Chapter 8

Vector Fields, Lie Derivatives, Integral Curves, Flows

Our goal in this chapter is to generalize the concept of a vector field to manifolds, and to promote some standard results about ordinary differential equations to manifolds.

8.1 Tangent and Cotangent Bundles

Let M be a C^k-manifold (with $k \geq 2$). Roughly speaking, a vector field on M is the assignment, $p \mapsto X(p)$, of a tangent vector $X(p) \in T_p(M)$, to a point $p \in M$.

Generally, we would like such assignments to have some smoothness properties when p varies in M, for example, to be C^l, for some l related to k.

Now, if the collection, $T(M)$, of all tangent spaces, $T_p(M)$, was a C^l-manifold, then it would be very easy to define what we mean by a C^l-vector field: We would simply require the map, $X\colon M \to T(M)$, to be C^l.

If M is a C^k-manifold of dimension n, then we can indeed make $T(M)$ into a C^{k-1}-manifold of dimension $2n$ and we now sketch this construction.

We find it most convenient to use Version 2 of the definition of tangent vectors, i.e., as equivalence classes of triples (U, φ, x), with $x \in \mathbb{R}^n$. Recall that (U, φ, x) and (V, ψ, y) are equivalent iff

$$(\psi \circ \varphi^{-1})'_{\varphi(p)}(x) = y.$$

First, we let $T(M)$ be the disjoint union of the tangent spaces $T_p(M)$, for all $p \in M$. See Figure 8.1.

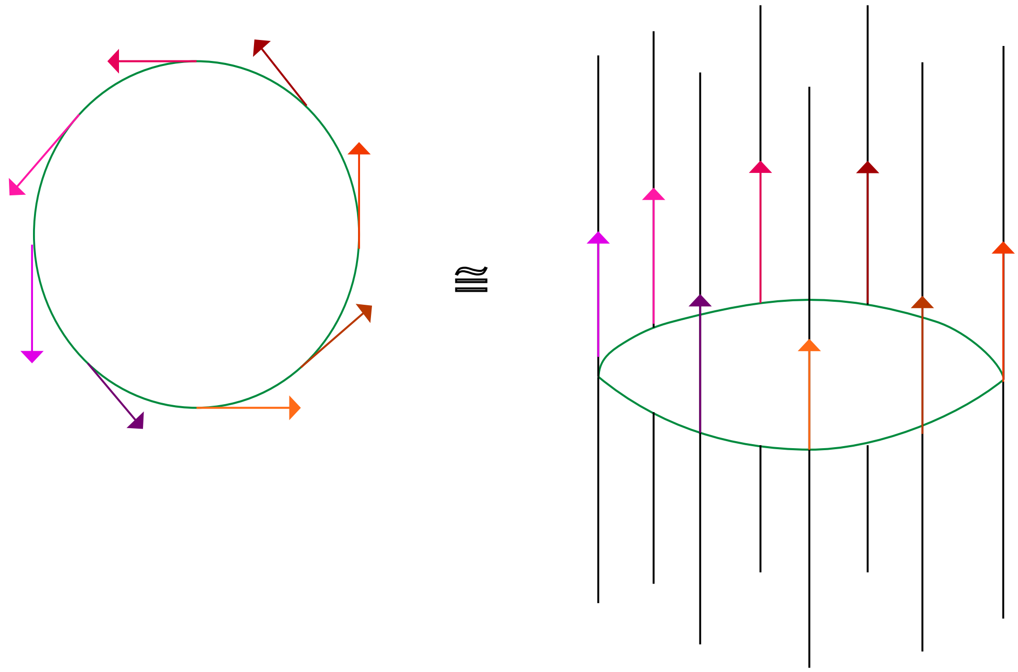

Figure 8.1: The tangent bundle of S^1.

Formally,

$$T(M) = \{(p, v) \mid p \in M, v \in T_p(M)\}.$$

There is a *natural projection*,

$$\pi \colon T(M) \to M, \quad \text{with} \quad \pi(p, v) = p.$$

We still have to give $T(M)$ a topology and to define a C^{k-1}-atlas.

For every chart, (U, φ), of M (with U open in M) we define the function, $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$, by

$$\widetilde{\varphi}(p, v) = (\varphi(p), \theta^{-1}_{U,\varphi,p}(v)),$$

where $(p, v) \in \pi^{-1}(U)$ and $\theta_{U,\varphi,p}$ is the isomorphism between $\mathbb{R}^n$ and $T_p(M)$ described just after Definition 7.12.

It is obvious that $\widetilde{\varphi}$ is a bijection between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$, an open subset of $\mathbb{R}^{2n}$. See Figure 8.2.

We give $T(M)$ the weakest topology that makes all the $\widetilde{\varphi}$ continuous, i.e., we take the collection of subsets of the form $\widetilde{\varphi}^{-1}(W)$, where W is any open subset of $\varphi(U) \times \mathbb{R}^n$, as a basis of the topology of $T(M)$.

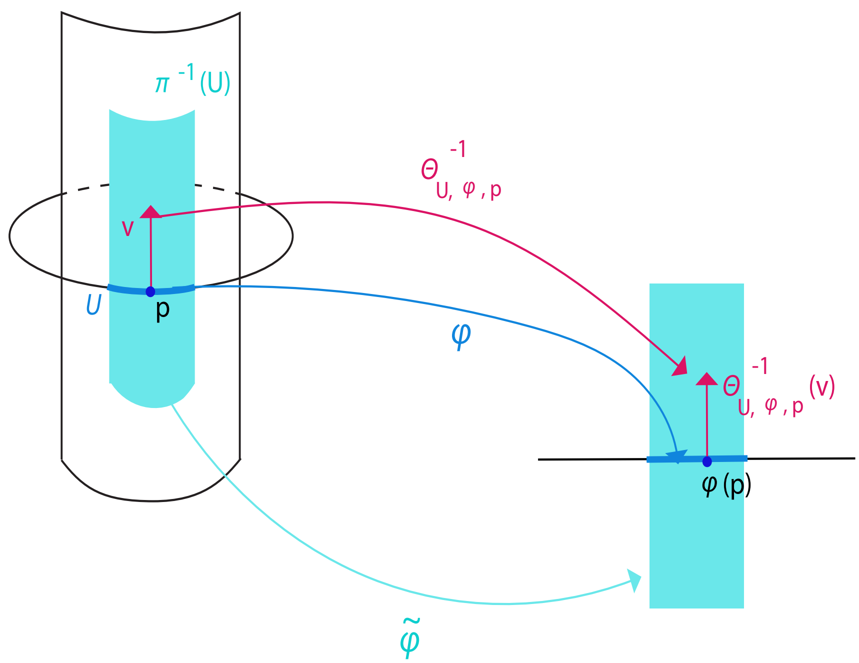

Figure 8.2: A chart for $T(S^1)$.

One easily checks that $T(M)$ is Hausdorff and second-countable in this topology.

If (U, φ) and (V, ψ) are overlapping charts, then the transition function

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

is given by

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(z, x) = (\psi \circ \varphi^{-1}(z), (\psi \circ \varphi^{-1})'_z(x)),$$

with $(z, x) \in \varphi(U \cap V) \times \mathbb{R}^n$.

It is clear that $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a C^{k-1}-map. Therefore, $T(M)$ is indeed a C^{k-1}-manifold of dimension $2n$, called the *tangent bundle*.

Remark: Even if the manifold M is naturally embedded in $\mathbb{R}^N$ (for some $N \geq n = \dim(M)$), it is not at all obvious how to view the tangent bundle, $T(M)$, as embedded in $\mathbb{R}^{N'}$, for some suitable N'. Hence, we see that the definition of an abtract manifold is unavoidable.

A similar construction can be carried out for the cotangent bundle.

In this case, we let $T^*(M)$ be the disjoint union of the cotangent spaces $T_p^*(M)$,

$$T^*(M) = \{(p, \omega) \mid p \in M, \omega \in T_p^*(M)\}.$$

We also have a natural projection, $\pi\colon T^*(M) \to M$.

We can define charts as follows:

For any chart, (U, φ), on M, we define the function $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(p, \omega) = \left(\varphi(p), \omega\left(\left(\frac{\partial}{\partial x_1}\right)_p\right), \ldots, \omega\left(\left(\frac{\partial}{\partial x_n}\right)_p\right)\right),$$

where $(p, \omega) \in \pi^{-1}(U)$ and the $\left(\frac{\partial}{\partial x_i}\right)_p$ are the basis of $T_p(M)$ associated with the chart (U, φ).

Again, one can make $T^*(M)$ into a C^{k-1}-manifold of dimension $2n$, called the *cotangent bundle*.

Another method using Version 3 of the definition of tangent vectors is presented in Section **??**.

For each chart (U, φ) on M, we obtain a chart

$$\widetilde{\varphi}^*\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$

on $T^*(M)$ given by

$$\widetilde{\varphi}^*(p, \omega) = (\varphi(p), \theta^*_{U,\varphi,\pi(\omega)}(\omega))$$

for all $(p, \omega) \in \pi^{-1}(U)$, where

$$\theta^*_{U,\varphi,p} = \iota \circ \theta^{\top}_{U,\varphi,p}\colon T_p^*(M) \to \mathbb{R}^n.$$

Here, $\theta^{\top}_{U,\varphi,p}\colon T_p^*(M) \to (\mathbb{R}^n)^*$ is obtained by dualizing the map, $\theta_{U,\varphi,p}\colon \mathbb{R}^n \to T_p(M)$, and $\iota\colon (\mathbb{R}^n)^* \to \mathbb{R}^n$ is the isomorphism induced by the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ and its dual basis.

For simplicity of notation, we also use the notation TM for $T(M)$ (resp. T^*M for $T^*(M)$).

Observe that for every chart, (U, φ), on M, there is a bijection

$$\tau_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

given by

$$\tau_U(p, v) = (p, \theta^{-1}_{U,\varphi,p}(v)).$$

Clearly, $pr_1 \circ \tau_U = \pi$, on $\pi^{-1}(U)$ as illustrated by the following commutative diagram:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\tau_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

Thus, locally, that is, over U, the bundle $T(M)$ looks like the product manifold $U \times \mathbb{R}^n$.

We say that $T(M)$ is *locally trivial* (over U) and we call τ_U a *trivializing map*.

For any $p \in M$, the vector space $\pi^{-1}(p) = \{p\} \times T_p(M) \cong T_p(M)$ is called the *fibre above p*.

Observe that the restriction of τ_U to $\pi^{-1}(p)$ is an isomorphism between $\{p\} \times T_p(M) \cong T_p(M)$ and $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$, for any $p \in M$.

Furthermore, for any two overlapping charts (U, φ) and (V, ψ), there is a function $g_{UV} \colon U \cap V \to \mathbf{GL}(n, \mathbb{R})$ such that

$$(\tau_U \circ \tau_V^{-1})(p, x) = (p, g_{UV}(p)(x))$$

for all $p \in U \cap V$ and all $x \in \mathbb{R}^n$, with $g_{UV}(p)$ given by

$$g_{UV}(p) = (\varphi \circ \psi^{-1})'_{\psi(p)}.$$

Obviously, $g_{UV}(p)$ is a linear isomorphism of $\mathbb{R}^n$ for all $p \in U \cap V$.

The maps $g_{UV}(p)$ are called the *transition functions* of the tangent bundle.

For example, if $M = S^n$, the n-sphere in $\mathbb{R}^{n+1}$, we have two charts given by the stereographic projection (U_N, σ_N) from the north pole, and the stereographic projection (U_S, σ_S) from the south pole (with $U_N = S^n - \{N\}$ and $U_S = S^n - \{S\}$), and on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\mathcal{I} = \sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

defined on $\varphi_N(U_N \cap U_S) = \varphi_S(U_N \cap U_S) = \mathbb{R}^n - \{0\}$, are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n);$$

that is, the inversion $\mathcal{I}$ of center $O = (0, \ldots, 0)$ and power 1.

We leave it as an exercice to prove that for every point $u \in \mathbb{R}^n - \{0\}$, we have

$$d\mathcal{I}_u(h) = \|u\|^{-2} \left(h - 2\frac{\langle u, h \rangle}{\|u\|^2}\, u \right),$$

the composition of the hyperplane reflection about the hyperplane $u^{\perp} \subseteq \mathbb{R}^n$ with the magnification of center O and ratio $\|u\|^{-2}$.

This is a *similarity transformation*. Therefore, the transition function g_{NS} (defined on $U_N \cap U_S$) of the tangent bundle TS^n is given by

$$g_{NS}(p)(h) = \|\sigma_S(p)\|^{-2} \left(h - 2\frac{\langle \sigma_S(p), h \rangle}{\|\sigma_S(p)\|^2}\, \sigma_S(p) \right).$$

All these ingredients are part of being a *vector bundle*.

For more on bundles, see Lang [33], Gallot, Hulin and Lafontaine [22], Lafontaine [31] or Bott and Tu [7].

When $M = \mathbb{R}^n$, observe that
$T(M) = M \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, i.e., the bundle $T(M)$ is (globally) trivial.

Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds, we can define the function, $dh\colon T(M) \to T(N)$, (also denoted Th, or h_*, or Dh) by setting

$$dh(u) = dh_p(u), \quad \text{iff} \quad u \in T_p(M).$$

We leave the next proposition as an exercise to the reader (A proof can be found in Berger and Gostiaux [6]).

Proposition 8.1. *Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds M and N (with $k \geq 1$), the map $dh\colon T(M) \to T(N)$ is a C^{k-1} map.*

We are now ready to define vector fields.

8.2 Vector Fields, Lie Derivative

Definition 8.1. Let M be a C^{k+1} manifold, with $k \geq 1$. For any open subset, U of M, a *vector field on U* is any *section X of $T(M)$ over U*, that is, any function $X\colon U \to T(M)$ such that $\pi \circ X = \mathrm{id}_U$ (i.e., $X(p) \in T_p(M)$, for every $p \in U$). We also say that X is a *lifting of U into $T(M)$*.

We say that X is a *C^k-vector field on U* iff X is a section over U and a C^k-map.

The set of C^k-vector fields over U is denoted $\Gamma^{(k)}(U, T(M))$; see Figure 8.3.

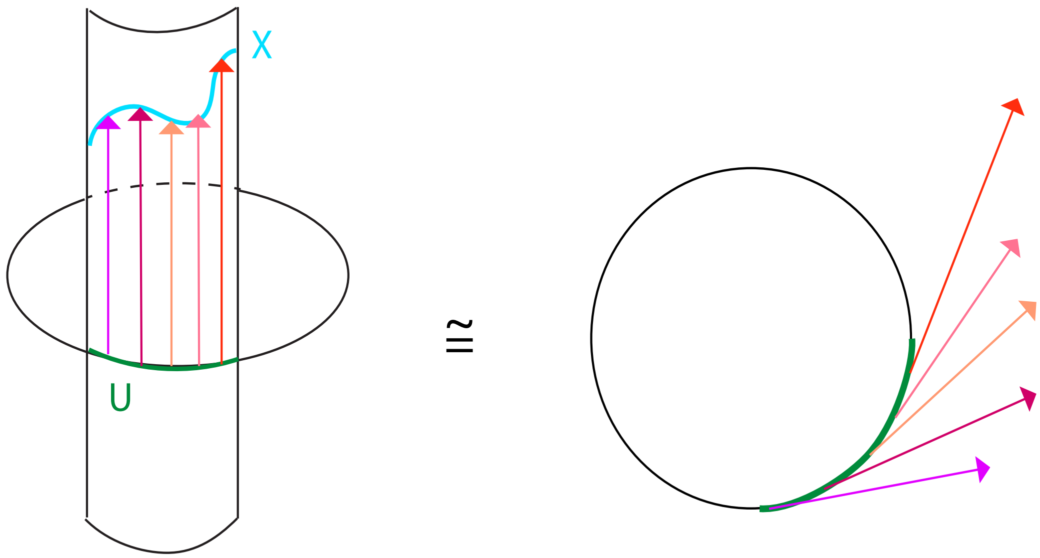

Figure 8.3: A vector field on S^1 represented as the section X in $T(S^1)$.

Given a curve, $\gamma\colon [a,b] \to M$, a *vector field X along γ* is any section of $T(M)$ over γ, i.e., a C^k-function, $X\colon [a,b] \to T(M)$, such that $\pi \circ X = \gamma$. We also say that *X lifts γ into $T(M)$*.

Clearly, $\Gamma^{(k)}(U, T(M))$ is a real vector space.

For short, the space $\Gamma^{(k)}(M, T(M))$ is also denoted by $\Gamma^{(k)}(T(M))$ (or $\mathfrak{X}^{(k)}(M)$, or even $\Gamma(T(M))$ or $\mathfrak{X}(M)$).

Remark: We can also define a *C^j-vector field on U* as a section, X, over U which is a C^j-map, where $0 \leq j \leq k$. Then, we have the vector space $\Gamma^{(j)}(U, T(M))$, *etc*.

If $M = \mathbb{R}^n$ and U is an open subset of M, then $T(M) = \mathbb{R}^n \times \mathbb{R}^n$ and a section of $T(M)$ over U is simply a function, X, such that

$$X(p) = (p, u), \quad \text{with} \quad u \in \mathbb{R}^n,$$

for all $p \in U$. In other words, X is defined by a function, $f\colon U \to \mathbb{R}^n$ (namely, $f(p) = u$).

This corresponds to the "old" definition of a vector field in the more basic case where the manifold, M, is just $\mathbb{R}^n$.

For any vector field $X \in \Gamma^{(k)}(U, T(M))$ and for any $p \in U$, we have $X(p) = (p, v)$ for some $v \in T_p(M)$, and it is convenient to denote the vector v by X_p so that $X(p) = (p, X_p)$.

In fact, *in most situations it is convenient to identify $X(p)$ with $X_p \in T_p(M)$, and we will do so from now on*.

This amounts to identifying the isomorphic vector spaces $\{p\} \times T_p(M)$ and $T_p(M)$.

Let us illustrate the advantage of this convention with the next definition.

Given any C^k-function, $f \in \mathcal{C}^k(U)$, and a vector field, $X \in \Gamma^{(k)}(U, T(M))$, we define the vector field, fX, by

$$(fX)_p = f(p)X_p, \quad p \in U.$$

Obviously, $fX \in \Gamma^{(k)}(U, T(M))$, which shows that $\Gamma^{(k)}(U, T(M))$ *is also a* $\mathcal{C}^k(U)$*-module*.

For any chart, (U, φ), on M it is easy to check that the map

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p, \quad p \in U,$$

is a C^k-vector field on U (with $1 \leq i \leq n$). This vector field is denoted $\left(\frac{\partial}{\partial x_i}\right)$ or $\frac{\partial}{\partial x_i}$.

Definition 8.2. Let M be a C^{k+1} manifold and let X be a C^k vector field on M. If U is any open subset of M and f is any function in $\mathcal{C}^k(U)$, then the *Lie derivative of f with respect to X*, denoted $X(f)$ or $L_X f$, is the function on U given by

$$X(f)(p) = X_p(f) = X_p(\mathbf{f}), \qquad p \in U.$$

Observe that

$$X(f)(p) = df_p(X_p),$$

where df_p is identified with the linear form in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}), \qquad v \in T_pM,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$ (see the discussion following Proposition 7.15).

The Lie derivative, $L_X f$, is also denoted $X[f]$.

As a special case, when (U, φ) is a chart on M, the vector field, $\frac{\partial}{\partial x_i}$, just defined above induces the function

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f, \quad f \in U,$$

denoted $\frac{\partial}{\partial x_i}(f)$ or $\left(\frac{\partial}{\partial x_i}\right) f$.

It is easy to check that $X(f) \in \mathcal{C}^{k-1}(U)$.

As a consequence, every vector field $X \in \Gamma^{(k)}(U, T(M))$ induces a linear map,

$$L_X \colon \mathcal{C}^k(U) \longrightarrow \mathcal{C}^{k-1}(U),$$

given by $f \mapsto X(f)$.

It is immediate to check that L_X has the Leibniz property, i.e.,

$$L_X(fg) = L_X(f)g + fL_X(g).$$

Linear maps with this property are called *derivations*.

Thus, we see that every vector field induces some kind of differential operator, namely, a derivation.

Unfortunately, not every derivation of the above type arises from a vector field, although this turns out to be true in the smooth case i.e., when $k = \infty$ (for a proof, see Gallot, Hulin and Lafontaine [22] or Lafontaine [31]).

In the rest of this section, unless stated otherwise, we assume that $k \geq 1$. The following easy proposition holds (c.f. Warner [52]):

Proposition 8.2. *Let X be a vector field on the C^{k+1}-manifold, M, of dimension n. Then, the following are equivalent:*

(a) X is C^k.

(b) If (U, φ) is a chart on M and if $f_1, \ldots, f_n$ are the functions on U uniquely defined by

$$X \restriction U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

then each f_i is a C^k-map.

(c) Whenever U is open in M and $f \in \mathcal{C}^k(U)$, then $X(f) \in \mathcal{C}^{k-1}(U)$.

Given any two C^k-vector field, X, Y, on M, for any function, $f \in \mathcal{C}^k(M)$, we defined above the function $X(f)$ and $Y(f)$.

Thus, we can form $X(Y(f))$ (resp. $Y(X(f))$), which are in $\mathcal{C}^{k-2}(M)$.

Unfortunately, even in the smooth case, there is generally no vector field, Z, such that

$$Z(f) = X(Y(f)), \quad \text{for all } f \in \mathcal{C}^k(M).$$

This is because $X(Y(f))$ (and $Y(X(f))$) involve second-order derivatives.

However, if we consider $X(Y(f)) - Y(X(f))$, then second-order derivatives cancel out and there is a unique vector field inducing the above differential operator.

Intuitively, $XY - YX$ measures the "failure of X and Y to commute."

Proposition 8.3. *Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, there is a unique C^{k-1}-vector field, $[X, Y]$, such that*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \textit{ for all } f \in \mathcal{C}^{k-1}(M).$$

Definition 8.3. Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, the *Lie bracket*, $[X, Y]$, of X and Y, is the C^{k-1} vector field defined so that

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \text{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

An an example, in $\mathbb{R}^3$, if X and Y are the two vector fields,

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial y},$$

then

$$[X, Y] = -\frac{\partial}{\partial z}.$$

We also have the following simple proposition whose proof is left as an exercise (or, see Do Carmo [16]):

Proposition 8.4. *Given any C^{k+1}-manifold, M, of dimension n, for any C^k-vector fields, X, Y, Z, on M, for all $f, g \in \mathcal{C}^k(M)$, we have:*

(a) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity).*

(b) $[X, X] = 0$.

(c) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

(d) $[-, -]$ *is bilinear.*

Consequently, for smooth manifolds ($k = \infty$), the space of vector fields, $\Gamma^{(\infty)}(T(M))$, is a vector space equipped with a bilinear operation, $[-, -]$, that satisfies the Jacobi identity.

This makes $\Gamma^{(\infty)}(T(M))$ a *Lie algebra*.

Let $h\colon M \to N$ be a diffeomorphism between two manifolds. Then, vector fields can be transported from N to M and conversely.

Definition 8.4. Let $h\colon M \to N$ be a diffeomorphism between two C^{k+1} manifolds. For every C^k vector field, Y, on N, the *pull-back of Y along h* is the vector field, h^*Y, on M, given by

$$(h^*Y)_p = dh^{-1}_{h(p)}(Y_{h(p)}), \qquad p \in M.$$

See Figure 8.4.

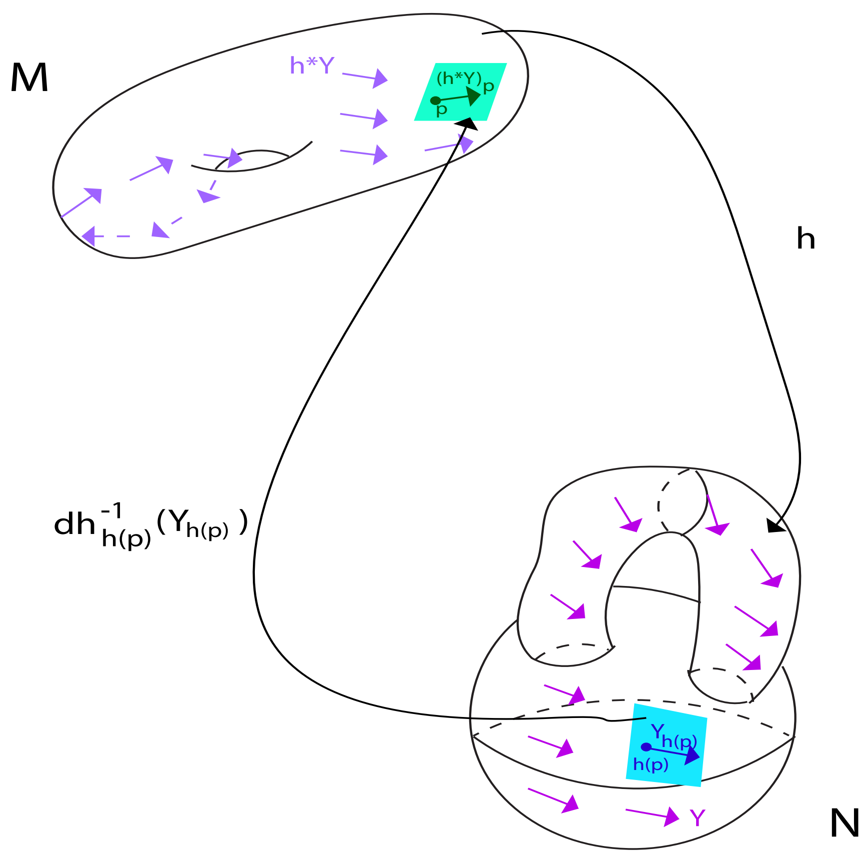

Figure 8.4: The pull-back of the vector field Y.

For every C^k vector field, X, on M, the *push-forward of X along h* is the vector field, h_*X, on N, given by

$$h_*X = (h^{-1})^*X,$$

that is, for every $p \in M$,

$$(h_*X)_{h(p)} = dh_p(X_p),$$

or equivalently,

$$(h_*X)_q = dh_{h^{-1}(q)}(X_{h^{-1}(q)}), \qquad q \in N.$$

See Figure 8.5.

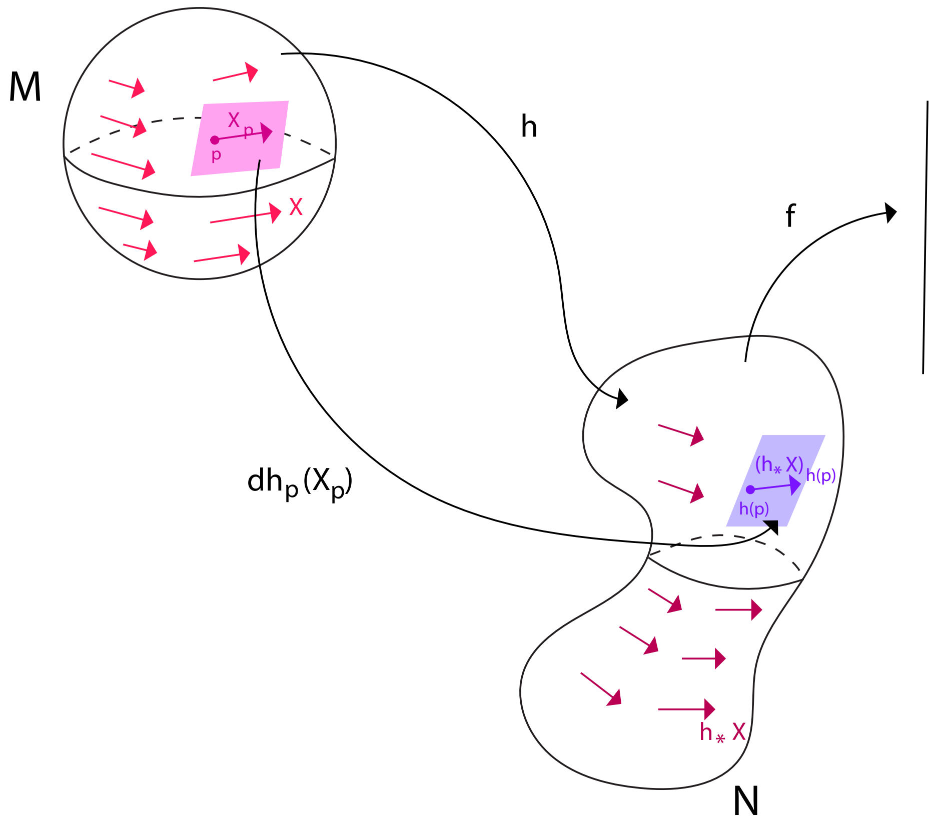

Figure 8.5: The push-forward of the vector field X.

It is not hard to check that

$$L_{h_*X}f = L_X(f \circ h) \circ h^{-1},$$

for any function $f \in C^k(N)$.

One more notion will be needed to when we deal with Lie algebras.

Definition 8.5. Let $h\colon M \to N$ be a C^{k+1}-map of manifolds. If X is a C^k vector field on M and Y is a C^k vector field on N, we say that *X and Y are h-related* iff

$$dh \circ X = Y \circ h.$$

Proposition 8.5. *Let $h\colon M \to N$ be a C^{k+1}-map of manifolds, let X and Y be C^k vector fields on M and let X_1, Y_1 be C^k vector fields on N. If X is h-related to X_1 and Y is h-related to Y_1, then $[X, Y]$ is h-related to $[X_1, Y_1]$.*

8.3 Integral Curves, Flow of a Vector Field, One-Parameter Groups of Diffeomorphisms

We begin with integral curves and (local) flows of vector fields on a manifold.

Definition 8.6. Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. An *integral curve (or trajectory) for X with initial condition p_0* is a C^{k-1}-curve, $\gamma\colon I \to M$, so that

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \text{for all } t \in I \quad \text{and} \quad \gamma(0) = p_0,$$

where $I = (a, b) \subseteq \mathbb{R}$ is an open interval containing 0.

What definition 8.6 says is that an integral curve, γ, with initial condition p_0 is a curve on the manifold M passing through p_0 and such that, for every point $p = \gamma(t)$ on this curve, the tangent vector to this curve at p, i.e., $\dot{\gamma}(t)$, coincides with the value, X_p, of the vector field X at p.

Given a vector field, X, as above, and a point $p_0 \in M$, is there an integral curve through p_0? Is such a curve unique? If so, how large is the open interval I?

We provide some answers to the above questions below.

Definition 8.7. Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. A *local flow for X at p_0* is a map,

$$\varphi \colon J \times U \to M,$$

where $J \subseteq \mathbb{R}$ is an open interval containing 0 and U is an open subset of M containing p_0, so that for every $p \in U$, the curve $t \mapsto \varphi(t, p)$ is an integral curve of X with initial condition p.

Thus, a local flow for X is a family of integral curves for all points in some small open set around p_0 such that these curves all have the same domain, J, independently of the initial condition, $p \in U$.

The following theorem is the main existence theorem of local flows.

This is a promoted version of a similar theorem in the classical theory of ODE's in the case where M is an open subset of $\mathbb{R}^n$.

Theorem 8.6. *(Existence of a local flow) Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. There is an open interval $J \subseteq \mathbb{R}$ containing 0 and an open subset $U \subseteq M$ containing p_0, so that there is a unique local flow $\varphi\colon J \times U \to M$ for X at p_0.*

What this means is that if $\varphi_1\colon J \times U \to M$ and $\varphi_2\colon J \times U \to M$ are both local flows with domain $J \times U$, then $\varphi_1 = \varphi_2$. Furthermore, φ is C^{k-1}.

Theorem 8.6 holds under more general hypotheses, namely, when the vector field satisfies some *Lipschitz* condition, see Lang [33] or Berger and Gostiaux [6].

Now, we know that for any initial condition, p_0, there is some integral curve through p_0.

However, there could be two (or more) integral curves $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ with initial condition p_0.

This leads to the natural question: How do γ_1 and γ_2 differ on $I_1 \cap I_2$? The next proposition shows they don't!

Proposition 8.7. *Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. If $\gamma_1\colon I_1 \to M$ and $\gamma_2\colon I_2 \to M$ are any two integral curves both with initial condition p_0, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$. See Figure 8.6.*

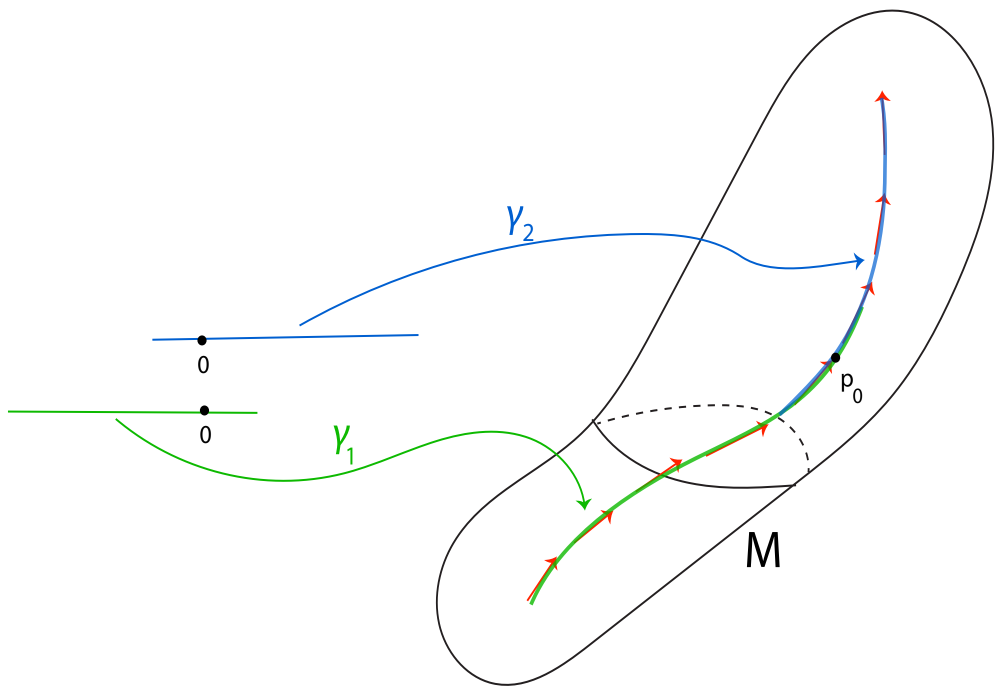

Figure 8.6: Two integral curves, γ_1 and γ_2, with initial condition p_0, which agree on the domain overlap $I_1 \cap I_2$.

Proposition 8.7 implies the important fact that there is a *unique maximal* integral curve with initial condition p.

Indeed, if $\{\gamma_j \colon I_j \to M\}_{j \in K}$ is the family of all integral curves with initial condition p (for some big index set, K), if we let $I(p) = \bigcup_{j \in K} I_j$, we can define a curve, $\gamma_p \colon I(p) \to M$, so that

$$\gamma_p(t) = \gamma_j(t), \quad \text{if} \quad t \in I_j.$$

Since γ_j and γ_l agree on $I_j \cap I_l$ for all $j, l \in K$, the curve γ_p is indeed well defined and it is clearly an integral curve with initial condition p with the largest possible domain (the open interval, $I(p)$).

The curve γ_p is called the *maximal integral curve with initial condition p* and it is also denoted by $\gamma(p, t)$.

Note that Proposition 8.7 implies that any two distinct integral curves are disjoint, i.e., do not intersect each other.

Consider the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and shown in Figure 8.7.

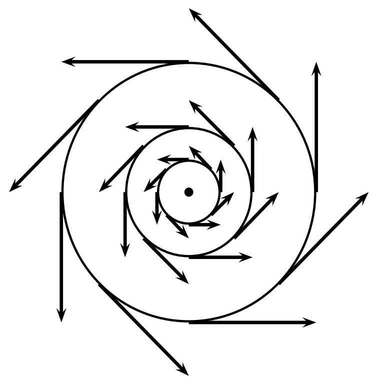

Figure 8.7: A vector field in $\mathbb{R}^2$

If we write $\gamma(t) = (x(t), y(t))$, the differential equation, $\dot{\gamma}(t) = X(\gamma(t))$, is expressed by

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t), \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we write $X = \binom{x}{y}$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the above equation is written as

$$X' = AX.$$

Now, as

$$e^{tA} = I + \frac{A}{1!}\,t + \frac{A^2}{2!}\,t^2 + \cdots + \frac{A^n}{n!}\,t^n + \cdots ,$$

we get

$$\frac{d}{dt}(e^{tA}) = A + \frac{A^2}{1!}\,t + \frac{A^3}{2!}\,t^2 + \cdots + \frac{A^n}{(n-1)!}\,t^{n-1} + \cdots = Ae^{tA},$$

so we see that $e^{tA}p$ is a solution of the ODE $X' = AX$ with initial condition $X = p$, and by uniqueness, $X = e^{tA}p$ is *the* solution of our ODE starting at $X = p$.

Thus, our integral curve, γ_p, through $p = \binom{x_0}{y_0}$ is the circle given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Observe that $I(p) = \mathbb{R}$, for every $p \in \mathbb{R}^2$.

Here is an example of a vector field on $M = \mathbb{R}$ that has integral curves not defined on the whole of $\mathbb{R}$.

Let X be the vector field on $\mathbb{R}$ given by

$$X(x) = (1 + x^2)\frac{\partial}{\partial x}.$$

It is easy to see that the maximal integral curve with initial condition $p_0 = 0$ is the curve $\gamma\colon (-\pi/2, \pi/2) \to \mathbb{R}$ given by

$$\gamma(t) = \tan t.$$

The following interesting question now arises: Given any $p_0 \in M$, if $\gamma_{p_0}\colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0 and, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$, then there is a maximal integral curve, $\gamma_{p_1}\colon I(p_1) \to M$, with initial condition p_1;

What is the relationship between γ_{p_0} and γ_{p_1}, if any?

The answer is given by

Proposition 8.8. *Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$) and let p_0 be a point on M. If $\gamma_{p_0}\colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$ and $\gamma_{p_1}\colon I(p_1) \to M$ is the maximal integral curve with initial condition p_1, then*

$$I(p_1) = I(p_0) - t_1 \quad \textit{and} \quad \gamma_{p_1}(t) = \gamma_{\gamma_{p_0}(t_1)}(t) = \gamma_{p_0}(t + t_1),$$

for all $t \in I(p_0) - t_1$ See Figure 8.8.

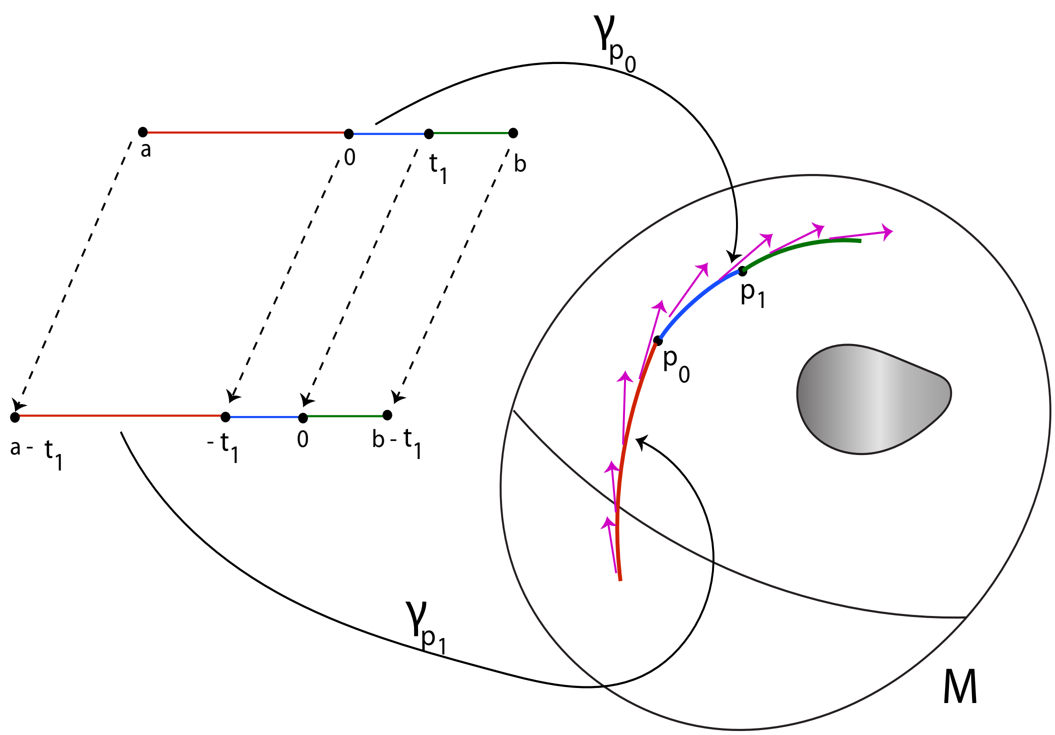

Figure 8.8: The integral curve γ_{p_1} is a reparametrization of γ_{p_0}.

Proposition 8.8 says that the traces $\gamma_{p_0}(I(p_0))$ and $\gamma_{p_1}(I(p_1))$ in M of the maximal integral curves γ_{p_0} and γ_{p_1} are identical; they only differ by a simple reparametrization ($u = t + t_1$).

It is useful to restate Proposition 8.8 by changing point of view.

So far, we have been focusing on integral curves, i.e., given any $p_0 \in M$, we let t vary in $I(p_0)$ and get an integral curve, γ_{p_0}, with domain $I(p_0)$.

Instead of holding $p_0 \in M$ fixed, we can hold $t \in \mathbb{R}$ fixed and consider the set

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\},$$

i.e., the set of points such that it is possible to "travel for t units of time from p" along the maximal integral curve, γ_p, with initial condition p (It is possible that $\mathcal{D}_t(X) = \emptyset$).

By definition, if $\mathcal{D}_t(X) \neq \emptyset$, the point $\gamma_p(t)$ is well defined, and so, we obtain a map, $\Phi_t^X \colon \mathcal{D}_t(X) \to M$, with domain $\mathcal{D}_t(X)$, given by

$$\Phi_t^X(p) = \gamma_p(t).$$

Definition 8.8. Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. For any $t \in \mathbb{R}$, let

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\}$$

and

$$\mathcal{D}(X) = \{(t, p) \in \mathbb{R} \times M \mid t \in I(p)\}$$

and let $\Phi^X \colon \mathcal{D}(X) \to M$ be the map given by

$$\Phi^X(t, p) = \gamma_p(t).$$

The map Φ^X is called the *(global) flow of X* and $\mathcal{D}(X)$ is called its *domain of definition*.

For any $t \in \mathbb{R}$ such that $\mathcal{D}_t(X) \neq \emptyset$, the map, $p \in \mathcal{D}_t(X) \mapsto \Phi^X(t, p) = \gamma_p(t)$, is denoted by Φ_t^X (i.e.,

$$\Phi_t^X(p) = \Phi^X(t, p) = \gamma_p(t)).$$

Observe that

$$\mathcal{D}(X) = \bigcup_{p \in M} (I(p) \times \{p\}).$$

Also, using the Φ_t^X notation, the property of Proposition 8.8 reads

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X, \tag{$*$}$$

whenever both sides of the equation make sense.

Indeed, the above says

$$\Phi_s^X(\Phi_t^X(p)) = \Phi_s^X(\gamma_p(t)) = \gamma_{\gamma_p(t)}(s) = \gamma_p(s+t) = \Phi_{s+t}^X(p).$$

Using the above property, we can easily show that the Φ_t^X are invertible. In fact, the inverse of Φ_t^X is Φ_{-t}^X.

Theorem 8.9. *Let X be a C^{k-1} vector field on a C^k-manifold, M, ($k \geq 2$). The following properties hold:*

(a) For every $t \in \mathbb{R}$, if $\mathcal{D}_t(X) \neq \emptyset$, then $\mathcal{D}_t(X)$ is open (this is trivially true if $\mathcal{D}_t(X) = \emptyset$).

(b) The domain, $\mathcal{D}(X)$, of the flow, Φ^X, is open and the flow is a C^{k-1} map, $\Phi^X \colon \mathcal{D}(X) \to M$.

(c) Each $\Phi_t^X \colon \mathcal{D}_t(X) \to \mathcal{D}_{-t}(X)$ is a C^{k-1}-diffeomorphism with inverse Φ_{-t}^X.

(d) For all $s, t \in \mathbb{R}$, the domain of definition of $\Phi_s^X \circ \Phi_t^X$ is contained but generally not equal to $\mathcal{D}_{s+t}(X)$. However, $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X) = \mathcal{D}_{s+t}(X)$ if s and t have the same sign. Moreover, on $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X)$, we have

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X.$$

We may omit the superscript, X, and write Φ instead of Φ^X if no confusion arises.

The reason for using the terminology flow in referring to the map Φ^X can be clarified as follows:

For any t such that $\mathcal{D}_t(X) \neq \emptyset$, every integral curve, γ_p, with initial condition $p \in \mathcal{D}_t(X)$, is defined on some open interval containing $[0, t]$, and we can picture these curves as "flow lines" along which the points p flow (travel) for a time interval t.

Then, $\Phi^X(t, p)$ is the point reached by "flowing" for the amount of time t on the integral curve γ_p (through p) starting from p.

Intuitively, we can imagine the flow of a fluid through M, and the vector field X is the field of velocities of the flowing particles.

Given a vector field, X, as above, it may happen that $\mathcal{D}_t(X) = M$, for all $t \in \mathbb{R}$.

In this case, namely, when $\mathcal{D}(X) = \mathbb{R} \times M$, we say that the vector field X is *complete*.

Then, the Φ_t^X are diffeomorphisms of M and they form a group.

The family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ a called a *1-parameter group of X*.

In this case, Φ^X induces a group homomorphism, $(\mathbb{R}, +) \longrightarrow \mathrm{Diff}(M)$, from the additive group $\mathbb{R}$ to the group of C^{k-1}-diffeomorphisms of M.

By abuse of language, even when it is **not** the case that $\mathcal{D}_t(X) = M$ for all t, the family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ is called a *local 1-parameter group of X*, even though it is *not* a group, because the composition $\Phi_s^X \circ \Phi_t^X$ may not be defined.

If we go back to the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

since the integral curve, $\gamma_p(t)$, through $p = \binom{x_0}{x_0}$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

the global flow associated with X is given by

$$\Phi^X(t, p) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} p,$$

and each diffeomorphism, Φ^X_t, is the rotation,

$$\Phi^X_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The 1-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is the group of rotations in the plane, $\mathbf{SO}(2)$.

More generally, if B is an $n\times n$ invertible matrix that has a real logarithm A (that is, if $e^A = B$), then the matrix A defines a vector field, X, in $\mathbb{R}^n$, with

$$X = \sum_{i,j=1}^{n} (a_{ij}x_j)\frac{\partial}{\partial x_i},$$

whose integral curves are of the form,

$$\gamma_p(t) = e^{tA}p,$$

and we have

$$\gamma_p(1) = Bp.$$

The one-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is given by $\{e^{tA}\}_{t\in\mathbb{R}}$.

When M is compact, it turns out that every vector field is complete, a nice and useful fact.

Proposition 8.10. *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. If M is compact, then X is complete, i.e., $\mathcal{D}(X) = \mathbb{R} \times M$. Moreover, the map $t \mapsto \Phi_t^X$ is a homomorphism from the additive group $\mathbb{R}$ to the group, $\mathrm{Diff}(M)$, of (C^{k-1}) diffeomorphisms of M.*

Remark: The proof of Proposition 8.10 also applies when X is a vector field with compact support (this means that the closure of the set $\{p \in M \mid X(p) \neq 0\}$ is compact).

If $h\colon M \to N$ is a diffeomorphism and X is a vector field on M, it can be shown that the local 1-parameter group associated with the vector field, $h_* X$, is

$$\{h \circ \Phi_t^X \circ h^{-1}\}_{t \in \mathbb{R}}.$$

A point $p \in M$ where a vector field vanishes, i.e., $X(p) = 0$, is called a *critical point of X*.

Critical points play a major role in the study of vector fields, in differential topology (e.g., the celebrated Poincaré–Hopf index theorem) and especially in Morse theory, but we won't go into this here.

Another famous theorem about vector fields says that every smooth vector field on a sphere of even dimension (S^{2n}) must vanish in at least one point (the so-called "hairy-ball theorem."

On S^2, it says that you can't comb your hair without having a singularity somewhere. Try it, it's true!).

Let us just observe that if an integral curve, γ, passes through a critical point, p, then γ is reduced to the point p, i.e., $\gamma(t) = p$, for all t.

Then, we see that if a maximal integral curve is defined on the whole of $\mathbb{R}$, either it is injective (it has no self-intersection), or it is simply periodic (i.e., there is some $T > 0$ so that $\gamma(t + T) = \gamma(t)$, for all $t \in \mathbb{R}$ and γ is injective on $[0, T[$), or it is reduced to a single point.

We conclude this section with the definition of the Lie derivative of a vector field with respect to another vector field.

Say we have two vector fields X and Y on M. For any $p \in M$, we can flow along the integral curve of X with initial condition p to $\Phi_t(p)$ (for t small enough) and then evaluate Y there, getting $Y(\Phi_t(p))$.

Now, this vector belongs to the tangent space $T_{\Phi_t(p)}(M)$, but $Y(p) \in T_p(M)$.

So to "compare" $Y(\Phi_t(p))$ and $Y(p)$, we bring back $Y(\Phi_t(p))$ to $T_p(M)$ by applying the tangent map, $d\Phi_{-t}$, at $\Phi_t(p)$, to $Y(\Phi_t(p))$. (Note that to alleviate the notation, we use the slight abuse of notation $d\Phi_{-t}$ instead of $d(\Phi_{-t})_{\Phi_t(p)}$.)

Then, we can form the difference $d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)$, divide by t and consider the limit as t goes to 0.

Definition 8.9. Let M be a C^{k+1} manifold. Given any two C^k vector fields, X and Y on M, for every $p \in M$, the *Lie derivative of Y with respect to X at p*, denoted $(L_X\, Y)_p$, is given by

$$\begin{aligned}(L_X\, Y)_p &= \lim_{t \longrightarrow 0} \frac{d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)}{t} \\ &= \frac{d}{dt}\left(d\Phi_{-t}(Y(\Phi_t(p)))\right)\bigg|_{t=0}.\end{aligned}$$

It can be shown that $(L_X\, Y)_p$ is our old friend, the Lie bracket, i.e.,

$$(L_X\, Y)_p = [X, Y]_p.$$

(For a proof, see Warner [52] or O'Neill [43]).

In terms of Definition 8.4, observe that

$$\begin{aligned}
(L_X\,Y)_p &= \lim_{t\longrightarrow 0} \frac{\big((\Phi_{-t})_* Y\big)(p) - Y(p)}{t} \\
&= \lim_{t\longrightarrow 0} \frac{\big(\Phi_t^* Y\big)(p) - Y(p)}{t} \\
&= \frac{d}{dt}\big(\Phi_t^* Y\big)(p)\bigg|_{t=0},
\end{aligned}$$

since $(\Phi_{-t})^{-1} = \Phi_t$.