \end{pmatrix}.$$

We also say that $f \circ g$ is represented by $(A, b)(B, c) = (AB, Ac + b)$.

(2) Prove that f is invertible iff A is invertible and that the matrix representing f^{-1} is

$$\begin{pmatrix} A^{-1} & -A^{-1}b \\ 0 & 1 \end{pmatrix}.$$

We also say that f^{-1} is represented by $(A, b)^{-1} = (A^{-1}, -A^{-1}b)$. Prove that if A is an orthogonal matrix, the matrix associated with f^{-1} is

$$\begin{pmatrix} A^{\top} & -A^{\top}b \\ 0 & 1 \end{pmatrix}.$$

Furthermore, denoting the columns of A as $A_1, \ldots, A_n$, prove that the vector $A^{\top}b$ is the column vector of components

$$(A_1 \cdot b, \ldots, A_n \cdot b).$$

(where $\cdot$ denotes the standard inner product of vectors)

(3) Given two affine frame $(a_0, \ldots, a_n)$ and $(a'_0, \ldots, a'_n)$ for E, any affine map $f: E \to E$ has a matrix representation (A, b) w.r.t. to $(a_0, \ldots, a_n)$ and $(a'_0, \ldots, a'_n)$ defined such that $b = \mathbf{a'_0 f(a_0)}$ is expressed over the basis $(\mathbf{a'_0a'_1}, \ldots, \mathbf{a'_0a'_n})$, and $a_{i\,j}$ is the ith coefficient of $f(\mathbf{a_0a_j})$ over the basis $(\mathbf{a'_0a'_1}, \ldots, \mathbf{a'_0a'_n})$. Given any three frames $(a_0, \ldots, a_n)$, $(a'_0, \ldots, a'_n)$, and $(a''_0, \ldots, a''_n)$, for any two affine maps $f: E \to E$ and $g: E \to E$, if f has the matrix representation (A, b) w.r.t. $(a_0, \ldots, a_n)$ and $(a'_0, \ldots, a'_n)$ and g has the matrix representation

(B,c) w.r.t. $(a'_0,\ldots,a'_n)$ and $(a''_0,\ldots,a''_n)$, prove that $g\circ f$ has the matrix representation $(B,c)(A,b)$ w.r.t. $(a_0,\ldots,a_n)$ and $(a''_0,\ldots,a''_n)$.

(4) Given two affine frame $(a_0,\ldots,a_n)$ and $(a'_0,\ldots,a'_n)$ for E, there is a unique affine map $h\colon E\to E$ such that $h(a_i)=a'_i$ for $i=0,\ldots,n$, and we let (P,ω) be its associated matrix representation with respect to the frame $(a_0,\ldots,a_n)$. Note that $\omega=\mathbf{a_0a'_0}$, and that $p_{i\,j}$ is the ith coefficient of $\mathbf{a'_0a'_j}$ over the basis $(\mathbf{a_0a_1},\ldots,\mathbf{a_0a_n})$. Observe that (P,ω) is also the matrix representation of id_E w.r.t. the frames $(a'_0,\ldots,a'_n)$ and $(a_0,\ldots,a_n)$, **in that order**. For any affine map $f\colon E\to E$, if f has the matrix representation (A,b) over the frame $(a_0,\ldots,a_n)$ and the matrix representation (A',b') over the frame $(a'_0,\ldots,a'_n)$, prove that

$$(A',b')=(P,\omega)^{-1}(A,b)(P,\omega).$$

Given any two affine maps $f\colon E\to E$ and $g\colon E\to E$, where f is invertible, for any affine frame $(a_0,\ldots,a_n)$ for E, if $(a'_0,\ldots,a'_n)$ is the affine frame image of $(a_0,\ldots,a_n)$ under f (i.e., $f(a_i)=a'_i$ for $i=0,\ldots,n$), letting (A,b) be the matrix representation of f w.r.t. the frame $(a_0,\ldots,a_n)$ and (B,c) be the matrix representation of g w.r.t. the frame $(a'_0,\ldots,a'_n)$ (**not** the frame $(a_0,\ldots,a_n)$), prove that $g\circ f$ is represented by the matrix $(A,b)(B,c)$ w.r.t. the frame $(a_0,\ldots,a_n)$.

Remark: Note that this is the **opposite** of what happens if f and g are both represented by matrices w.r.t. the "fixed" frame $(a_0,\ldots,a_n)$, where $g\circ f$ is represented by the matrix $(B,c)(A,b)$. The frame $(a'_0,\ldots,a'_n)$ can be viewed as a "moving" frame. The above has applications in robotics, for example to rotation matrices expressed in terms of Euler angles, or "roll, pitch, and yaw".

Problem B4 (20 pts). Let S be any nonempty subset of an affine space E. Given some point $a\in S$, we say that S is *star-shaped with respect to* a iff the line segment $[a,x]$ is contained in S for every $x\in S$, i.e. $(1-\lambda)a+\lambda x\in S$ for all λ such that $0\le\lambda\le 1$. We say that S is *star-shaped* iff it is star-shaped w.r.t. to some point $a\in S$.

(1) Prove that every nonempty convex set is star-shaped.

(2) Show that there are star-shaped subsets that are not convex. Show that there are nonempty subsets that are not star-shaped (give an example in $\mathbb{A}^n$, $n=1,2,3$).

(3) Given a star-shaped subset S of E, let $N(S)$ be the set of all points $a\in S$ such that S is star-shaped with respect to a. Prove that $N(S)$ is convex.

Problem B5 (50 pts). (a) Let E be a vector space, and let U and V be two subspaces of E so that they form a direct sum $E=U\oplus V$. Recall that this means that every vector $x\in E$ can be written as $x=u+v$, for some unique $u\in U$ and some unique $v\in V$. Define the function $p_U\colon E\to U$ (resp. $p_V\colon E\to V$) so that $p_U(x)=u$ (resp. $p_V(x)=v$), where $x=u+v$, as explained above. Check that that p_U and p_V are linear.

(b) Now assume that E is an affine space (nontrivial), and let U and V be affine subspaces such that $\overrightarrow{E} = \overrightarrow{U} \oplus \overrightarrow{V}$. Pick any $\Omega \in V$, and define $q_U \colon E \to \overrightarrow{U}$ (resp. $q_V \colon E \to \overrightarrow{V}$, with $\Omega \in U$) so that

$$q_U(a) = p_{\overrightarrow{U}}(\mathbf{\Omega a}) \quad \text{(resp.} \quad q_V(a) = p_{\overrightarrow{V}}(\mathbf{\Omega a})), \quad \text{for every } a \in E.$$

Prove that q_U does not depend on the choice of $\Omega \in V$ (resp. q_V does not depend on the choice of $\Omega \in U$). Define the map $p_U \colon E \to U$ (resp. $p_V \colon E \to V$) so that

$$p_U(a) = a - q_V(a) \quad \text{(resp.} \quad p_V(a) = a - q_U(a)), \quad \text{for every } a \in E.$$

Prove that p_U (resp. p_V) is affine.

The map p_U (resp. p_V) is called the *projection onto U parallel to V (resp. projection onto V parallel to U)*.

(c) Let $(a_0, \ldots, a_n)$ be $n+1$ affinely independent points in $\mathbb{A}^n$, and let $\Delta(a_0, \ldots, a_n)$ denote the convex hull of $(a_0, \ldots, a_n)$ (an n-simplex). Prove that if $f \colon \mathbb{A}^n \to \mathbb{A}^n$ is an affine map sending $\Delta(a_0, \ldots, a_n)$ inside itself, i.e.,

$$f(\Delta(a_0, \ldots, a_n)) \subseteq \Delta(a_0, \ldots, a_n),$$

then, f has some fixed point $b \in \Delta(a_0, \ldots, a_n)$, i.e.,

$$f(b) = b.$$

Hint: Proceed by induction on n. First, treat the case $n = 1$. The affine map is determined by $f(a_0)$ and $f(a_1)$, which are affine combinations of a_0 and a_1. There is an explicit formula for some fixed point of f. For the induction step, compose f with some suitable projections.

Problem B6 (40 pts). Let A be a nonempty convex subset of $\mathbb{A}^n$. A function $f \colon A \to \mathbb{R}$ is *convex* if

$$f((1-\lambda)a + \lambda b) \leq (1-\lambda)f(a) + \lambda f(b)$$

for all $a, b \in A$ and for all $\lambda \in [0, 1]$.

(a) If f is convex, prove that

$$f\Big(\sum_{i \in I} \lambda_i a_i\Big) \leq \sum_{i \in I} \lambda_i f(a_i)$$

for every finite convex combination in A, i.e., any finite family $(a_i)_{i \in I}$ of points in A and any family $(\lambda_i)_{i \in I}$ with $\sum_{i \in I} \lambda_i = 1$ and $\lambda_i \geq 0$ for all $i \in I$.

(b) Let $f \colon A \to \mathbb{R}$ be a convex function and assume that A is convex and compact and that f is continuous. Prove that f achieves its maximum in some extremal point of A.

Problem B7 (100 pts). (a) Let A be any subset of $\mathbb{A}^n$. Prove that if A is compact, then its convex hull $\mathcal{C}(A)$ is also compact.

(b) Give a proof of the following version of Helly's theorem using Corollary 1.10 of the notes on convex sets (Convex sets: A deeper look):

Given any affine space E of dimension m, for every family $\{K_1, \ldots, K_n\}$ of n convex and compact subsets of E, if $n \geq m+2$ and the intersection $\bigcap_{i \in I} K_i$ of any $m+1$ of the K_i is nonempty (where $I \subseteq \{1, \ldots, n\}$, $|I| = m+1$), then $\bigcap_{i=1}^{n} K_i$ is nonempty.

Hint: First, prove that the general case can be reduced to the case where $n = m+2$.

(c) Use (b) to prove Helly's theorem without the assumption that the K_i are compact.

You will need to construct some nonempty compacts $C_i \subseteq K_i$. For this, you will need to prove that the convex hull of finitely many points is compact.

(d) Prove that Helly's theorem holds even if the family $(K_i)_{I \in I}$ is infinite, provided that the K_i are convex and compact.

TOTAL: 290 points.