# Chapter 9

# Partitions of Unity, Covering Maps $\circledast$

## 9.1 Partitions of Unity

To study manifolds, it is often necessary to construct various objects such as functions, vector fields, Riemannian metrics, volume forms, etc., by gluing together items constructed on the domains of charts.

Partitions of unity are a crucial technical tool in this gluing process.

The first step is to define "bump functions" (also called plateau functions). For any $r > 0$, we denote by $B(r)$ the open ball

$$B(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\},$$

and by $\overline{B(r)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq r\}$, its closure.

Given a topological space, $X$, for any function, $f\colon X \to \mathbb{R}$, the *support of $f$*, denoted $\operatorname{supp} f$, is the closed set

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

**Proposition 9.1.** *There is a smooth function, $b\colon \mathbb{R}^n \to \mathbb{R}$, so that*

$$b(x) = \begin{cases} 1 & \text{if } x \in \overline{B(1)} \\ 0 & \text{if } x \in \mathbb{R}^n - B(2). \end{cases}$$

*See Figures 9.1 and 9.2.*

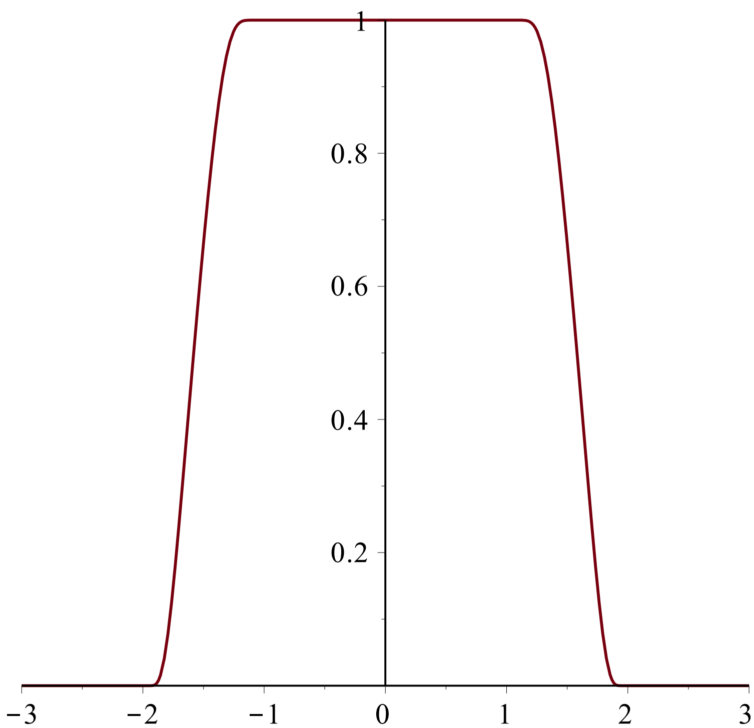


Figure 9.1: The graph of $b\colon \mathbb{R} \to \mathbb{R}$ used in Proposition 9.1.

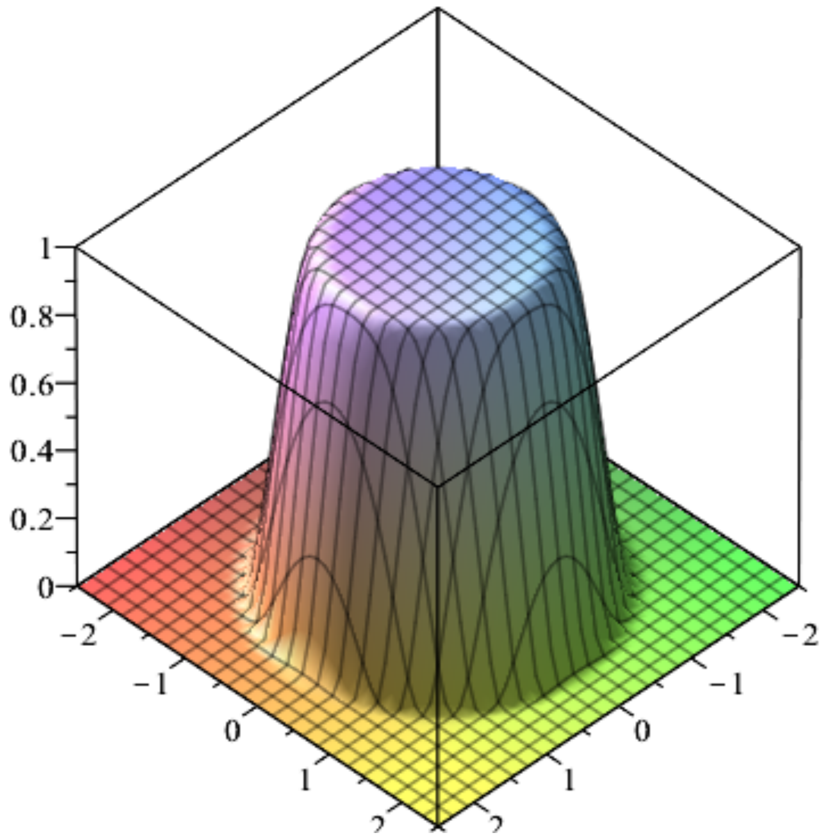


Figure 9.2: The graph of $b\colon \mathbb{R}^2 \to \mathbb{R}$ used in Proposition 9.1.

Proposition 9.1 yields the following useful technical result:

**Proposition 9.2.** *Let $M$ be a smooth manifold. For any open subset, $U \subseteq M$, any $p \in U$ and any smooth function, $f\colon U \to \mathbb{R}$, there exist an open subset, $V$, with $p \in V$ and a smooth function, $\widetilde{f}\colon M \to \mathbb{R}$, defined on the whole of $M$, so that $\overline{V}$ is compact,*

$$\overline{V} \subseteq U, \qquad \operatorname{supp} \widetilde{f} \subseteq U$$

*and*

$$\widetilde{f}(q) = f(q), \qquad \textit{for all } q \in \overline{V}.$$

If $X$ is a (Hausdorff) topological space, a family, $\{U_\alpha\}_{\alpha\in I}$, of subsets $U_\alpha$ of $X$ is a *cover* (or *covering*) of $X$ iff $X = \bigcup_{\alpha\in I} U_\alpha$.

A cover, $\{U_\alpha\}_{\alpha\in I}$, such that each $U_\alpha$ is open is an *open cover*.

If $\{U_\alpha\}_{\alpha\in I}$ is a cover of $X$, for any subset, $J \subseteq I$, the subfamily $\{U_\alpha\}_{\alpha\in J}$ is a *subcover* of $\{U_\alpha\}_{\alpha\in I}$ if $X = \bigcup_{\alpha\in J} U_\alpha$, i.e., $\{U_\alpha\}_{\alpha\in J}$ is still a cover of $X$.

Given a cover $\{V_\beta\}_{\beta\in J}$, we say that a family $\{U_\alpha\}_{\alpha\in I}$ is a *refinement* of $\{V_\beta\}_{\beta\in J}$ if it is a cover and if there is a function, $h\colon I \to J$, so that $U_\alpha \subseteq V_{h(\alpha)}$, for all $\alpha \in I$.

A family $\{U_\alpha\}_{\alpha\in I}$ of subsets of $X$ is *locally finite* iff for every point, $p \in X$, there is some open subset, $U$, with $p \in U$, so that $U \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in I$.

A space, $X$, is *paracompact* iff every open cover has an open locally finite refinement.

**Remark:** Recall that a space, $X$, is *compact* iff it is Hausdorff and if every open cover has a *finite* subcover. Thus, the notion of paracompactness (due to Jean Dieudonné) is a generalization of the notion of compactness.

Recall that a topological space, $X$, is *second-countable* if it has a countable basis, i.e., if there is a countable family of open subsets, $\{U_i\}_{i\geq 1}$, so that every open subset of $X$ is the union of some of the $U_i$'s.

A topological space, $X$, if *locally compact* iff it is Hausdorff and for every $a \in X$, there is some compact subset, $K$, and some open subset, $U$, with $a \in U$ and $U \subseteq K$.

As we will see shortly, every locally compact and second-countable topological space is paracompact.

It is important to observe that every manifold (even not second-countable) is locally compact.

**Definition 9.1.** Let $M$ be a (smooth) manifold. A *partition of unity on $M$* is a family, $\{f_i\}_{i\in I}$, of smooth functions on $M$ (the index set $I$ may be uncountable) such that

(a) The family of supports, $\{\mathrm{supp}\, f_i\}_{i\in I}$, is locally finite.

(b) For all $i \in I$ and all $p \in M$, we have $0 \leq f_i(p) \leq 1$, and
$$\sum_{i\in I} f_i(p) = 1, \quad \text{for every } p \in M.$$

Note that condition (b) implies that $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a cover of $M$. If $\{U_\alpha\}_{\alpha\in J}$ is a cover of $M$, we say that the partition of unity $\{f_i\}_{i\in I}$ is *subordinate* to the cover $\{U_\alpha\}_{\alpha\in J}$ if $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a refinement of $\{U_\alpha\}_{\alpha\in J}$.

When $I = J$ and $\mathrm{supp}\, f_i \subseteq U_i$, we say that $\{f_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in I}$ *with the same index set as the partition of unity*.

In Definition 9.1, by (a), for every $p \in M$, there is some open set, $U$, with $p \in U$ and $U$ meets only finitely many of the supports, supp $f_i$.

So, $f_i(p) \neq 0$ for only finitely many $i \in I$ and the infinite sum $\sum_{i \in I} f_i(p)$ is well defined.

**Proposition 9.3.** *Let $X$ be a topological space which is second-countable and locally compact (thus, also Hausdorff). Then, $X$ is paracompact. Moreover, every open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

**Remarks:**

1. Proposition 9.3 implies that a second-countable, locally compact (Hausdorff) topological space is the union of countably many compact subsets. Thus, $X$ is *countable at infinity*, a notion that we already encountered in Proposition 5.11 and Theorem 5.14.

2. A manifold that is countable at infinity has a countable open cover by domains of charts. It follows that $M$ is second-countable. Thus, for manifolds, second-countable is equivalent to countable at infinity.

Recall that we are assuming that our manifolds are Hausdorff and second-countable.

**Theorem 9.4.** *Let $M$ be a smooth manifold and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for $M$. Then, there is a countable partition of unity, $\{f_i\}_{i \geq 1}$, subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ and the support, $\mathrm{supp}\, f_i$, of each $f_i$ is compact.*

*If one does not require compact supports, then there is a partition of unity, $\{f_\alpha\}_{\alpha \in I}$, subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ with at most countably many of the $f_\alpha$ not identically zero. (In the second case, $\mathrm{supp}\, f_\alpha \subseteq U_\alpha$.)*

We close this section by stating a famous theorem of Whitney whose proof uses partitions of unity.

**Theorem 9.5.** *(Whitney, 1935) Any smooth manifold (Hausdorff and second-countable), $M$, of dimension $n$ is diffeomorphic to a closed submanifold of $\mathbb{R}^{2n+1}$.*

For a proof, see Hirsch [23], Chapter 2, Section 2, Theorem 2.14.

## 9.2 Covering Maps and Universal Covering Manifolds

Covering maps are an important technical tool in algebraic topology and more generally in geometry.

We begin with covering maps.

**Definition 9.2.** A map, $\pi\colon M \to N$, between two smooth manifolds is a *covering map* (or *cover*) iff

(1) The map $\pi$ is smooth and surjective.

(2) For any $q \in N$, there is some open subset, $V \subseteq N$, so that $q \in V$ and
$$\pi^{-1}(V) = \bigcup_{i \in I} U_i,$$
where the $U_i$ are pairwise disjoint open subsets, $U_i \subseteq M$, and $\pi\colon U_i \to V$ is a diffeomorphism for every $i \in I$. We say that $V$ is *evenly covered*.

The manifold, $M$, is called a *covering manifold* of $N$. See Figure 9.3.

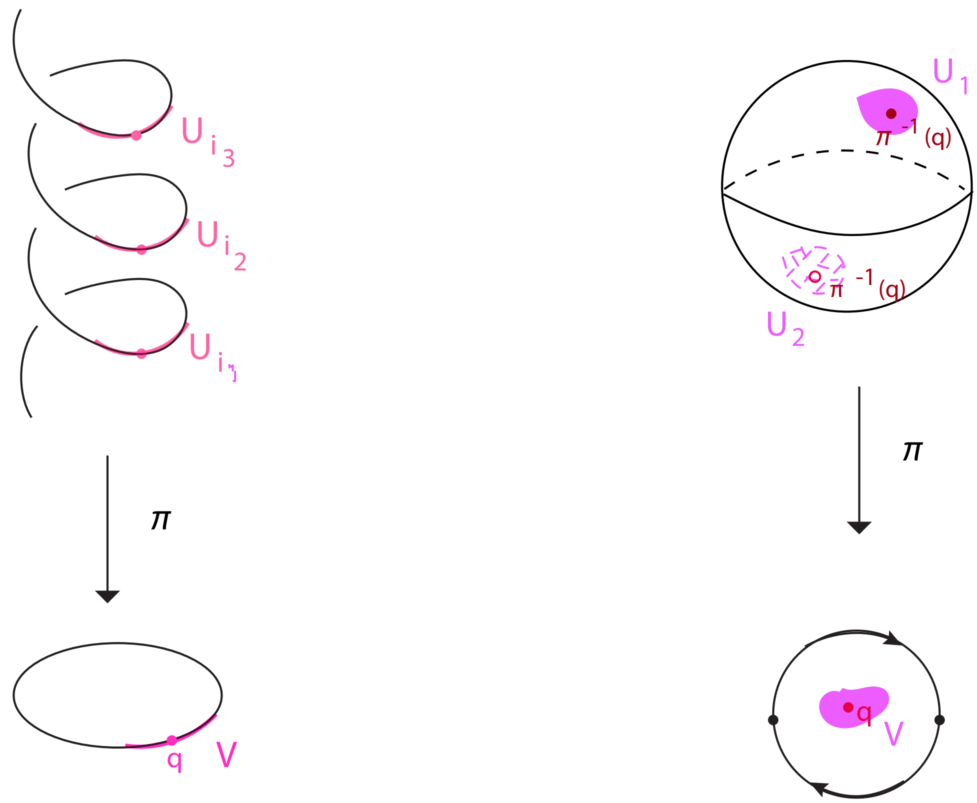


Figure 9.3: Two examples of a covering map. The left illustration is $\pi\colon \mathbb{R} \to S^1$ with $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$, while the right illustration is the 2-fold antipodal covering of $\mathbb{RP}^2$ by $S^2$.

A *homomorphism* of coverings, $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$, is a smooth map, $\phi\colon M_1 \to M_2$, so that

$$\pi_1 = \pi_2 \circ \phi,$$

that is, the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & M_2 \arrow[dl, "\pi_2"] \\ & N & \end{tikzcd}.$$

We say that the coverings $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are *equivalent* iff there is a homomorphism, $\phi\colon M_1 \to M_2$, between the two coverings and $\phi$ is a diffeomorphism.

As usual, the inverse image, $\pi^{-1}(q)$, of any element $q \in N$ is called the *fibre over $q$*, the space $N$ is called the *base* and $M$ is called the *covering space*.

As $\pi$ is a covering map, each fibre is a discrete space.

Note that a homomorphism maps each fibre $\pi_1^{-1}(q)$ in $M_1$ to the fibre $\pi_2^{-1}(\phi(q))$ in $M_2$, for every $q \in M_1$.

**Proposition 9.6.** *Let $\pi\colon M \to N$ be a covering map. If $N$ is connected, then all fibres, $\pi^{-1}(q)$, have the same cardinality for all $q \in N$. Furthermore, if $\pi^{-1}(q)$ is not finite then it is countably infinite.*

When the common cardinality of fibres is finite it is called the *multiplicity* of the covering (or the number of *sheets*).

For any integer, $n > 0$, the map, $z \mapsto z^n$, from the unit circle $S^1 = \mathbf{U}(1)$ to itself is a covering with $n$ sheets. The map,

$$t\colon \ \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

is a covering, $\mathbb{R} \to S^1$, with infinitely many sheets.

It is also useful to note that a covering map, $\pi\colon M \to N$, is a local diffeomorphism (which means that $d\pi_p\colon T_pM \to T_{\pi(p)}N$ is a bijective linear map for every $p \in M$).

The crucial property of covering manifolds is that curves in $N$ can be lifted to $M$, in a unique way.

**Definition 9.3.** Let $\pi\colon M \to N$ be a covering map, and let $P$ be a Hausdorff topological space. For any map $\phi\colon P \to N$, a *lift of $\phi$ through $\pi$* is a map $\widetilde{\phi}\colon P \to M$ so that

$$\phi = \pi \circ \widetilde{\phi},$$

as in the following commutative diagram.

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \end{tikzcd}$$

We would like to state three propositions regarding covering spaces.

However, two of these propositions use the notion of a simply connected manifold.

Intuitively, a manifold is simply connected if it has no "holes."

More precisely, a manifold is simply connected if it has a trivial fundamental group.

A fundamental group is a homotopic loop group.

Therefore, given topological spaces $X$ and $Y$, we need to define a homotopy between two continuous functions $f\colon X \to Y$ and $g\colon X \to Y$.

**Definition 9.4.** Let $X$ and $Y$ be topological spaces, $f: X \to Y$ and $g: X \to Y$ be two continuous functions, and let $I = [0,1]$. We say that *$f$ is homotopic to $g$* if there exists a continuous function $F: X \times I \to Y$ (where $X \times I$ is given the product topology) such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map $F$ is a *homotopy from $f$ to $g$*, and this is denoted $f \sim_F g$. If $f$ and $g$ agree on $A \subseteq X$, i.e. $f(a) = g(a)$ whenever $a \in A$, we say *$f$ is homotopic to $g$ relative $A$*, and this is denoted $f \sim_F g$ rel $A$.

A homotopy provides a means of continuously deforming $f$ into $g$ through a family $\{f_t\}$ of continuous functions $f_t: X \to Y$ where $t \in [0,1]$ and $f_0(x) = f(x)$ and $f_1(x) = g(x)$ for all $x \in X$.

For example, let $D$ be the unit disk in $\mathbb{R}^2$ and consider two continuous functions $f\colon I \to D$ and $g\colon I \to D$.

Then $f \sim_F g$ via the straight line homotopy $F\colon I \times I \to D$, where $F(x,t) = (1-t)f(x) + tg(x)$.

**Proposition 9.7.** *Let $X$ and $Y$ be topological spaces and let $A \subseteq X$. Homotopy (or homotopy rel $A$) is an equivalence relation on the set of all continuous functions from $X$ to $Y$.*

The next two propositions show that homotopy behaves well with respect to composition.

**Proposition 9.8.** *Let $X$, $Y$, and $Z$ be topological spaces and let $A \subseteq X$. For any continuous functions $f\colon X \to Y$, $g\colon X \to Y$, and $h\colon Y \to Z$, if $f \sim_F g$ rel $A$, then $h \circ f \sim_{h\circ F} h \circ g$ rel $A$ as maps from $X$ to $Z$.*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

**Proposition 9.9.** *Let $X$, $Y$, and $Z$ be topological spaces and let $B \subseteq Y$. For any continuous functions $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon Y \to Z$, if $g \sim_G h$ rel $B$, then $g \circ f \sim_F h \circ f$ rel $f^{-1}B$, where $F(x,t) = G(f(x),t)$.*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z. \end{tikzcd}$$

In order to define the fundamental group of a topological space $X$, we recall the definition of a loop.

**Definition 9.5.** Let $X$ be a topological space, $p$ be a point in $X$, and let $I = [0,1]$. We say $\alpha$ is a *loop based at $p = \alpha(0)$* if $\alpha$ is a continuous map $\alpha\colon I \to X$ with $\alpha(0) = \alpha(1)$.

Given a topological space $X$, choose a point $p \in X$ and form $S$, the set of all loops in $X$ based at $p$.

By applying Proposition 9.7, we know that the relation of homotopy relative to $\{0, 1\}$ is an equivalence relation on $S$. This leads to the following definition.

**Definition 9.6.** Let $X$ be a topological space, $p$ be a point in $X$, and let $\alpha$ be a loop in $X$ based at $p$. The set of all loops homotopic to $\alpha$ relative to $\{0, 1\}$ is *the homotopy class of $\alpha$* and is denoted $\langle\alpha\rangle$.

**Definition 9.7.** Given two loops $\alpha$ and $\beta$ in a topological space $X$ based at $p$, the *product* $\alpha \cdot \beta$ is a loop in $X$ based at $p$ defined by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} < t \leq 1. \end{cases}$$

The product of loops gives rise to the product of homotopy classes where

$$\langle \alpha \rangle \cdot \langle \beta \rangle = \langle \alpha \cdot \beta \rangle.$$

We leave it the reader to check that the multiplication of homotopy classes is well defined and associative, namely $\langle \alpha \cdot \beta \rangle \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot \langle \beta \cdot \gamma \rangle$ whenever $\alpha$, $\beta$, and $\gamma$ are loops in $X$ based at $p$.

Let $\langle e \rangle$ be the homotopy class of the constant loop in $X$ based at $p$, and define the inverse of $\langle \alpha \rangle$ as $\langle \alpha \rangle^{-1} = \langle \alpha^{-1} \rangle$, where $\alpha^{-1}(t) = \alpha(1 - t)$.

With these conventions, the product operation between homotopy classes gives rise to a group. In particular,

**Proposition 9.10.** *Let $X$ be a topological space and let $p$ be a point in $X$. The set of homotopy classes of loops in $X$ based at $p$ is a group with multiplication given by* $\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha \cdot \beta\rangle$

**Definition 9.8.** Let $X$ be a topological space and $p$ a point in $X$. The group of homotopy classes of loops in $X$ based at $p$ is the *fundamental group of $X$ based at $p$*, and is denoted by $\pi_1(X, p)$.

If we assume $X$ is path connected, we can show that $\pi_1(X, p) \cong \pi_1(X, q)$ for any points $p$ and $q$ in $X$. Therefore, when $X$ is path connected, we simply write $\pi_1(X)$.

For example, it can be shown that $\pi_1(S^1) = \mathbb{Z}$, and for the 2-torus $\mathbb{T}^2$, $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$.

These are abelian groups, but in general the fundamental group is not abelian.

A simple example is a compact surface of genus 2, that is, the result of gluing two tori along a disc.

In this case the fundamental group $\pi_1(M_2)$ is the quotient of the free group on four generators $\{a_1, b_1, a_2, b_2\}$ by the subgroup generated by

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}.$$

**Definition 9.9.** If $X$ is path connected topological space and $\pi_1(X) = \langle e \rangle$, (which is also denoted as $\pi_1(X) = (0)$), we say $X$ is *simply connected*.

In other words, every loop in $X$ can be shrunk in a continuous manner within $X$ to its basepoint.

Examples of simply connected spaces include $R^n$ and $S^n$ whenever $n \geq 2$.

On the other hand, the torus and the circle are not simply connected. See Figures 9.4 and 9.5.

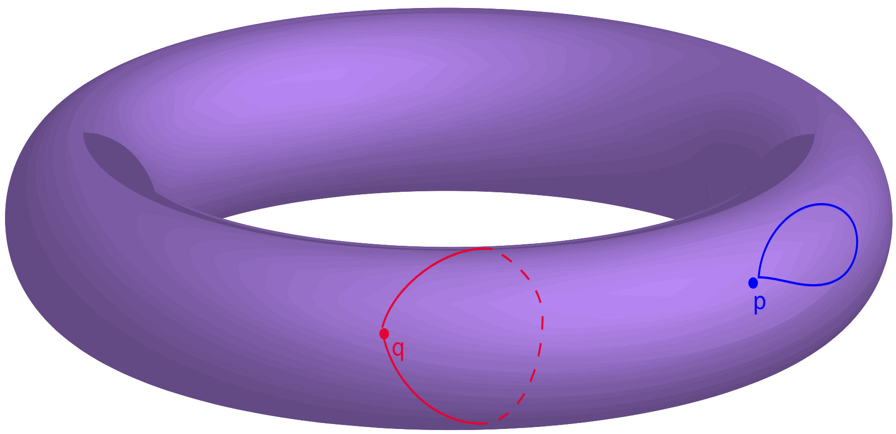


Figure 9.4: The torus is not simply connected. The loop at $p$ is homotopic to a point, but the loop at $q$ is not.

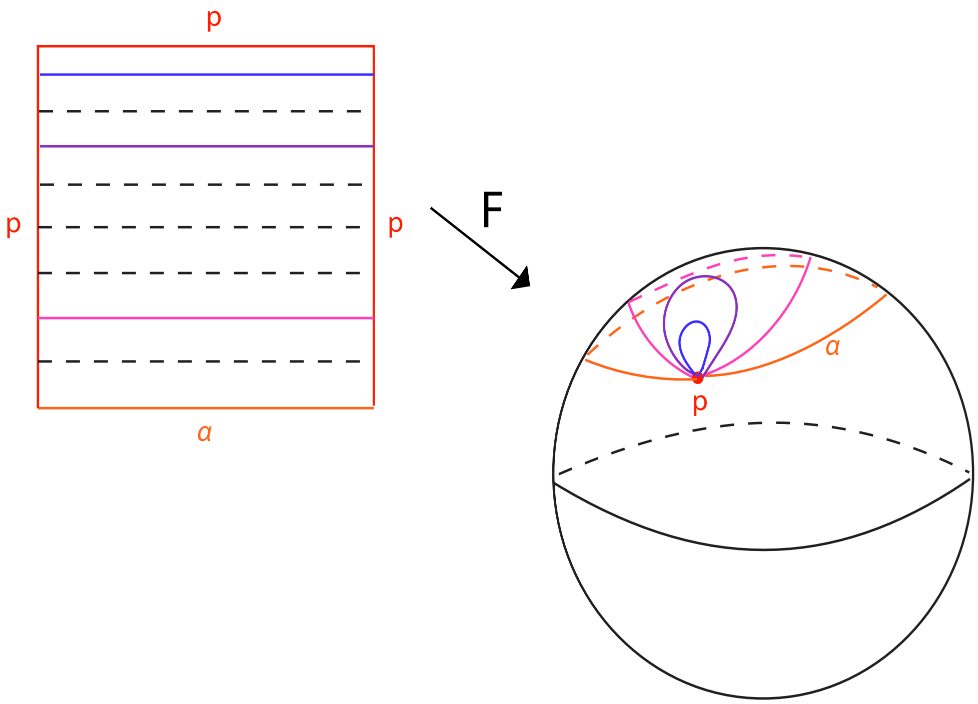


Figure 9.5: The unit sphere $S^2$ is simply connected since every loop can be continuously deformed to a point. This deformation is represented by the map $F\colon I \times I \to S^2$ where $F(x,0) = \alpha$ and $F(x,1) = p$.

We now state without proof the following results:

**Proposition 9.11.** *If $\pi\colon M \to N$ is a covering map, then for every smooth curve, $\alpha\colon I \to N$, in $N$ (with $0 \in I$) and for any point, $q \in M$, such that $\pi(q) = \alpha(0)$, there is a unique smooth curve, $\widetilde{\alpha}\colon I \to M$, lifting $\alpha$ through $\pi$ such that $\widetilde{\alpha}(0) = q$. See Figure 9.6.*

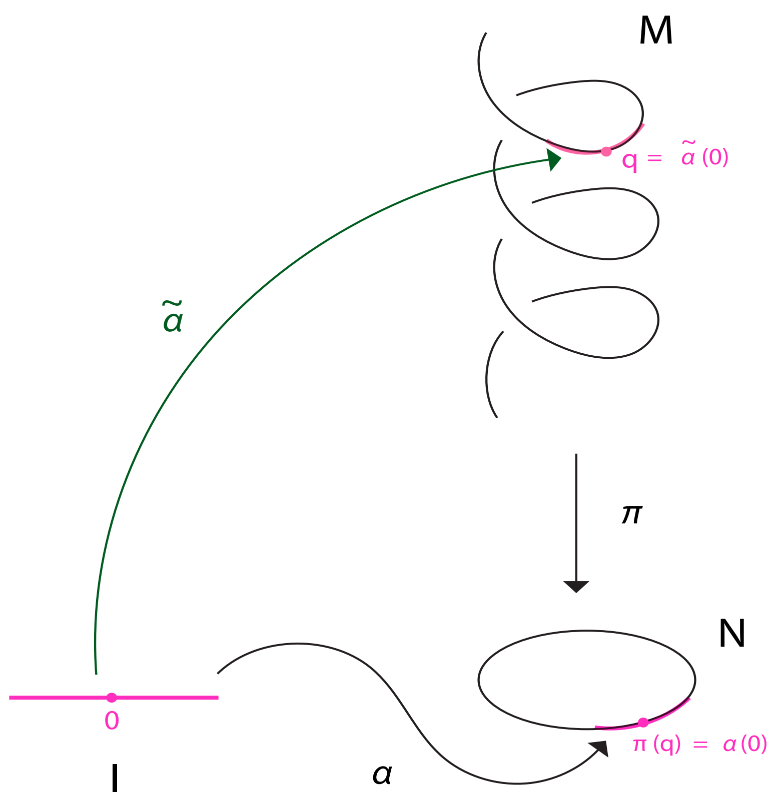


Figure 9.6: The lift of a curve $\alpha$ when $\pi\colon \mathbb{R} \to S^1$ is $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$.

**Proposition 9.12.** *Let $\pi\colon M \to N$ be a covering map and let $\phi\colon P \to N$ be a smooth map. For any $p_0 \in P$, any $q_0 \in M$ and any $r_0 \in N$ with $\pi(q_0) = \phi(p_0) = r_0$, the following properties hold:*

*(1) If $P$ is connected then there is at most one lift, $\widetilde{\phi}\colon P \to M$, of $\phi$ through $\pi$ such that $\widetilde{\phi}(p_0) = q_0$.*

*(2) If $P$ is simply connected, then such a lift exists.*

$$\begin{tikzcd} & M \ni q_0 \arrow[d, "\pi"] \\ p_0 \in P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \ni r_0 \end{tikzcd}$$

**Theorem 9.13.** *Every connected manifold, $M$, possesses a simply connected covering map, $\pi\colon \widetilde{M} \to M$, that is, with $\widetilde{M}$ simply connected. Any two simply connected coverings of $N$ are equivalent.*

In view of Theorem 9.13, it is legitimate to speak of *the* simply connected cover, $\widetilde{M}$, of $M$, also called *universal covering* (or *cover*) of $M$.

It can be shown that $\pi_1(\mathbf{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$, so $\mathbf{SO}(3)$ is not simply-connected (but it is path-connected).

The universal cover of $\mathbf{SO}(3)$ is the group $\mathbf{SU}(2)$ of unit quaternions.

More generally, for $n \geq 3$, $\mathbf{SO}(n)$ is path-connected and $\pi_1(\mathbf{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$, so $\mathbf{SO}(n)$ is not simply-connected.

The universal cover of $\mathbf{SO}(n)$ is a group denoted $\mathbf{Spin}(n)$ and called a *spin group*. It is a matrix Lie group.

The group $\mathbf{SL}(2,\mathbb{R})$ is path-connected and $\pi_1(\mathbf{SL}(2,\mathbb{R})) = \mathbb{Z}$, so $\mathbf{SL}(2,\mathbb{R})$ is not simply-connected.

The universal cover of $\mathbf{SL}(2,\mathbb{R})$, often denoted $\widetilde{S}$, is a Lie group but *not* a matrix Lie group.

For $n \geq 3$, the group $\mathbf{SL}(n,\mathbb{R})$ is path-connected and $\pi_1(\mathbf{SL}(n,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, so $\mathbf{SL}(n,\mathbb{R})$ is not simply-connected.

On the other hand, $\mathbf{SL}(n,\mathbb{C})$ is path-connected and simply-connected for all $n \geq 1$.

For more on all this, see Fulton and Harris [17] (Chapters 10, 11, 23).

Given any point, $p \in M$, let $\pi_1(M, p)$ denote the fundamental group of $M$ with basepoint $p$.

If $\phi\colon M \to N$ is a smooth map, for any $p \in M$, if we write $q = \phi(p)$, then we have an *induced group homomorphism*

$$\phi_*\colon \pi_1(M, p) \to \pi_1(N, q)$$

defined as follows. For every loop $\gamma$ in $M$ based at $p$, the map $f \circ \gamma$ is a loop based at $q = \varphi(p)$ in $N$, so let

$$\phi_*([\gamma]) = [f \circ \gamma].$$

It is easily verified that the map $\phi_*$ is well-defined, that is, does not depend on the choice of the loop $\gamma$ in the homotopy class $[\gamma] \in \pi_1(M, p)$, and that it is a group homomorphism.

**Proposition 9.14.** *If $\pi\colon M \to N$ is a covering map, for every $p \in M$, if $q = \pi(p)$, then the induced homomorphism, $\pi_*\colon \pi_1(M, p) \to \pi_1(N, q)$, is injective.*

**Proposition 9.15.** *Let $\pi\colon M \to N$ be a covering map and let $\phi\colon P \to N$ be a smooth map. For any $p_0 \in P$, any $q_0 \in M$ and any $r_0 \in N$ with $\pi(q_0) = \phi(p_0) = r_0$, if $P$ is connected, then a lift, $\widetilde{\phi}\colon P \to M$, of $\phi$ such that $\widetilde{\phi}(p_0) = q_0$ exists iff*

$$\phi_*(\pi_1(P, p_0)) \subseteq \pi_*(\pi_1(M, q_0)),$$

*as illustrated in the diagram below*

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} & \pi_1(M, q_0) \arrow[d, "\pi_*"] \\ \pi_1(P, p_0) \arrow[ur, dashed] \arrow[r, "\phi_*"'] & \pi_1(N, r_0) \end{tikzcd}$$

**Basic Assumption**: For any covering, $\pi\colon M \to N$, if $N$ is connected then we also assume that $M$ is connected.

Using Proposition 9.14, we get

**Proposition 9.16.** *If $\pi\colon M \to N$ is a covering map and $N$ is simply connected, then $\pi$ is a diffeomorphism (recall that $M$ is connected); thus, $M$ is diffeomorphic to the universal cover, $\widetilde{N}$, of $N$.*

The following proposition shows that the universal covering of a space covers every other covering of that space. This justifies the terminology "*universal covering*."

**Proposition 9.17.** *Say $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are two coverings of $N$, with $N$ connected. Every homomorphism, $\phi\colon M_1 \to M_2$, between these two coverings is a covering map. As a consequence, if $\pi\colon \widetilde{N} \to N$ is a universal covering of $N$, then for every covering, $\pi'\colon M \to N$, of $N$, there is a covering, $\phi\colon \widetilde{N} \to M$, of $M$.*

The notion of deck-transformation group of a covering is also useful because it yields a way to compute the fundamental group of the base space.

**Definition 9.10.** If $\pi\colon M \to N$ is a covering map, a *deck-transformation* is any diffeomorphism, $\phi\colon M \to M$, such that $\pi = \pi \circ \phi$, that is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi"] \\ & N & \end{tikzcd}.$$

Note that deck-transformations are just automorphisms of the covering map.

The commutative diagram of Definition 9.10 means that a deck transformation permutes the elements of every fibre. It is immediately verified that the set of deck transformations of a covering map is a group denoted $\Gamma_\pi$ (or simply, $\Gamma$), called the *deck-transformation group* of the covering.

Observe that any deck transformation $\phi$ is a lift of $\pi$ through $\pi$ as shown below.

$$\begin{tikzcd} & M \ni p \arrow[d, "\pi"] \\ p \in M \arrow[ur, "\phi"] \arrow[r, "\pi"'] & N \ni q \end{tikzcd}$$

Consequently, if $M$ is connected, by Proposition 9.12 (1), every deck-transformation is determined by its value at a single point, say $p$.

So, the deck-transformations are determined by their action on each point of any fixed fibre, $\pi^{-1}(q)$, with $q \in N$.

Since the fibre $\pi^{-1}(q)$ is countable, $\Gamma$ is also countable, that is, a discrete Lie group.

Moreover, if $M$ is compact, as each fibre, $\pi^{-1}(q)$, is compact and discrete, it must be finite and so, the deck-transformation group is also finite.

It can also be shown that $\Gamma_\pi$ *operates without fixed points*, which means that if $\phi \in \Gamma_\pi$ is not the identity map, then $\phi$ has not fixed points.

The following proposition gives a useful method for determining the fundamental group of a manifold.

**Proposition 9.18.** *If $\pi\colon \widetilde{M} \to M$ is the universal covering of a connected manifold, $M$, then the deck-transformation group, $\widetilde{\Gamma}$, is isomorphic to the fundamental group, $\pi_1(M)$, of $M$.*

**Remark:** When $\pi\colon \widetilde{M} \to M$ is the universal covering of $M$, it can be shown that the group $\widetilde{\Gamma}$ acts simply and transitively on every fibre, $\pi^{-1}(q)$.

This means that for any two elements, $x, y \in \pi^{-1}(q)$, there is a *unique* deck-transformation, $\phi \in \widetilde{\Gamma}$ such that $\phi(x) = y$.

So, there is a bijection between $\pi_1(M) \cong \widetilde{\Gamma}$ and the fibre $\pi^{-1}(q)$.

Proposition 9.13 together with previous observations implies that if the universal cover of a connected (compact) manifold is compact, then $M$ has a finite fundamental group.

We will use this fact later, in particular, in the proof of Myers' Theorem.