Chapter 7

Manifolds, Tangent Spaces, Cotangent Spaces, Submanifolds

7.1 Charts and Manifolds

In Chapter 1 we defined the notion of a manifold embedded in some ambient space, $\mathbb{R}^N$.

In order to maximize the range of applications of the theory of manifolds it is necessary to generalize the concept of a manifold to spaces that are not a priori embedded in some $\mathbb{R}^N$.

The basic idea is still that, whatever a manifold is, it is a topological space that can be covered by a collection of open subsets, U_α, where each U_α is isomorphic to some "standard model," *e.g.*, some open subset of Euclidean space, $\mathbb{R}^n$.

Of course, manifolds would be very dull without functions defined on them and between them.

This is a general fact learned from experience: *Geometry arises not just from spaces but from spaces and interesting classes of functions between them*.

In particular, we still would like to "do calculus" on our manifold and have good notions of curves, tangent vectors, differential forms, etc.

The small drawback with the more general approach is that the definition of a tangent vector is more abstract.

We can still define the notion of a curve on a manifold, but such a curve does not live in any given $\mathbb{R}^n$, so it it not possible to define tangent vectors in a simple-minded way using derivatives.

Instead, we have to resort to the notion of chart. This is not such a strange idea.

For example, a geography atlas gives a set of maps of various portions of the earth and this provides a very good description of what the earth is, without actually imagining the earth embedded in 3-space.

Given $\mathbb{R}^n$, recall that the *projection functions*, $pr_i \colon \mathbb{R}^n \to \mathbb{R}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n.$$

For technical reasons, in particular to ensure that partitions of unity exist (a crucial tool in manifold theory) from now on, all topological spaces under consideration will be assumed to be Hausdorff and second-countable (which means that the topology has a countable basis).

Definition 7.1. Given a topological space, M, a *chart* (or *local coordinate map*) is a pair, (U, φ), where U is an open subset of M and $\varphi\colon U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geq 1$).

For any $p \in M$, a chart, (U, φ), is a *chart at p* iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of p* w.r.t. the chart.

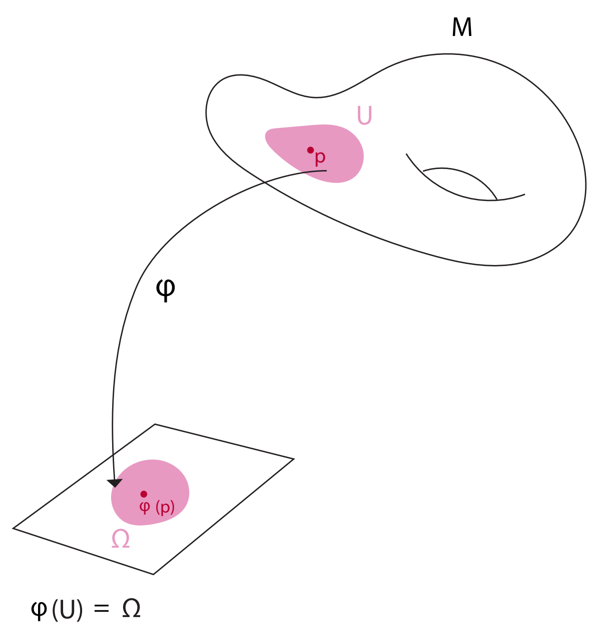

Figure 7.1: A chart (U, φ) on M.

The inverse, (Ω, φ^{-1}), of a chart is called a *local parametrization*.

Given any two charts, (U_i, φ_i) and (U_j, φ_j), if $U_i \cap U_j \neq \emptyset$, we have the *transition maps*, $\varphi_i^j \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_j^i \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, defined by

$$\varphi_i^j = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_j^i = \varphi_i \circ \varphi_j^{-1}.$$

Clearly, $\varphi_j^i = (\varphi_i^j)^{-1}$.

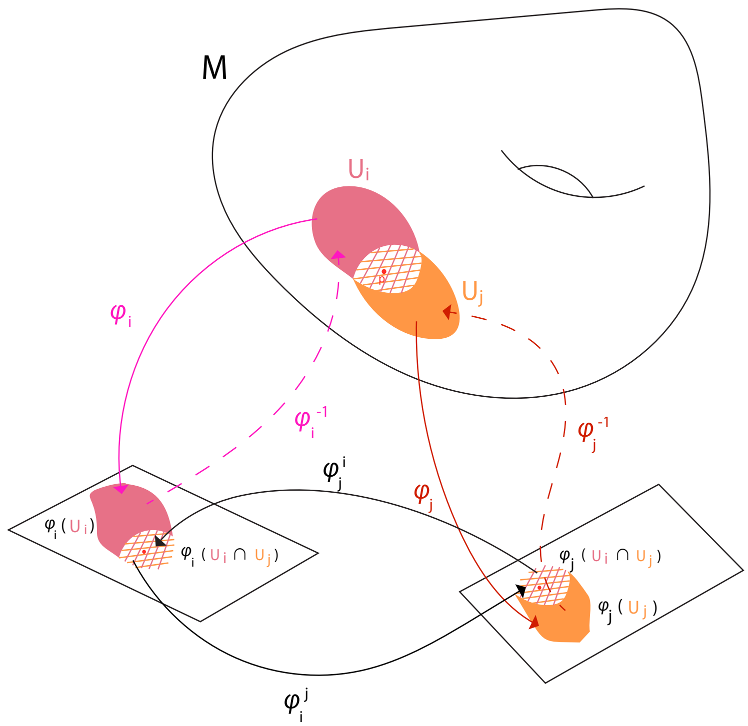

Figure 7.2: The transition maps φ_i^j and φ_j^i.

Observe that the transition maps φ_i^j (resp. φ_j^i) are maps between *open subsets of* $\mathbb{R}^n$.

This is good news! Indeed, the whole arsenal of calculus is available for functions on $\mathbb{R}^n$, and we will be able to promote many of these results to manifolds by imposing suitable conditions on transition functions.

Definition 7.2. Given a topological space, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$ for all i;

(2) The U_i cover M, i.e.,
$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism. When $k = \infty$, the φ_i^j are smooth diffeomorphisms.

We must insure that we have enough charts in order to carry out our program of generalizing calculus on $\mathbb{R}^n$ to manifolds.

For this, we must be able to add new charts whenever necessary, provided that they are consistent with the previous charts in an existing atlas.

Technically, given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the atlas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$).

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* iff every chart of one is compatible with the other atlas.

This is equivalent to saying that the union of the two atlases is still an atlas.

It is immediately verified that compatibility induces an equivalence relation on C^k n-atlases on M.

In fact, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of atlases compatible with $\mathcal{A}$.

Definition 7.3. Given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k-manifold of dimension n* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M. Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on* $\mathbb{R}^n$. When $k = \infty$, we say that M is a *smooth manifold*.

Remark: It might have been better to use the terminology *abstract manifold* rather than manifold, to emphasize the fact that the space M is not a priori a subspace of $\mathbb{R}^N$, for some suitable N.

We can allow $k = 0$ in the above definitions. Condition (3) in Definition 7.2 is void, since a C^0-diffeomorphism is just a homeomorphism, but φ_i^j is always a homeomorphism.

In this case, M is called a *topological manifold of dimension n*.

We do not require a manifold to be connected but we require all the components to have the same dimension, n.

Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same. This is quite easy to show if $k \geq 1$ but for $k = 0$, this requires a deep theorem of Brouwer.

What happens if $n = 0$? In this case, every one-point subset of M is open, so every subset of M is open, i.e., M is any (countable if we assume M to be second-countable) set with the discrete topology!

Observe that since $\mathbb{R}^n$ is locally compact and locally connected, so is every manifold.

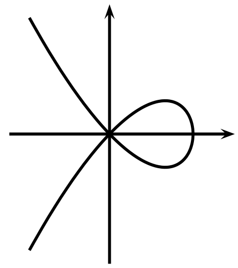

Figure 7.3: A nodal cubic; not a manifold

In order to get a better grasp of the notion of manifold it is useful to consider examples of non-manifolds.

First, consider the curve in $\mathbb{R}^2$ given by the zero locus of the equation

$$y^2 = x^2 - x^3,$$

namely, the set of points

$$M_1 = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2 - x^3\}.$$

This curve showed in Figure 7.3 and called a *nodal cubic* is also defined as the parametric curve

$$\begin{aligned} x &= 1 - t^2 \\ y &= t(1 - t^2). \end{aligned}$$

We claim that M_1 is not even a topological manifold. The problem is that the nodal cubic has a self-intersection at the origin.

If M_1 was a topological manifold, then there would be a connected open subset, $U \subseteq M_1$, containing the origin, $O = (0, 0)$, namely the intersection of a small enough open disc centered at O with M_1, and a local chart, $\varphi\colon U \to \Omega$, where Ω is some connected open subset of $\mathbb{R}$ (that is, an open interval), since φ is a homeomorphism.

However, $U - \{O\}$ consists of four disconnected components and $\Omega - \varphi(O)$ of two disconnected components, contradicting the fact that φ is a homeomorphism.

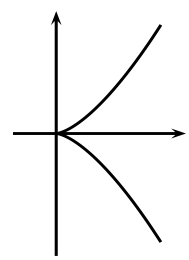

Figure 7.4: A Cuspidal Cubic

Let us now consider the curve in $\mathbb{R}^2$ given by the zero locus of the equation

$$y^2 = x^3,$$

namely, the set of points

$$M_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3\}.$$

This curve showed in Figure 7.4 and called a *cuspidal cubic* is also defined as the parametric curve

$$\begin{aligned} x &= t^2 \\ y &= t^3. \end{aligned}$$

Consider the map, $\varphi\colon M_2 \to \mathbb{R}$, given by

$$\varphi(x, y) = y^{1/3}.$$

Since $x = y^{2/3}$ on M_2, we see that φ^{-1} is given by

$$\varphi^{-1}(t) = (t^2, t^3)$$

and clearly, φ is a homeomorphism, so M_2 is a topological manifold.

However, in the atlas consisting of the single chart, $\{\varphi\colon M_2 \to \mathbb{R}\}$, the space M_2 is also a smooth manifold!

Indeed, as there is a single chart, condition (3) of Definition 7.2 holds vacuously.

This fact is somewhat unexpected because the cuspidal cubic is usually not considered smooth at the origin, since the tangent vector of the parametric curve, $c\colon t \mapsto (t^2, t^3)$, at the origin is the zero vector (the velocity vector at t, is $c'(t) = (2t, 3t^2)$).

However, this apparent paradox has to do with the fact that, as a parametric curve, M_2 is *not immersed* in $\mathbb{R}^2$ since c' is not injective (see Definition 7.21 (a)), whereas as an abstract manifold, with this single chart, M_2 is diffeomorphic to $\mathbb{R}$.

Now, we also have the chart, $\psi \colon M_2 \to \mathbb{R}$, given by

$$\psi(x, y) = y,$$

with ψ^{-1} given by

$$\psi^{-1}(u) = (u^{2/3}, u).$$

Then, observe that

$$\varphi \circ \psi^{-1}(u) = u^{1/3},$$

a map that is *not* differentiable at $u = 0$. Therefore, the atlas $\{\varphi \colon M_2 \to \mathbb{R}, \psi \colon M_2 \to \mathbb{R}\}$ is not C^1 and thus, with respect to that atlas, M_2 is not a C^1-manifold.

The example of the cuspidal cubic shows a peculiarity of the definition of a C^k (or C^∞) manifold:

If a space, M, happens to be a topological manifold because it has an atlas consisting of a single chart, then it is automatically a smooth manifold!

In particular, if $f\colon U \to \mathbb{R}^m$ is any *continuous* function from some open subset, U, of $\mathbb{R}^n$, to $\mathbb{R}^m$, then the graph, $\Gamma(f) \subseteq \mathbb{R}^{n+m}$, of f given by

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} \mid x \in U\}$$

is a smooth manifold with respect to the atlas consisting of the single chart, $\varphi\colon \Gamma(f) \to U$, given by

$$\varphi(x, f(x)) = x,$$

with its inverse, $\varphi^{-1}\colon U \to \Gamma(f)$, given by

$$\varphi^{-1}(x) = (x, f(x)).$$

The notion of a submanifold using the concept of "adapted chart" (see Definition 7.20 in Section 7.6) gives a more satisfactory treatment of C^k (or smooth) submanifolds of $\mathbb{R}^n$.

The example of the cuspidal cubic also shows clearly that whether a topological space is a C^k-manifold or a smooth manifold depends on the choice of atlas.

In some cases, M does not come with a topology in an obvious (or natural) way and a slight variation of Definition 7.2 is more convenient in such a situation:

Definition 7.4. Given a set, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) Each U_i is a subset of M and $\varphi_i \colon U_i \to \varphi_i(U_i)$ is a bijection onto an open subset, $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the set $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$ and the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism.

Then, the notion of a chart being compatible with an atlas and of two atlases being compatible is just as before and we get a new definition of a manifold, analogous to Definition 7.2.

But, this time, we give M the topology in which the open sets are arbitrary unions of domains of charts (the U_i's in a maximal atlas).

It is not difficult to verify that the axioms of a topology are verified and M is indeed a topological space with this topology.

It can also be shown that when M is equipped with the above topology, then the maps $\varphi_i \colon U_i \to \varphi_i(U_i)$ are homeomorphisms, so M is a manifold according to Definition 7.3.

We require M to be Hausdorff and second-countable with this topology.

Thus, we are back to the original notion of a manifold where it is assumed that M is already a topological space.

If the underlying topological space of a manifold is compact, then M has some finite atlas.

Also, if $\mathcal{A}$ is some atlas for M and (U, φ) is a chart in $\mathcal{A}$, for any (nonempty) open subset, $V \subseteq U$, we get a chart, $(V, \varphi \upharpoonright V)$, and it is obvious that this chart is compatible with $\mathcal{A}$.

Thus, $(V, \varphi \upharpoonright V)$ is also a chart for M. This observation shows that if U is any open subset of a C^k-manifold, M, then U is also a C^k-manifold whose charts are the restrictions of charts on M to U.

Interesting manifolds often occur as the result of a quotient construction.

For example, real projective spaces and Grassmannians are obtained this way.

In this situation, the natural topology on the quotient object is the quotient topology but, unfortunately, even if the original space is Hausdorff, the quotient topology may not be.

Therefore, it is useful to have criteria that insure that a quotient topology is Hausdorff (or second-countable). We will present two criteria.

First, let us review the notion of quotient topology.

Definition 7.5. Given any topological space, X, and any set, Y, for any surjective function, $f\colon X \to Y$, we define the *quotient topology on Y determined by f* (also called the *identification topology on Y determined by f*), by requiring a subset, V, of Y to be open if $f^{-1}(V)$ is an open set in X.

Given an equivalence relation R on a topological space X, if $\pi\colon X \to X/R$ is the projection sending every $x \in X$ to its equivalence class $[x]$ in X/R, the space X/R equipped with the quotient topology determined by π is called the *quotient space of X modulo R*.

Thus a set, V, of equivalence classes in X/R is open iff $\pi^{-1}(V)$ is open in X, which is equivalent to the fact that $\bigcup_{[x]\in V}[x]$ is open in X.

It is immediately verified that Definition 7.5 defines topologies and that $f\colon X \to Y$ and $\pi\colon X \to X/R$ are continuous when Y and X/R are given these quotient topologies.

One should be careful that if X and Y are topological spaces and $f\colon X \to Y$ is a continuous surjective map, Y *does not* necessarily have the quotient topology determined by f.

Indeed, it may not be true that a subset V of Y is open when $f^{-1}(V)$ is open. However, this will be true in two important cases.

Definition 7.6. A continuous map, $f\colon X \to Y$, is an *open map* (or simply *open*) if $f(U)$ is open in Y whenever U is open in X, and similarly, $f\colon X \to Y$, is a *closed map* (or simply *closed*) if $f(F)$ is closed in Y whenever F is closed in X.

Then, Y has the quotient topology induced by the continuous surjective map f if either f is open or f is closed.

If $\cdot\colon G \times X \to X$ is an action of a group G on a topological space X and if for every $g \in G$, the map from X to itself given by $x \mapsto g \cdot x$ is continuous, then it can be show that the projection $\pi\colon X \to X/G$ is an open map.

Furthermore, if G is a finite group, then π is a closed map.

Unfortunately, the Hausdorff separation property is not necessarily preserved under quotient.

Nevertheless, it is preserved in some special important cases.

Proposition 7.1. *Let X and Y be topological spaces, let $f\colon X \to Y$ be a continuous surjective map, and assume that X is compact and that Y has the quotient topology determined by f. Then Y is Hausdorff iff f is a closed map.*

Another simple criterion uses continuous open maps.

Proposition 7.2. *Let $f\colon X \to Y$ be a surjective continuous map between topological spaces. If f is an open map then Y is Hausdorff iff the set*

$$\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

Note that the hypothesis of Proposition 7.2 implies that Y has the quotient topology determined by f.

A special case of Proposition 7.2 is discussed in Tu.

Given a topological space, X, and an equivalence relation, R, on X, we say that R is *open* if the projection map, $\pi\colon X \to X/R$, is an open map, where X/R is equipped with the quotient topology.

Then, if R is an open equivalence relation on X, the topological quotient space X/R is Hausdorff iff R is closed in $X \times X$.

The following proposition yields a sufficient condition for second-countability:

Proposition 7.3. *If X is a topological space and R is an open equivalence relation on X, then for any basis, $\{B_\alpha\}$, for the topology of X, the family $\{\pi(B_\alpha)\}$ is a basis for the topology of X/R, where $\pi\colon X \to X/R$ is the projection map. Consequently, if X is second-countable, then so is X/R.*

Example 7.1. The sphere S^n.

Using the stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $\sigma_N \colon S^n - \{N\} \to \mathbb{R}^n$ and $\sigma_S \colon S^n - \{S\} \to \mathbb{R}^n$, where $N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ (the north pole) and $S = (0, \cdots, 0, -1) \in \mathbb{R}^{n+1}$ (the south pole) be the maps called respectively *stereographic projection from the north pole* and *stereographic projection from the south pole* given by

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

and

$$\sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

Thus, if we let $U_N = S^n - \{N\}$ and $U_S = S^n - \{S\}$, we see that U_N and U_S are two open subsets covering S^n, both homeomorphic to $\mathbb{R}^n$.

Furthermore, it is easily checked that on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n - \{O\}$, so we conclude that (U_N, σ_N) and (U_S, σ_S) form a smooth atlas for S^n.

Example 7.2. Smooth manifolds in $\mathbb{R}^N$.

Any m-dimensional manifold M in $\mathbb{R}^N$ is a smooth manifold, because by Lemma 4.2, the inverse maps $\varphi^{-1}\colon U \to \Omega$ of the parametrizations $\varphi\colon \Omega \to U$ are charts that yield smooth transition functions.

In particular, by Theorem 4.8, any linear Lie group is a smooth manifold.

Example 7.3. The projective space $\mathbb{RP}^n$.

To define an atlas on $\mathbb{RP}^n$ it is convenient to view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

Given any $p = [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$, we call $(x_1, \ldots, x_{n+1})$ the *homogeneous coordinates* of p.

It is customary to write $(x_1 \colon \cdots : x_{n+1})$ instead of $[x_1, \ldots, x_{n+1}]$. (Actually, in most books, the indexing starts with 0, i.e., homogeneous coordinates for $\mathbb{RP}^n$ are written as $(x_0 \colon \cdots : x_n)$.)

Now, $\mathbb{RP}^n$ can also be viewed as the quotient of the sphere, S^n, under the equivalence relation where any two antipodal points, x and $-x$, are identified.

It is not hard to show that the projection $\pi \colon S^n \to \mathbb{RP}^n$ is both open and closed.

Since S^n is compact and second-countable, we can apply our previous results to prove that under the quotient topology, $\mathbb{RP}^n$ is Hausdorff, second-countable, and compact.

We define charts in the following way. For any i, with $1 \leq i \leq n+1$, let

$$U_i = \{(x_1\colon \cdots : x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Observe that U_i is well defined, because if $(y_1\colon \cdots : y_{n+1}) = (x_1\colon \cdots : x_{n+1})$, then there is some $\lambda \neq 0$ so that $y_i = \lambda z_i$, for $i = 1, \ldots, n+1$.

We can define a homeomorphism, φ_i, of U_i onto $\mathbb{R}^n$, as follows:

$$\varphi_i(x_1\colon \cdots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

where the ith component is omitted. Again, it is clear that this map is well defined since it only involves ratios.

We can also define the maps, ψ_i, from $\mathbb{R}^n$ to $U_i \subseteq \mathbb{RP}^n$, given by

$$\psi_i(x_1, \ldots, x_n) = (x_1 \colon \cdots : x_{i-1} \colon 1 \colon x_i \colon \cdots : x_n),$$

where the 1 goes in the ith slot, for $i = 1, \ldots, n+1$.

One easily checks that φ_i and ψ_i are mutual inverses, so the φ_i are homeomorphisms. On the overlap, $U_i \cap U_j$, (where $i \neq j$), as $x_j \neq 0$, we have

$$(\varphi_j \circ \varphi_i^{-1})(x_1, \ldots, x_n) = \\ \left(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

(We assumed that $i < j$; the case $j < i$ is similar.) This is clearly a smooth function from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

As the U_i cover $\mathbb{RP}^n$, see conclude that the (U_i, φ_i) are $n+1$ charts making a smooth atlas for $\mathbb{RP}^n$.

Intuitively, the space $\mathbb{RP}^n$ is obtained by gluing the open subsets U_i on their overlaps. Even for $n = 3$, this is not easy to visualize!

Example 7.4. The Grassmannian $G(k, n)$.

Recall that $G(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, also called k-planes.

Every k-plane, W, is the linear span of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; furthermore, $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ both span W iff there is an invertible $k \times k$-matrix, $\Lambda = (\lambda_{ij})$, such that

$$v_j = \sum_{i=1}^{k} \lambda_{ij} u_i, \quad 1 \leq j \leq k.$$

Obviously, there is a bijection between the collection of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$ and the collection of $n \times k$ matrices of rank k.

Furthermore, two $n \times k$ matrices A and B of rank k represent the same k-plane iff

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda.$$

(Note the analogy with projective spaces where two vectors u, v represent the same point iff $v = \lambda u$ for some invertible $\lambda \in \mathbb{R}$.)

The set of $n \times k$ matrices of rank k is a subset of $\mathbb{R}^{n \times k}$, in fact, an open subset.

One can show that the equivalence relation on $n \times k$ matrices of rank k given by

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda,$$

is open and that the graph of this equivalence relation is closed.

By Proposition 7.2, the Grassmannian $G(k, n)$ is Hausdorff and second-countable.

We can define the domain of charts (according to Definition 7.2) on $G(k,n)$ as follows: For every subset, $S = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, let U_S be the subset of $n \times k$ matrices, A, of rank k whose rows of index in $S = \{i_1, \ldots, i_k\}$ form an invertible $k \times k$ matrix denoted A_S.

Observe that the $k \times k$ matrix consisting of the rows of the matrix AA_S^{-1} whose index belong to S is the identity matrix, I_k.

Therefore, we can define a map, $\varphi_S \colon U_S \to \mathbb{R}^{(n-k)\times k}$, where $\varphi_S(A) =$ the $(n-k)\times k$ matrix obtained by deleting the rows of index in S from AA_S^{-1}.

We need to check that this map is well defined, i.e., that it does not depend on the matrix, A, representing W.

Let us do this in the case where $S = \{1, \ldots, k\}$, which is notationally simpler. The general case can be reduced to this one using a suitable permutation.

If $B = A\Lambda$, with Λ invertible, if we write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

as $B = A\Lambda$, we get $B_1 = A_1\Lambda$ and $B_2 = A_2\Lambda$, from which we deduce that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} B_1^{-1} = \begin{pmatrix} I_k \\ B_2 B_1^{-1} \end{pmatrix} = \\ \begin{pmatrix} I_k \\ A_2 \Lambda \Lambda^{-1} A_1^{-1} \end{pmatrix} = \begin{pmatrix} I_k \\ A_2 A_1^{-1} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} A_1^{-1}.$$

Therefore, our map is indeed well-defined.

It is clearly injective and we can define its inverse ψ_S as follows:

Let π_S be the permutation of $\{1, \ldots, n\}$ sending $\{1, \ldots, k\}$ to S defined such that if $S = \{i_1 < \cdots < i_k\}$, then $\pi_S(j) = i_j$ for $j = 1, \ldots, k$, and if $\{h_1 < \cdots < h_{n-k}\} = \{1, \ldots, n\} - S$, then $\pi_S(k + j) = h_j$ for $j = 1, \ldots, n - k$ (this is a k-shuffle).

If P_S is the permutation matrix associated with π_S, for any $(n - k) \times k$ matrix M, let

$$\psi_S(M) = P_S \begin{pmatrix} I_k \\ M \end{pmatrix}.$$

The effect of ψ_S is to "insert into M" the rows of the identity matrix I_k as the rows of index from S.

At this stage, we have charts that are bijections from subsets, U_S, of $G(k, n)$ to open subsets, namely, $\mathbb{R}^{(n-k)\times k}$.

Then, the reader can check that the transition map $\varphi_T \circ \varphi_S^{-1}$ from $\varphi_S(U_S \cap U_T)$ to $\varphi_T(U_S \cap U_T)$ is given by

$$M \mapsto (P_3 + P_4 M)(P_1 + P_2 M)^{-1},$$

where

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = P_T^{-1} P_S$$

is the matrix of the permutation $\pi_T^{-1} \circ \pi_S$.

This map is smooth, as it is given by determinants, and so, the charts (U_S, φ_S) form a smooth atlas for $G(k, n)$.

Finally, it is easy to check that the conditions of Definition 7.2 are satisfied, so the atlas just defined makes $G(k, n)$ into a topological space and a smooth manifold.

The Grassmannian $G(k, n)$ is actually compact. To see this, observe that if W is any k-plane, then using the Gram-Schmidt orthonormalization procedure, every basis $B = (b_1, \ldots, b_k)$ for W yields an orthonormal basis $U = (u_1, \ldots, u_k)$ and there is an invertible matrix, Λ, such that

$$U = B\Lambda,$$

where the the columns of B are the b_js and the columns of U are the u_js.

The matrices U have orthonormal columns and are characterized by the equation

$$U^\top U = I_k.$$

Consequently, the space of such matrices is closed and clearly bounded in $\mathbb{R}^{n\times k}$ and thus, compact.

The Grassmannian $G(k, n)$ is the quotient of this space under our usual equivalence relation and $G(k, n)$ is the image of a compact set under the projection map, which is clearly continuous, so $G(k, n)$ is compact.

Remark: The reader should have no difficulty proving that the collection of k-planes represented by matrices in U_S is precisely the set of k-planes, W, supplementary to the $(n-k)$-plane spanned by the $n-k$ canonical basis vectors $e_{j_{k+1}}, \ldots, e_{j_n}$ (i.e., $\mathrm{span}(W \cup \{e_{j_{k+1}}, \ldots, e_{j_n}\}) = \mathbb{R}^n$, where $S = \{i_1, \ldots, i_k\}$ and $\{j_{k+1}, \ldots, j_n\} = \{1, \ldots, n\} - S$).

Example 7.5. Product Manifolds.

Let M_1 and M_2 be two C^k-manifolds of dimension n_1 and n_2, respectively.

The topological space, $M_1 \times M_2$, with the product topology (the opens of $M_1 \times M_2$ are arbitrary unions of sets of the form $U \times V$, where U is open in M_1 and V is open in M_2) can be given the structure of a C^k-manifold of dimension $n_1 + n_2$ by defining charts as follows:

For any two charts, (U_i, φ_i) on M_1 and (V_j, ψ_j) on M_2, we declare that $(U_i \times V_j, \varphi_i \times \psi_j)$ is a chart on $M_1 \times M_2$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^{n_1+n_2}$ is defined so that

$$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q)), \quad \text{for all } (p, q) \in U_i \times V_j.$$

See Figure 7.5.

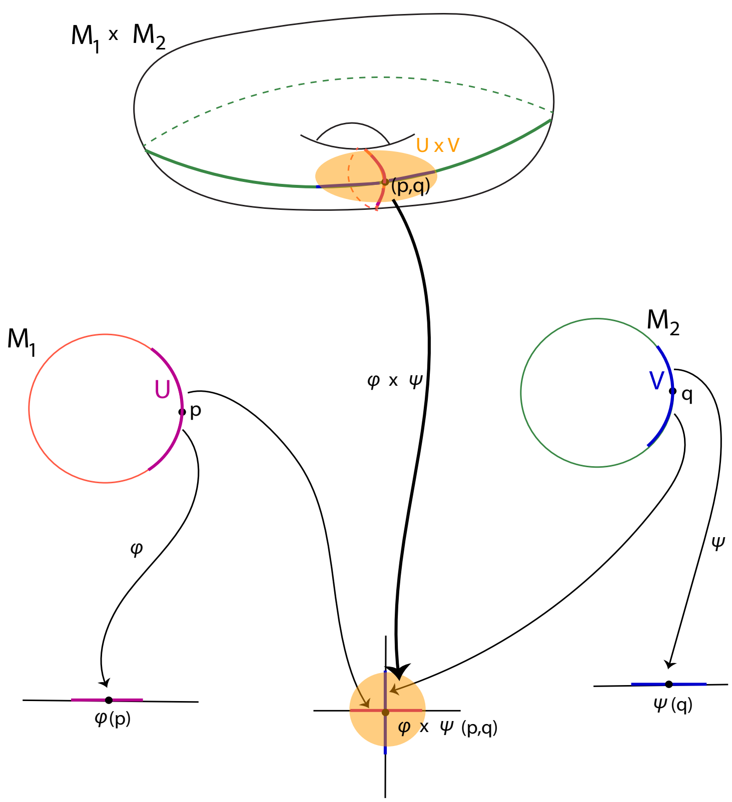

Figure 7.5: A chart for the torus as the product manifold $S^1 \times S^1$.

We define C^k-maps between manifolds as follows:

Definition 7.7. Given any two C^k-manifolds M and N of dimension m and n respectively, a *C^k-map* is a continuous function $h\colon M \to N$ satisfying the following property: For every $p \in M$, there is some chart (U, φ) at p and some chart (V, ψ) at $q = h(p)$, with $h(U) \subseteq V$ and

$$\psi \circ h \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \psi(V)$$

a C^k-function. See Figure 7.6.

It is easily shown that Definition 7.7 does not depend on the choice of charts. In particular, if $N = \mathbb{R}$, we obtain a *C^k-function on M*.

The requirement in Definition 7.7 that $h\colon M \to N$ should be continuous is actually redundant.

Other definitions of a C^k-map appear in the literature, some requiring continuity.

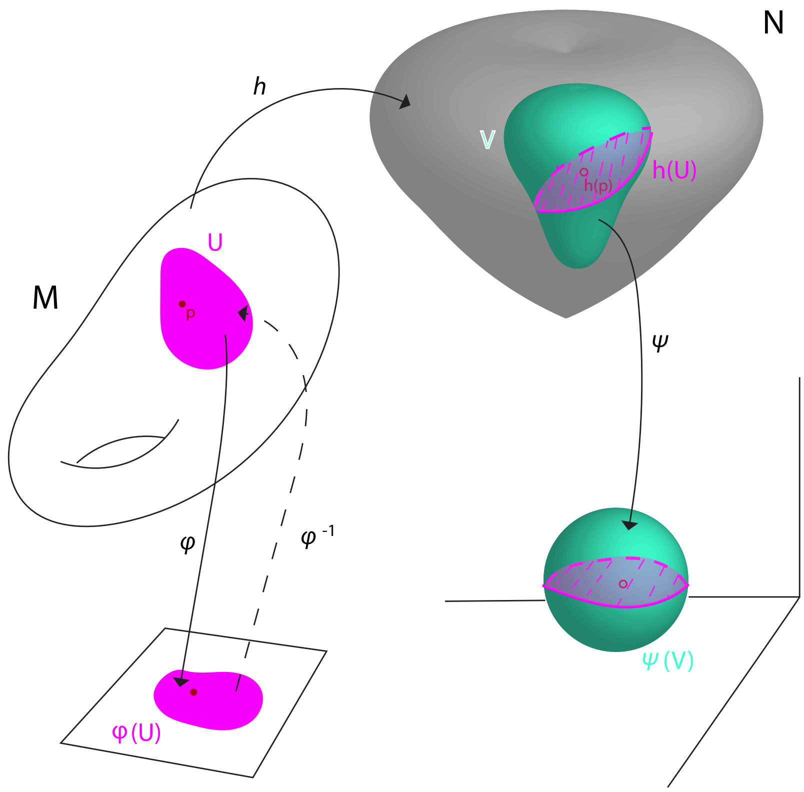

Figure 7.6: The C^k map from M to N, where M is a 2-dimensional manifold and N is a 3-dimensional manifold.

The following proposition from Berger and Gostiaux [6] (Theorem 2.3.3) helps clarifying how these definitions relate.

Proposition 7.4. *Let $h\colon M \to N$ be a function between two manifolds M and N. The following equivalences hold.*

(1) The map h is continuous, and for every $p \in M$, for every chart (U, φ) at p and every chart (V, ψ) at $h(p)$, the function $\psi \circ h \circ \varphi^{-1}$ from $\varphi(U \cap h^{-1}(V))$ to $\psi(V)$ is a C^k-function.

(2) The map h is continuous, and for every $p \in M$, for every chart (U, φ) at p and every chart (V, ψ) at $h(p)$, if $h(U) \subseteq V$, then the function $\psi \circ h \circ \varphi^{-1}$ from $\varphi(U)$ to $\psi(V)$ is a C^k-function.

(3) For every $p \in M$, there is some chart (U, φ) at p and some chart (V, ψ) at $q = h(p)$ with $h(U) \subseteq V$, such that the function $\psi \circ h \circ \varphi^{-1}$ from $\varphi(U)$ to $\psi(V)$ is a C^k-function.

Observe that Condition (3) states exactly the conditions of Definition 7.7, with the continuity requirement omitted.

Condition (1) is used by many texts. The continuity of h is required to ensure that $h^{-1}(V)$ is an open set.

The implication (2) $\Rightarrow$ (3) also requires the continuity of h.

Even though the continuity requirement in Definition 7.7 is redundant, it seems to us that it does not hurt to emphasize that C^k-maps are continuous.

In the special case where $N = \mathbb{R}$, we obtain the notion of a *C^k-function on M*.

One checks immediately that a function $f\colon M \to \mathbb{R}$ is a C^k-map iff the following condition holds.

Definition 7.8. A function $f\colon M \to \mathbb{R}$ is a *C^k-map* iff for every $p \in M$, there is some chart (U, φ) at p so that

$$f \circ \varphi^{-1} \colon \varphi(U) \longrightarrow \mathbb{R}$$

is a C^k-function. See Figure 7.7.

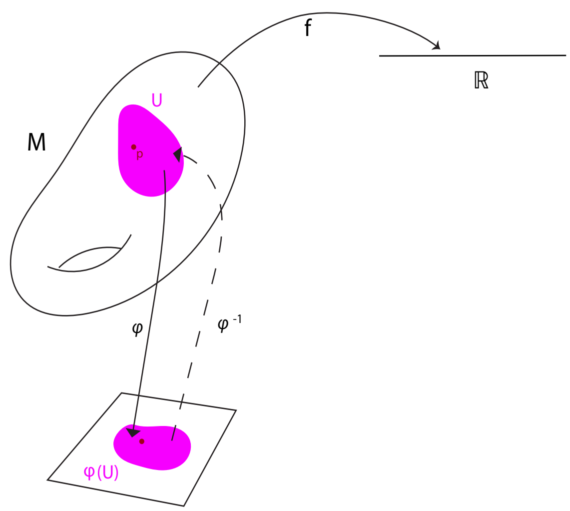

Figure 7.7: A schematic illustration of a C^k-function on the torus M.

If U is an open subset of M, the set of C^k-functions on U is denoted by $\mathcal{C}^k(U)$.

In particular, $\mathcal{C}^k(M)$ denotes the set of C^k-functions on the manifold, M. Observe that $\mathcal{C}^k(U)$ is a commutative ring.

On the other hand, if M is an open interval of $\mathbb{R}$, say $M = (a, b)$, then $\gamma\colon (a, b) \to N$ is called a *C^k-curve* in N.

One checks immediately that a function $\gamma\colon (a, b) \to N$ is a C^k-map iff the following condition holds.

Definition 7.9. A function $\gamma\colon (a, b) \to N$ is a C^k-map iff for every $q \in N$, there is some chart (V, ψ) at q and some open subinterval (c, d) of (a, b), so that $\gamma((c, d)) \subseteq V$ and

$$\psi \circ \gamma\colon (c, d) \longrightarrow \psi(V)$$

is a C^k-function. See Figure 7.8.

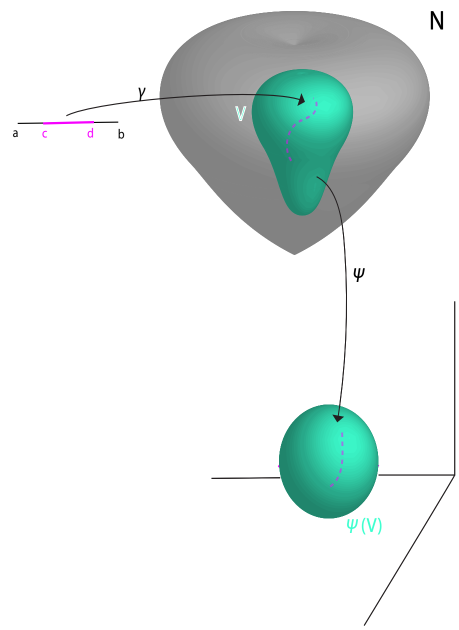

Figure 7.8: A schematic illustration of a C^k-curve in the solid spheroid N.

It is clear that the composition of C^k-maps is a C^k-map. A C^k-map, $h\colon M \to N$, between two manifolds is a *C^k-diffeomorphism* iff h has an inverse, $h^{-1}\colon N \to M$ (i.e., $h^{-1} \circ h = \mathrm{id}_M$ and $h \circ h^{-1} = \mathrm{id}_N$), and both h and h^{-1} are C^k-maps (in particular, h and h^{-1} are homeomorphisms).

Next, we define tangent vectors.

7.2 Tangent Vectors, Tangent Spaces

Let M be a C^k manifold of dimension n, with $k \geq 1$.

The purpose of the next three sections is to define the tangent space $T_p(M)$, at a point p of a manifold M.

We provide three definitions of the notion of a tangent vector to a manifold and prove their equivalence.

The first definition uses equivalence classes of curves on a manifold and is the most intuitive.

The second definition makes heavy use of the charts and of the transition functions.

It is also quite intuitive and it is easy to see that that it is equivalent to the first definition.

The second definition is the most convenient one to define the manifold structure of the tangent bundle $T(M)$ (see Section 8.1).

The third definition (given in the next section) is based on the view that a tangent vector v, at p, induces a differential operator on functions f, defined locally near p;

namely, the map $f \mapsto v(f)$ is a linear form satisfying an additional property akin to the rule for taking the derivative of a product (the Leibniz property).

Such linear forms are called *point-derivations*. This third definition is more intrinsic than the first two but more abstract.

However, for any point p on the manifold M and for any chart whose domain contains p, there is a convenient basis of the tangent space $T_p(M)$.

The third definition is also the most convenient one to define vector fields.

A few technical complications arise when M is not a smooth manifold (when $k \neq \infty$) but these are easily overcome using "stationary germs."

As pointed out by Serre, the relationship between the first definition and the third definition of the tangent space at p is best described by a nondegenerate pairing which shows that $T_p(M)$ is the dual of the space of point-derivations at p that vanish on stationary germs. This pairing is presented in Section 7.4.

The most intuitive method to define tangent vectors is to use curves.

Let $p \in M$ be any point on M and let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a C^1-curve passing through p, that is, with $\gamma(0) = p$.

Unfortunately, if M is not embedded in any $\mathbb{R}^N$, the derivative $\gamma'(0)$ does not make sense.

However, for any chart, (U, φ), at p, the map $\varphi \circ \gamma$ is a C^1-curve in $\mathbb{R}^n$ and the tangent vector $v = (\varphi \circ \gamma)'(0)$ is well defined.

The trouble is that different curves may yield the same v!

To remedy this problem, we define an equivalence relation on curves through p as follows:

Definition 7.10. Given a C^k manifold, M, of dimension n, for any $p \in M$, two C^1-curves, $\gamma_1 \colon (-\epsilon_1, \epsilon_1) \to M$ and $\gamma_2 \colon (-\epsilon_2, \epsilon_2) \to M$, through p (i.e., $\gamma_1(0) = \gamma_2(0) = p$) are *equivalent* iff there is some chart, (U, φ), at p so that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

See Figure 7.9.

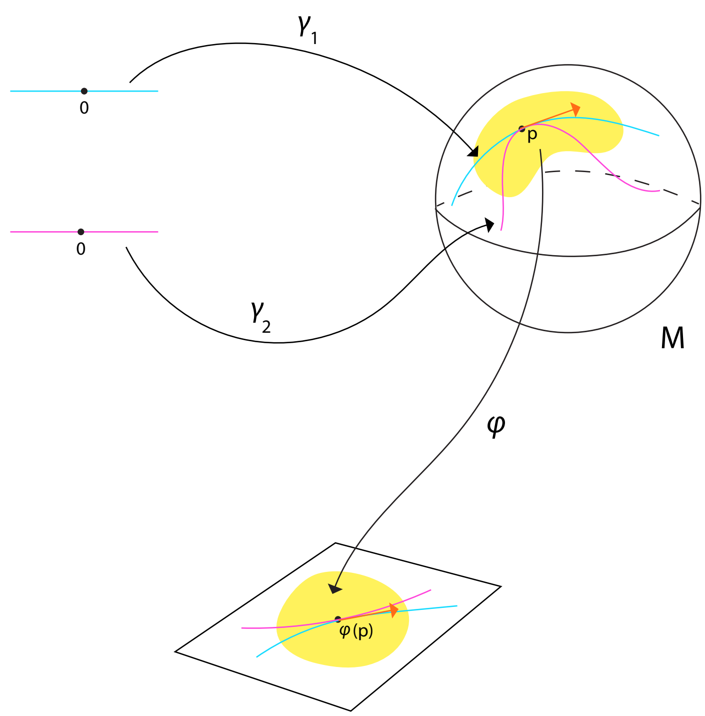

Figure 7.9: Equivalent curves γ_1, in blue, and γ_2, in pink.

The problem is that this definition seems to depend on the choice of the chart. Fortunately, this is not the case.

Definition 7.11. (Tangent Vectors, Version 1) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any equivalence class of C^1-curves through p on M, modulo the equivalence relation defined in Definition 7.10. The set of all tangent vectors at p is denoted by $T_p(M)$.

It is obvious that $T_p(M)$ is a vector space.

Observe that the map that sends a curve, $\gamma\colon (-\epsilon,\epsilon) \to M$, through p (with $\gamma(0) = p$) to its tangent vector, $(\varphi \circ \gamma)'(0) \in \mathbb{R}^n$ (for any chart (U, φ), at p), induces a map, $\overline{\varphi}\colon T_p(M) \to \mathbb{R}^n$.

It is easy to check that $\overline{\varphi}$ is a linear bijection (by definition of the equivalence relation on curves through p).

This shows that $T_p(M)$ is a vector space of dimension $n =$ dimension of M.

In particular, if M is an n-dimensional smooth manifold in $\mathbb{R}^N$ and if γ is a curve in M through p, then $\gamma'(0) = u$ is well defined as a vector in $\mathbb{R}^N$, and the equivalence class of all curves γ through p such that $(\varphi \circ \gamma)'(0)$ is the same vector in some chart $\varphi\colon U \to \Omega$ can be identified with u.

Thus, the tangent space T_pM to M at p is isomorphic to

$$\{\gamma'(0) \mid \gamma\colon (-\epsilon, \epsilon) \to M \text{ is a } C^1\text{-curve with } \gamma(0) = p\}.$$

In the special case of a linear Lie group G, Proposition 4.10 shows that the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism from some open subset of $\mathfrak{g}$ containing 0 to some open subset of G containing I.

For every $g \in G$, since L_g is a diffeomorphism, the map $L_g \circ \exp\colon L_g(\mathfrak{g}) \to G$ is a diffeomorphism from some open subset of $L_g(\mathfrak{g})$ containing 0 to some open subset of G containing g. Furthermore,

$$L_g(\mathfrak{g}) = g\mathfrak{g} = \{gX \mid X \in \mathfrak{g}\}.$$

Thus, we obtain smooth parametrizations of G whose inverses are charts on G, and since by definition of $\mathfrak{g}$, for every $X \in \mathfrak{g}$, the curve $\gamma(t) = ge^{tX}$ is a curve through g in G such that $\gamma'(0) = gX$, we see that *the tangent space T_gG to G at g is isomorphic to $g\mathfrak{g}$*.

One should observe that unless $M = \mathbb{R}^n$, in which case, for any $p, q \in \mathbb{R}^n$, the tangent space $T_q(M)$ is naturally isomorphic to the tangent space $T_p(M)$ by the translation $q - p$, for an arbitrary manifold, there is no relationship between $T_p(M)$ and $T_q(M)$ when $p \neq q$.

The second way of defining tangent vectors has the advantage that it makes it easier to define tangent bundles (see Section 8.1).

Definition 7.12. (Tangent Vectors, Version 2) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, consider the triples, (U, φ, u), where (U, φ) is any chart at p and u is any vector in $\mathbb{R}^n$.

Say that two such triples (U, φ, u) and (V, ψ, v) are *equivalent* iff

$$(\psi \circ \varphi^{-1})'_{\varphi(p)}(u) = v.$$

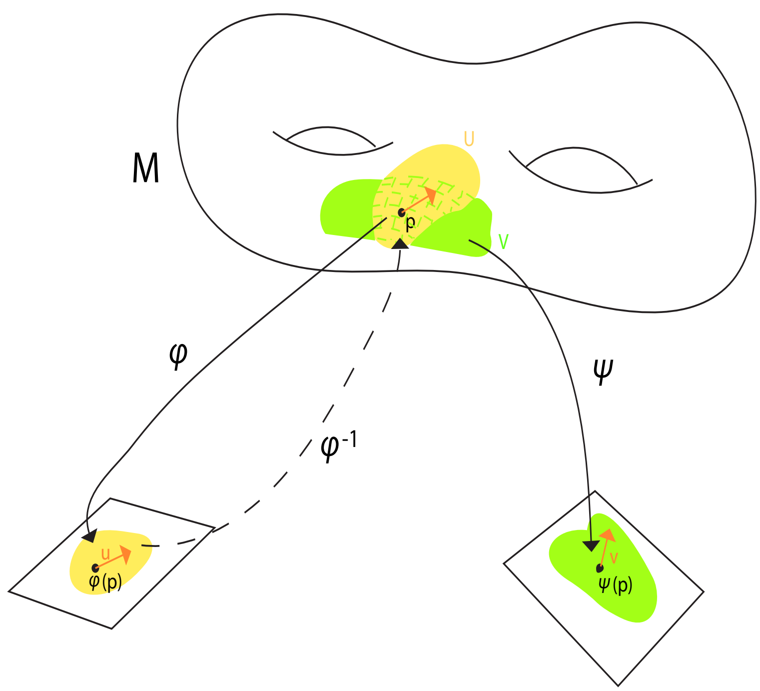

Figure 7.10: Two equivalent tangent vector u and v.

A *tangent vector* to M at p is an equivalence class of triples, $[(U, \varphi, u)]$, for the above equivalence relation.

The intuition behind Definition 7.12 is quite clear: The vector u is considered as a tangent vector to $\mathbb{R}^n$ at $\varphi(p)$.

If (U, φ) is a chart on M at p, we can define a natural isomorphism, $\theta_{U,\varphi,p} \colon \mathbb{R}^n \to T_p(M)$, between $\mathbb{R}^n$ and $T_p(M)$, as follows: For any $u \in \mathbb{R}^n$,

$$\theta_{U,\varphi,p} \colon u \mapsto [(U, \varphi, u)].$$

One immediately check that the above map is indeed linear and a bijection.

For simplicity of notation, we also use the notation T_pM for $T_p(M)$ (resp. T_p^*M for $T_p^*(M)$).

The equivalence of this definition with the definition in terms of curves (Definition 7.11) is easy to prove.

Proposition 7.5. *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For every $p \in M$, for every chart, (U, φ), at p, if $x = [\gamma]$ is any tangent vector (version 1) given by some equivalence class of C^1-curves $\gamma\colon (-\epsilon, +\epsilon) \to M$ through p (i.e., $p = \gamma(0)$), then the map*

$$x \mapsto [(U, \varphi, (\varphi \circ \gamma)'(0))]$$

is an isomorphism between $T_p(M)$-version 1 and $T_p(M)$-version 2.

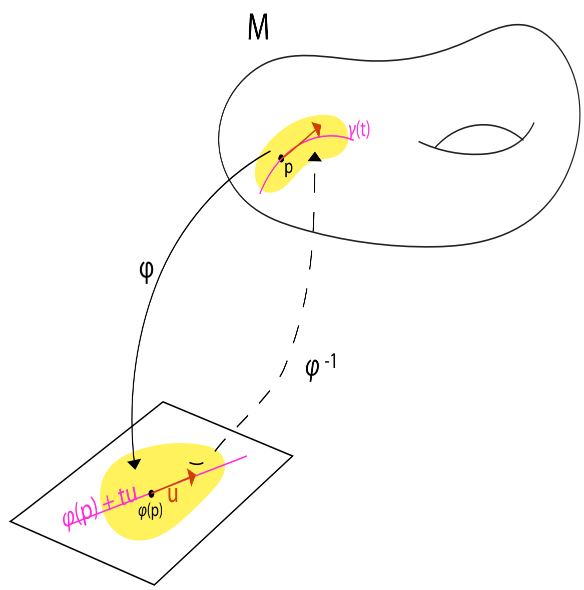

Figure 7.11: The tangent vector u is in one-to-one correspondence with the line through $\varphi(p)$ with direction u.

7.3 Tangent Vectors as Derivations

One of the defects of the above definitions of a tangent vector is that it has no clear relation to the C^k-differential structure of M.

In particular, the definition does not seem to have anything to do with the functions defined locally at p.

There is another way to define tangent vectors that reveals this connection more clearly. Moreover, such a definition is more intrinsic, i.e., does not refer explicitly to charts.

As a first step, consider the following: Let (U, φ) be a chart at $p \in M$ (where M is a C^k-manifold of dimension n, with $k \geq 1$) and let $x_i = pr_i \circ \varphi$, the ith local coordinate $(1 \leq i \leq n)$.

For any real-valued function f defined on $U \ni p$, set

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n.$$

(Here, $(\partial g / \partial X_i)|_y$ denotes the partial derivative of a function $g \colon \mathbb{R}^n \to \mathbb{R}$ with respect to the ith coordinate, evaluated at y.)

We would expect that the function that maps f to the above value is a linear map on the set of functions defined locally at p, but there is technical difficulty:

The set of real-valued functions defined locally at p is **not** a vector space!

To see this, observe that if f is defined on an open $U \ni p$ and g is defined on a different open $V \ni p$, then we do not know how to define $f + g$.

The problem is that we need to identify functions that agree on a smaller open subset. This leads to the notion of *germs*.

Definition 7.13. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *locally defined function at p* is a pair, (U, f), where U is an open subset of M containing p and f is a real-valued function defined on U.

Two locally defined functions, (U, f) and (V, g), at p are *equivalent* iff there is some open subset, $W \subseteq U \cap V$, containing p so that

$$f \restriction W = g \restriction W.$$

The equivalence class of a locally defined function at p, denoted $[f]$ or $\mathbf{f}$, is called a *germ at p*.

One should check that the relation of Definition 7.13 is indeed an equivalence relation.

For example, for every $a \in (-1, 1)$, the locally defined functions $(\mathbb{R} - \{1\}, 1/(1-x))$ and $((-1, 1), \sum_{n=0}^{\infty} x^n)$ at a are equivalent.

Of course, the value at p of all the functions, f, in any germ, $\mathbf{f}$, is $f(p)$. Thus, we set $\mathbf{f}(p) = f(p)$.

We can define addition of germs, multiplication of a germ by a scalar, and multiplication of germs as follows.

If (U, f) and (V, g) are two locally defined functions at p, we define $(U \cap V, f + g)$, $(U \cap V, fg)$ and $(U, \lambda f)$ as the locally defined functions at p given by $(f + g)(q) = f(q) + g(q)$ and $(fg)(q) = f(q)g(q)$ for all $q \in U \cap V$, and $(\lambda f)(q) = \lambda f(q)$ for all $q \in U$, with $\lambda \in \mathbb{R}$.

If $\mathbf{f} = [f]$ and $\mathbf{g} = [g]$ are two germs at p, and then

$$\begin{aligned} [f] + [g] &= [f + g] \\ \lambda[f] &= [\lambda f] \\ [f][g] &= [fg]. \end{aligned}$$

Therefore, the germs at p form a ring.

The *ring of germs of C^k-functions at p is denoted $\mathcal{O}^{(k)}_{M,p}$*.

When $k = \infty$, we usually drop the superscript ∞.

Remark: Most readers will most likely be puzzled by the notation $\mathcal{O}^{(k)}_{M,p}$.

In fact, it is standard in algebraic geometry, but it is not as commonly used in differential geometry.

For any open subset, U, of a manifold, M, the ring, $\mathcal{C}^k(U)$, of C^k-functions on U is also denoted $\mathcal{O}^{(k)}_M(U)$ (certainly by people with an algebraic geometry bent!).

Then, it turns out that the map $U \mapsto \mathcal{O}^{(k)}_M(U)$ is a *sheaf*, denoted $\mathcal{O}^{(k)}_M$, and the ring $\mathcal{O}^{(k)}_{M,p}$ is the *stalk* of the sheaf $\mathcal{O}^{(k)}_M$ at p.

Such rings are called *local rings*.

Roughly speaking, all the "local" information about M at p is contained in the local ring $\mathcal{O}^{(k)}_{M,p}$. (This is to be taken with a grain of salt. In the C^k-case where $k < \infty$, we also need the "stationary germs," as we will see shortly.)

Now that we have a rigorous way of dealing with functions locally defined at p, observe that the map

$$v_i \colon f \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f$$

yields the same value for all functions f in a germ $\mathbf{f}$ at p.

Furthermore, the above map is linear on $\mathcal{O}^{(k)}_{M,p}$. More is true.

(1) For any two functions f, g locally defined at p, we have

$$\left(\frac{\partial}{\partial x_i}\right)_p (fg) = \left(\left(\frac{\partial}{\partial x_i}\right)_p f\right) g(p) + f(p) \left(\frac{\partial}{\partial x_i}\right)_p g.$$

(2) If $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, then

$$\left(\frac{\partial}{\partial x_i}\right)_p f = 0.$$

The first property says that v_i is a *point-derivation*.

As to the second property, when $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, we say that *$f$ is stationary at p*.

It is easy to check (using the chain rule) that being stationary at p does not depend on the chart, (U, φ), at p or on the function chosen in a germ, $\mathbf{f}$. Therefore, the notion of a stationary germ makes sense:

Definition 7.14. We say that a germ $\mathbf{f}$ at $p \in M$ is a *stationary germ* iff $(f \circ \varphi^{-1})'(\varphi(p)) = 0$ for some chart, (U, φ), at p and some function, f, in the germ, $\mathbf{f}$.

The C^k-stationary germs form a subring of $\mathcal{O}_{M,p}^{(k)}$ (but not an ideal!) denoted $\mathcal{S}_{M,p}^{(k)}$.

Remarkably, it turns out that the set of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ is isomorphic to the tangent space $T_p(M)$.

First, we prove that this space has $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ as a basis.

Proposition 7.6. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n functions, $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$, defined on $\mathcal{O}^{(k)}_{M,p}$ by*

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)} \qquad 1 \leq i \leq n,$$

are linear forms that vanish on $\mathcal{S}^{(k)}_{M,p}$.

Every linear form L on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$ can be expressed in a unique way as

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

where $\lambda_i \in \mathbb{R}$. Therefore, the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

form a basis of the vector space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

To define our third version of tangent vectors, we need to define point-derivations.

Definition 7.15. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *derivation at p in M* or *point-derivation on $\mathcal{O}_{M,p}^{(k)}$* is a linear form v, on $\mathcal{O}_{M,p}^{(k)}$, such that

$$v(\mathbf{fg}) = v(\mathbf{f})\mathbf{g}(p) + \mathbf{f}(p)v(\mathbf{g}),$$

for all germs $\mathbf{f}, \mathbf{g} \in \mathcal{O}_{M,p}^{(k)}$. The above is called the *Leibniz property*.

Let $\mathcal{D}_p^{(k)}(M)$ denote the set of point-derivations on $\mathcal{O}_{M,p}^{(k)}$.

As expected, point-derivations vanish on constant functions.

Proposition 7.7. *Every point-derivation, v, on $\mathcal{O}^{(k)}_{M,p}$, vanishes on germs of constant functions.*

Recall that we observed earlier that the $\left(\frac{\partial}{\partial x_i}\right)_p$ are point-derivations at p. Therefore, we have

Proposition 7.8. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ are exactly the point-derivations on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.*

Remarks: Proposition 7.8 says that any linear form on $\mathcal{O}_{M,p}^{(k)}$ that vanishes on $\mathcal{S}_{M,p}^{(k)}$ belongs to $\mathcal{D}_p^{(k)}(M)$, the set of point-derivations on $\mathcal{O}_{M,p}^{(k)}$.

However, in general, when $k \neq \infty$, a point-derivation on $\mathcal{O}_{M,p}^{(k)}$ does *not* necessarily vanish on $\mathcal{S}_{M,p}^{(k)}$.

We will see in Proposition 7.12 that this is true for $k = \infty$.

Here is now our third definition of a tangent vector.

Definition 7.16. (Tangent Vectors, Version 3) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any point-derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$, the subspace of stationary germs.

Let us consider the simple case where $M = \mathbb{R}$. In this case, for every $x \in \mathbb{R}$, the tangent space, $T_x(\mathbb{R})$, is a one-dimensional vector space isomorphic to $\mathbb{R}$ and $\left(\frac{\partial}{\partial t}\right)_x = \frac{d}{dt}\Big|_x$ is a basis vector of $T_x(\mathbb{R})$.

For every C^k-function, f, locally defined at x, we have

$$\left(\frac{\partial}{\partial t}\right)_x f = \frac{df}{dt}\bigg|_x = f'(x).$$

Thus, $\left(\frac{\partial}{\partial t}\right)_x$ is: compute the derivative of a function at x.

We now prove the equivalence of version 1 and version 3 of the definitions of a tangent vector.

Proposition 7.9. *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For any $p \in M$, let u be any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma\colon (-\epsilon, +\epsilon) \to M$, through p (i.e., $p = \gamma(0)$). Then, the map L_u defined on $\mathcal{O}_{M,p}^{(k)}$ by*

$$L_u(\mathbf{f}) = (f \circ \gamma)'(0)$$

is a point-derivation that vanishes on $\mathcal{S}_{M,p}^{(k)}$.

Furthermore, the map $u \mapsto L_u$ defined above is an isomorphism between $T_p(M)$ and the space of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.

We show in the next section that the the space of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ is isomorphic to $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$ (the dual of the quotient space $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$).

Even though this is just a restatement of Proposition 7.6, we state the following proposition because of its practical usefulness:

Proposition 7.10. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n tangent vectors,*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p,$$

form a basis of T_pM.

When M is a smooth manifold, things get a little simpler.

Indeed, it turns out that in this case, every point-derivation vanishes on stationary germs.

To prove this, we recall the following result from calculus (see Warner [52]):

Proposition 7.11. *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is a C^k-function ($k \geq 2$) on a convex open U, about $p \in \mathbb{R}^n$, then for every $q \in U$, we have*

$$g(q) = g(p) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_p (q_i - p_i)$$
$$+ \sum_{i,j=1}^{n} (q_i - p_i)(q_j - p_j) \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{(1-t)p+tq} dt.$$

In particular, if $g \in C^\infty(U)$, then the integral as a function of q is C^∞.

Proposition 7.12. *Let M be any C^∞-manifold of dimension n. For any $p \in M$, any point-derivation on $\mathcal{O}^{(\infty)}_{M,p}$ vanishes on $\mathcal{S}^{(\infty)}_{M,p}$, the ring of stationary germs. Consequently, $T_p(M) = \mathcal{D}^{(\infty)}_p(M)$.*

Proposition 7.12 shows that in the case of a smooth manifold, in Definition 7.16, we can omit the requirement that point-derivations vanish on stationary germs, since this is automatic.

Remark: In the case of smooth manifolds ($k = \infty$) some authors, including Morita [41] (Chapter 1, Definition 1.32) and O'Neil [43] (Chapter 1, Definition 9), define derivations as linear derivations with domain $\mathcal{C}^\infty(M)$, the set of all smooth funtions on the entire manifold, M.

This definition is simpler in the sense that it does not require the definition of the notion of germ but it is not local, because it is not obvious that if v is a point-derivation at p, then $v(f) = v(g)$ whenever $f, g \in \mathcal{C}^\infty(M)$ agree locally at p.

In fact, if two smooth locally defined functions agree near p it may not be possible to extend both of them to the whole of M.

However, it can be proved that this property is local because on smooth manifolds, "bump functions" exist (see Section 9.1, Proposition 9.2).

Unfortunately, this argument breaks down for C^k-manifolds with $k < \infty$ and in this case the ring of germs at p can't be avoided.

7.4 Tangent and Cotangent Spaces Revisited ⊛

The space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ turns out to be isomorphic to the dual of the quotient space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, and this fact shows that the dual $(T_pM)^*$ of the tangent space T_pM, called the *cotangent space* to M at p, can be viewed as the quotient space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$.

This provides a fairly intrinsic definition of the cotangent space to M at p. For notational simplicity, we write T^*_pM instead of $(T_pM)^*$.

As the subspace of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ is isomorphic to the dual $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$ of the space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, we see that the linear forms

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$$

also form a basis of $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$.

There is a conceptually clearer way to define a canonical isomorphism between $T_p(M)$ and the dual of $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ in terms of a *nondegenerate pairing* between $T_p(M)$ and $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$

This pairing is described by Serre in [49] (Chapter III, Section 8) for analytic manifolds and can be adapted to our situation.

Define the map, $\omega\colon T_p(M) \times \mathcal{O}_{M,p}^{(k)} \to \mathbb{R}$, so that

$$\omega([\gamma], \mathbf{f}) = (f \circ \gamma)'(0),$$

for all $[\gamma] \in T_p(M)$ and all $\mathbf{f} \in \mathcal{O}_{M,p}^{(k)}$ (with $f \in \mathbf{f}$).

It is easy to check that the above expression does not depend on the representatives chosen in the equivalences classes $[\gamma]$ and $\mathbf{f}$ and that ω is bilinear.

However, as defined, ω is degenerate because $\omega([\gamma], \mathbf{f}) = 0$ if $\mathbf{f}$ is a stationary germ.

Thus, we are led to consider the pairing with domain $T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})$ given by

$$\omega([\gamma], [\mathbf{f}]) = (f \circ \gamma)'(0),$$

where $[\gamma] \in T_p(M)$ and $[\mathbf{f}] \in \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, which we also denote $\omega \colon T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}) \to \mathbb{R}$.

Proposition 7.13. *The map* $\omega \colon T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}) \to \mathbb{R}$ *defined so that*

$$\omega([\gamma], [\mathbf{f}]) = (f \circ \gamma)'(0),$$

for all $[\gamma] \in T_p(M)$ *and all* $[\mathbf{f}] \in \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$*, is a non-degenerate pairing (with* $f \in \mathbf{f}$*).*

Consequently, there is a canonical isomorphism between $T_p(M)$ *and* $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$ *and a canonical isomorphism between* $T^*_p(M)$ *and* $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$.

In view of Proposition 7.13, we can identify $T_p(M)$ with $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$ and $T_p^*(M)$ with $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$.

Remark: Also recall that if E is a finite dimensional space, the map $i_E\colon E \to E^{**}$ defined so that, for any $v \in E$,

$$v \mapsto \widetilde{v}, \quad \text{where} \quad \widetilde{v}(f) = f(v), \quad \text{for all } f \in E^*$$

is a linear isomorphism.

Observe that we can view $\omega(u, \mathbf{f}) = \omega([\gamma], [\mathbf{f}])$ as the result of computing the directional derivative of the locally defined function $f \in \mathbf{f}$ in the direction u (given by a curve γ).

Proposition 7.13 also suggests the following definition:

Definition 7.17. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the *tangent space at p*, denoted $T_p(M)$, is the space of point-derivations on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

Thus, $T_p(M)$ can be identified with $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$.

The space $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ is called the *cotangent space at p*; it is isomorphic to the dual $T^*_p(M)$, of $T_p(M)$.

Observe that if $x_i = pr_i \circ \varphi$, as

$$\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{i,j},$$

the images of $x_1, \ldots, x_n$ in $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ constitute the dual basis of the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

Given any C^k-function f, on U, we denote the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ by df_p.

This is the *differential of f at p*.

Using the isomorphism between $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ and $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^{**}$ described above, df_p corresponds to the linear map in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}), \quad \text{for all } v \in T_p(M).$$

With this notation, we see that $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T_p^*(M)$, and this basis is dual to the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

For simplicity of notation, we often omit the subscript p unless confusion arises.

Remark: Strictly speaking, a tangent vector, $v \in T_p(M)$, is defined on the space of germs, $\mathcal{O}^{(k)}_{M,p}$, at p. However, it is often convenient to define v on C^k-functions, $f \in \mathcal{C}^k(U)$, where U is some open subset containing p. This is easy: set

$$v(f) = v(\mathbf{f}).$$

Given any chart, (U, φ), at p, since v can be written in a unique way as

$$v = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

we get

$$v(f) = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p f.$$

This shows that $v(f)$ is the *directional derivative of f in the direction v*.

It is also possible to define $T_p(M)$ just in terms of $\mathcal{O}^{(\infty)}_{M,p}$.

Let $\mathfrak{m}_{M,p} \subseteq \mathcal{O}^{(\infty)}_{M,p}$ be the ideal of germs that vanish at p.

Then, we also have the ideal $\mathfrak{m}^2_{M,p}$, which consists of all finite linear combinations of products of two elements in $\mathfrak{m}_{M,p}$, and it turns out that $T^*_p(M)$ is isomorphic to $\mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}$ (see Warner [52], Lemma 1.16).

Actually, if we let $\mathfrak{m}^{(k)}_{M,p} \subseteq \mathcal{O}^{(k)}_{M,p}$ denote the ideal of C^k-germs that vanish at p and $\mathfrak{s}^{(k)}_{M,p} \subseteq \mathcal{S}^{(k)}_{M,p}$ denote the ideal of stationary C^k-germs that vanish at p, adapting Warner's argument, we can prove the following proposition:

Proposition 7.14. *We have the inclusion,* $(\mathfrak{m}^{(k)}_{M,p})^2 \subseteq \mathfrak{s}^{(k)}_{M,p}$ *and the isomorphism*

$$(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^* \cong (\mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p})^*.$$

As a consequence, $T_p(M) \cong (\mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p})^*$ *and* $T^*_p(M) \cong \mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p}$.

When $k = \infty$, Proposition 7.11 shows that every stationary germ that vanishes at p belongs to $\mathfrak{m}^2_{M,p}$.

Therefore, when $k = \infty$, we have

$$\mathfrak{s}^{(\infty)}_{M,p} = \mathfrak{m}^2_{M,p}$$

and so, we obtain the result quoted above (from Warner):

$$T^*_p(M) = \mathcal{O}^{(\infty)}_{M,p}/\mathcal{S}^{(\infty)}_{M,p} \cong \mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}.$$

Remarks:

(1) The isomorphism

$$(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^* \cong (\mathfrak{m}_{M,p}^{(k)}/\mathfrak{s}_{M,p}^{(k)})^*$$

yields another proof that the linear forms in $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$ are point-derivations, using the argument from Warner [52] (Lemma 1.16).

(2) The ideal $\mathfrak{m}_{M,p}^{(k)}$ is in fact the unique *maximal ideal* of $\mathcal{O}_{M,p}^{(k)}$.

This is because if $\mathbf{f} \in \mathcal{O}_{M,p}^{(k)}$ does not vanish at p, then $\mathbf{1}/\mathbf{f}$ belongs to $\mathcal{O}_{M,p}^{(k)}$, and any proper ideal containing $\mathfrak{m}_{M,p}^{(k)}$ and $\mathbf{f}$ would be equal to $\mathcal{O}_{M,p}^{(k)}$, which is absurd.

Thus, $\mathcal{O}_{M,p}^{(k)}$ is a local ring (in the sense of commutative algebra) called the *local ring of germs of C^k-functions at p*. These rings play a crucial role in algebraic geometry.

7.5 Tangent Maps

After having explored thorougly the notion of tangent vector, we show how a C^k-map, $h\colon M \to N$, between C^k manifolds, induces a linear map, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, for every $p \in M$.

We find it convenient to use Version 3 of the definition of a tangent vector. So, let $u \in T_p(M)$ be a point-derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$.

We would like $dh_p(u)$ to be a point-derivation on $\mathcal{O}^{(k)}_{N,h(p)}$ that vanishes on $\mathcal{S}^{(k)}_{N,h(p)}$.

Now, for every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$, if $g \in \mathbf{g}$ is any locally defined function at $h(p)$, it is clear that $g \circ h$ is locally defined at p and is C^k and that if $g_1, g_2 \in \mathbf{g}$ then $g_1 \circ h$ and $g_2 \circ h$ are equivalent.

The germ of all locally defined functions at p of the form $g \circ h$, with $g \in \mathbf{g}$, will be denoted $\mathbf{g} \circ h$.

Then, we set

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ h).$$

Moreover, if $\mathbf{g}$ is a stationary germ at $h(p)$, then for some chart, (V, ψ) on N at $q = h(p)$, we have $(g \circ \psi^{-1})'(\psi(q)) = 0$ and, for some chart (U, φ) at p on M, we get

$$(g \circ h \circ \varphi^{-1})'(\varphi(p)) = (g \circ \psi^{-1})(\psi(q))((\psi \circ h \circ \varphi^{-1})'(\varphi(p))) \\ = 0,$$

which means that $\mathbf{g} \circ h$ is stationary at p.

Therefore, $dh_p(u) \in T_{h(p)}(M)$. It is also clear that dh_p is a linear map.

Definition 7.18. (Using Version 3 of a tangent vector) Given any two C^k-manifolds, M and N, of dimension m and n, respectively, for any C^k-map, $h\colon M \to N$ and for every $p \in M$, the *differential of h at p* or *tangent map* $dh_p\colon T_p(M) \to T_{h(p)}(N)$ (also denoted $T_ph\colon T_p(M) \to T_{h(p)}(N)$), is the linear map defined so that

$$dh_p(u)(\mathbf{g}) = T_ph(u)(\mathbf{g}) = u(\mathbf{g} \circ h),$$

for every $u \in T_p(M)$ and every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$.

The linear map dh_p ($= T_ph$) is sometimes denoted h'_p or D_ph.

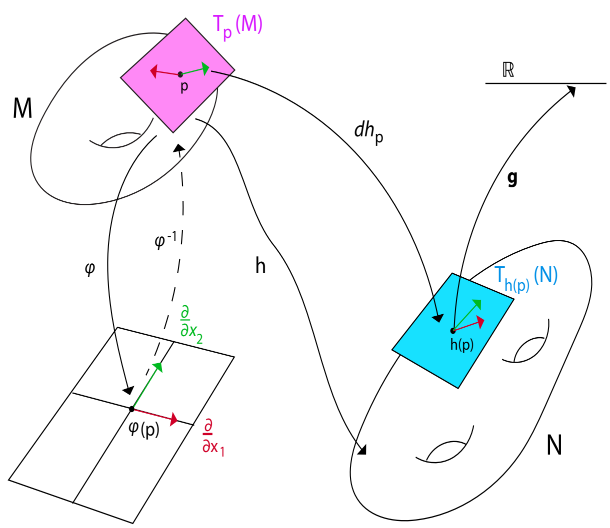

Figure 7.12: The tangent map $dh_p(u)(\mathbf{g}) = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p g \circ h$.

The chain rule is easily generalized to manifolds.

Proposition 7.15. *Given any two C^k-maps $f\colon M \to N$ and $g\colon N \to P$ between smooth C^k-manifolds, for any $p \in M$, we have*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

In the special case where $N = \mathbb{R}$, a C^k-map between the manifolds M and $\mathbb{R}$ is just a C^k-function on M.

It is interesting to see what $T_p f$ is explicitly. Since $N = \mathbb{R}$, germs (of functions on $\mathbb{R}$) at $t_0 = f(p)$ are just germs of C^k-functions, $g\colon \mathbb{R} \to \mathbb{R}$, locally defined at t_0.

Then, for any $u \in T_p(M)$ and every germ $\mathbf{g}$ at t_0,

$$T_p f(u)(\mathbf{g}) = u(\mathbf{g} \circ f).$$

If we pick a chart, (U, φ), on M at p, we know that the $\left(\frac{\partial}{\partial x_i}\right)_p$ form a basis of $T_p(M)$, with $1 \leq i \leq n$.

Therefore, it is enough to figure out what $T_p f(u)(\mathbf{g})$ is when $u = \left(\frac{\partial}{\partial x_i}\right)_p$.

In this case,

$$T_p f\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left.\frac{\partial(g \circ f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}.$$

Using the chain rule, we find that

$$T_p f\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left(\frac{\partial}{\partial x_i}\right)_p f \left.\frac{dg}{dt}\right|_{t_0}.$$

Therefore, we have

$$T_p f(u) = u(\mathbf{f}) \left.\frac{d}{dt}\right|_{t_0}.$$

This shows that we can identify $T_p f$ with the linear form in $T_p^*(M)$ defined by

$$df_p(u) = u(\mathbf{f}), \qquad u \in T_p M,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$.

This is consistent with our previous definition of df_p as the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ (as $T_p(M)$ is isomorphic to $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$).

Proposition 7.16. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n linear maps,*

$$(dx_1)_p, \ldots, (dx_n)_p,$$

form a basis of $T_p^* M$, where $(dx_i)_p$, the differential of x_i at p, is identified with the linear form in $T_p^* M$ such that

$$(dx_i)_p(v) = v(\mathbf{x_i}), \quad \textit{for every } v \in T_p M$$

(by identifying $T_\lambda \mathbb{R}$ with $\mathbb{R}$).

In preparation for the definition of the flow of a vector field (which will be needed to define the exponential map in Lie group theory), we need to define the tangent vector to a curve on a manifold.

Given a C^k-curve, $\gamma\colon (a, b) \to M$, on a C^k-manifold, M, for any $t_0 \in (a, b)$, we would like to define the tangent vector to the curve γ at t_0 as a tangent vector to M at $p = \gamma(t_0)$.

We do this as follows: Recall that $\frac{d}{dt}\big|_{t_0}$ is a basis vector of $T_{t_0}(\mathbb{R}) = \mathbb{R}$.

So, define the *tangent vector to the curve γ at t_0*, denoted $\dot{\gamma}(t_0)$ (or $\gamma'(t_0)$, or $\frac{d\gamma}{dt}(t_0)$), by

$$\dot{\gamma}(t_0) = d\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right).$$

We find it necessary to define curves (in a manifold) whose domain is not an open interval.

A map, $\gamma\colon [a, b] \to M$, is a *$C^k$-curve in M* if it is the restriction of some C^k-curve, $\widetilde{\gamma}\colon (a - \epsilon, b + \epsilon) \to M$, for some (small) $\epsilon > 0$.

Note that for such a curve (if $k \geq 1$) the tangent vector, $\dot{\gamma}(t)$, is defined for all $t \in [a, b]$.

A continuous curve, $\gamma\colon [a, b] \to M$, is *piecewise C^k* iff there a sequence, $a_0 = a, a_1, \ldots, a_m = b$, so that the restriction, γ_i, of γ to each $[a_i, a_{i+1}]$ is a C^k-curve, for $i = 0, \ldots, m - 1$.

This implies that $\gamma_i'(a_{i+1})$ and $\gamma_{i+1}'(a_{i+1})$ are defined for $i = 0, \ldots, m-1$, but there may be a *jump* in the tangent vector to γ at a_{i+1}, that is, we may have
$\gamma_i'(a_{i+1}) \neq \gamma_{i+1}'(a_{i+1})$.

Sometimes, especially in the case of a linear Lie group, it is more convenient to define the tangent map in terms of Version 1 of a tangent vector.

Given any C^k-map $h\colon M \to N$, for every $p \in M$, it is easy to show that for every equivalence class $u = [\gamma]$ of curves through p in M, all curves of the form $h \circ \gamma$ (with $\gamma \in u$) through $h(p)$ in N belong to the same equivalence class, and can make the following definition.

Definition 7.19. (Using Version 1 of a tangent vector) Given any two C^k-manifolds M and N, of dimension m and n respectively, for any C^k-map $h\colon M \to N$ and for every $p \in M$, the *differential of h at p* or *tangent map* $dh_p\colon T_p(M) \to T_{h(p)}(N)$ (also denoted $T_ph\colon T_p(M) \to T_{h(p)}(N)$), is the linear map defined such that for every equivalence class $u = [\gamma]$ of curves γ in M with $\gamma(0) = p$,

$$dh_p(u) = T_ph(u) = v,$$

where v is the equivalence class of all curves through $h(p)$ in N of the form $h \circ \gamma$, with $\gamma \in u$. See Figure 7.13.

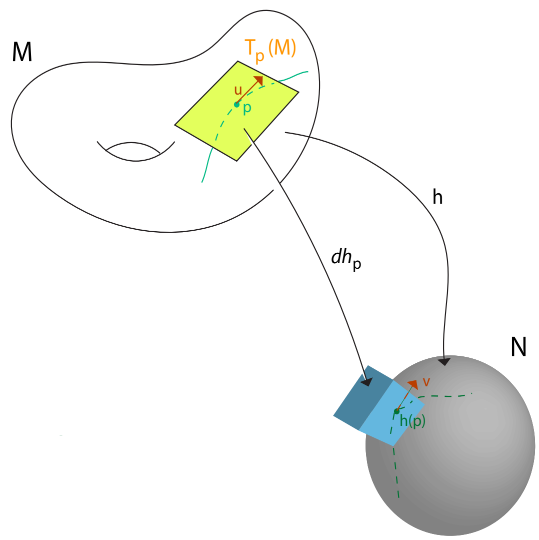

Figure 7.13: The tangent map $dh_p(u) = v$ defined via equivalent curves.

If M is a manifold in $\mathbb{R}^{N_1}$ and N is a manifold in $\mathbb{R}^{N_2}$ (for some $N_1, N_2 \geq 1$), then $\gamma'(0) \in \mathbb{R}^{N_1}$ and $(h \circ \gamma)'(0) \in \mathbb{R}^{N_2}$, so in this case the definition of $dh_p = Th_p$ is just Definition 4.9; namely, for any curve γ in M such that $\gamma(0) = p$ and $\gamma'(0) = u$,

$$dh_p(u) = Th_p(u) = (h \circ \gamma)'(0).$$

For example, consider the linear Lie group $\mathbf{SO}(3)$, pick any vector $u \in \mathbb{R}^3$, and let $f\colon \mathbf{SO}(3) \to \mathbb{R}^3$ be given by

$$f(R) = Ru, \quad R \in \mathbf{SO}(3).$$

To compute $df_R\colon T_R\mathbf{SO}(3) \to T_{Ru}\mathbb{R}^3$, since $T_R\mathbf{SO}(3) = R\mathfrak{so}(3)$ and $T_{Ru}\mathbb{R}^3 = \mathbb{R}^3$, pick any tangent vector $RB \in R\mathfrak{so}(3) = T_R\mathbf{SO}(3)$ (where B is any 3×3 skew symmetric matrix), let $\gamma(t) = Re^{tB}$ be the curve through R such that $\gamma'(0) = RB$, and compute

$$df_R(RB) = (f(\gamma(t)))'(0) = (Re^{tB}u)'(0) = RBu.$$

Therefore, we see that

$$df_R(X) = Xu, \quad X \in T_R\mathbf{SO}(3) = R\mathfrak{so}(3).$$

If we express the skew symmetric matrix $B \in \mathfrak{so}(3)$ as $B = \omega_\times$ for some vector $\omega \in \mathbb{R}^3$, then we have

$$df_R(R\omega_\times) = R\omega_\times u = R(\omega \times u).$$

Using the isomorphism of the Lie algebras $(\mathbb{R}^3, \times)$ and $\mathfrak{so}(3)$, the tangent map df_R is given by

$$df_R(R\omega) = R(\omega \times u).$$

Here is another example inspired by an optimization problem investigated by Taylor and Kriegman.

Pick any two vectors $u, v \in \mathbb{R}^3$, and let $f\colon \mathbf{SO}(3) \to \mathbb{R}$ be the function given by

$$f(R) = (u^\top R v)^2.$$

To compute $df_R\colon T_R\mathbf{SO}(3) \to T_{f(R)}\mathbb{R}$, since $T_R\mathbf{SO}(3) = R\mathfrak{so}(3)$ and $T_{f(R)}\mathbb{R} = \mathbb{R}$, again pick any tangent vector $RB \in R\mathfrak{so}(3) = T_R\mathbf{SO}(3)$ (where B is any 3×3 skew symmetric matrix), let $\gamma(t) = Re^{tB}$ be the curve through R such that $\gamma'(0) = RB$, and compute

$$\begin{aligned}
df_R(RB) &= (f(\gamma(t)))'(0) \\
&= ((u^\top Re^{tB} v)^2)'(0) \\
&= u^\top RBvu^\top Rv + u^\top Rvu^\top RBv \\
&= 2u^\top RBvu^\top Rv.
\end{aligned}$$

Therefore,

$$df_R(X) = 2u^\top X v u^\top R v, \quad X \in R\mathfrak{so}(3).$$

Unlike the case of functions defined on vector spaces, in order to define the gradient of f, a function defined on $\mathbf{SO}(3)$, a "nonflat" manifold, we need to pick a Riemannian metric on $\mathbf{SO}(3)$.

We will explain how to do this in Chapter 11.

7.6 Submanifolds, Immersions, Embeddings

Although the notion of submanifold is intuitively rather clear, technically, it is a bit tricky.

In fact, the reader may have noticed that many different definitions appear in books and that it is not obvious at first glance that these definitions are equivalent.

What is important is that a submanifold, N, of a given manifold, M, not only have the topology induced M but also that the charts of N be somewhow induced by those of M.

(Recall that if X is a topological space and Y is a subset of X, then the *subspace topology on Y* or *topology induced by X on Y*, has for its open sets all subsets of the form $Y \cap U$, where U is an arbitary open subset of X.).

Given m, n, with $0 \leq m \leq n$, we can view $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ using the inclusion

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{\underbrace{(0, \ldots, 0)}_{n-m}\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n,$$

given by

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m}).$$

Definition 7.20. Given a C^k-manifold, M, of dimension n, a subset, N, of M is an *m-dimensional submanifold of M* (where $0 \leq m \leq n$) iff for every point, $p \in N$, there is a chart (U, φ) of M (in the maximal atlas for M), with $p \in U$, so that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\}).$$

(We write $0_{n-m} = \underbrace{(0, \ldots, 0)}_{n-m}$.)

The subset, $U \cap N$, of Definition 7.20 is sometimes called a *slice* of (U, φ) and we say that (U, φ) is *adapted to N* (See O'Neill [43] or Warner [52]).

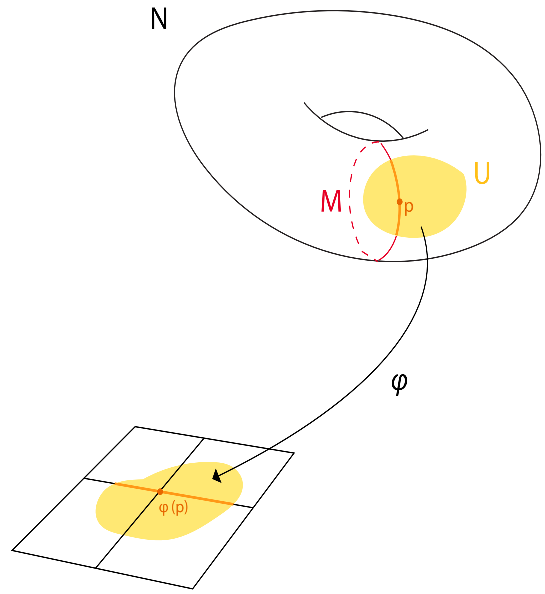

Figure 7.14: The red circle M is a 1-dimensional submanifold of the torus N.

Other authors, including Warner [52], use the term submanifold in a broader sense than us and they use the word *embedded submanifold* for what is defined in Definition 7.20.

The following proposition has an almost trivial proof but it justifies the use of the word submanifold:

Proposition 7.17. *Given a C^k-manifold, M, of dimension n, for any submanifold, N, of M of dimension $m \leq n$, the family of pairs $(U \cap N, \varphi \upharpoonright U \cap N)$, where (U, φ) ranges over the charts over any atlas for M, is an atlas for N, where N is given the subspace topology. Therefore, N inherits the structure of a C^k-manifold.*

In fact, every chart on N arises from a chart on M in the following precise sense:

Proposition 7.18. *Given a C^k-manifold, M, of dimension n and a submanifold, N, of M of dimension $m \leq n$, for any $p \in N$ and any chart, (W, η), of N at p, there is some chart, (U, φ), of M at p so that*

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\})$$

and

$$\varphi \upharpoonright U \cap N = \eta \upharpoonright U \cap N,$$

where $p \in U \cap N \subseteq W$.

It is also useful to define more general kinds of "submanifolds."

Definition 7.21. Let $h\colon N \to M$ be a C^k-map of manifolds.

(a) The map h is an *immersion* of N into M iff dh_p is injective for all $p \in N$.

(b) The set $h(N)$ is an *immersed submanifold* of M iff h is an injective immersion.

(c) The map h is an *embedding* of N into M iff h is an injective immersion such that the induced map, $N \longrightarrow h(N)$, is a homeomorphism, where $h(N)$ is given the subspace topology (equivalently, h is an open map from N into $h(N)$ with the subspace topology). We say that $h(N)$ (with the subspace topology) is an *embedded submanifold* of M.

(d) The map h is a *submersion* of N into M iff dh_p is surjective for all $p \in N$.

Again, we warn our readers that certain authors (such as Warner [52]) call $h(N)$, in (b), a submanifold of M! We prefer the terminology *immersed submanifold*.

The notion of immersed submanifold arises naturally in the framework of Lie groups.

Indeed, the fundamental correspondence between Lie groups and Lie algebras involves Lie subgroups that are not necessarily closed.

But, as we will see later, subgroups of Lie groups that are also submanifolds are always closed.

It is thus necessary to have a more inclusive notion of submanifold for Lie groups and the concept of immersed submanifold is just what's needed.

Immersions of $\mathbb{R}$ into $\mathbb{R}^3$ are parametric curves and immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$ are parametric surfaces. These have been extensively studied, for example, see DoCarmo [15], Berger and Gostiaux [6] or Gallier [21].

Immersions (i.e., subsets of the form $h(N)$, where N is an immersion) are generally neither injective immersions (i.e., subsets of the form $h(N)$, where N is an injective immersion) nor embeddings (or submanifolds).

For example, immersions can have self-intersections, as the plane curve (nodal cubic) shown in Figure 7.15 and given by: $x = t^2 - 1; y = t(t^2 - 1)$.

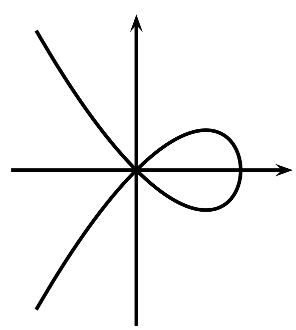

Figure 7.15: A nodal cubic; an immersion, but not an immersed submanifold.

Injective immersions are generally not embeddings (or submanifolds) because $h(N)$ may not be homeomorphic to N.

An example is given by the Lemniscate of Bernoulli shown in Figure 7.16, an injective immersion of $\mathbb{R}$ into $\mathbb{R}^2$:

$$x = \frac{t(1+t^2)}{1+t^4},$$
$$y = \frac{t(1-t^2)}{1+t^4}.$$

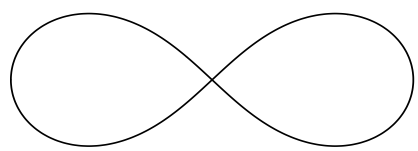

Figure 7.16: Lemniscate of Bernoulli; an immersed submanifold, but not an embedding.

When $t = 0$, the curve passes through the origin.

When $t \mapsto -\infty$, the curve tends to the origin from the left and from above, and when $t \mapsto +\infty$, the curve tends tends to the origin from the right and from below.

Therefore, the inverse of the map defining the Lemniscate of Bernoulli is not continuous at the origin.

Another interesting example is the immersion of $\mathbb{R}$ into the 2-torus, $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$, given by

$$t \mapsto (\cos t, \sin t, \cos ct, \sin ct),$$

where $c \in \mathbb{R}$.

One can show that the image of $\mathbb{R}$ under this immersion is closed in T^2 iff c is rational. Moreover, the image of this immersion is dense in T^2 but not closed iff c is irrational.

The above example can be adapted to the torus in $\mathbb{R}^3$: One can show that the immersion given by

$$t \mapsto ((2+\cos t)\cos(\sqrt{2}\,t), (2+\cos t)\sin(\sqrt{2}\,t), \sin t),$$

is dense but not closed in the torus (in $\mathbb{R}^3$) given by

$$(s,t) \mapsto ((2+\cos s)\cos t, (2+\cos s)\sin t, \sin s),$$

where $s,t \in \mathbb{R}$.

There is, however, a close relationship between submanifolds and embeddings.

Proposition 7.19. *If N is a submanifold of M, then the inclusion map, $j\colon N \to M$, is an embedding. Conversely, if $h\colon N \to M$ is an embedding, then $h(N)$ with the subspace topology is a submanifold of M and h is a diffeomorphism between N and $h(N)$.*

In summary, embedded submanifolds and (our) submanifolds coincide.

Some authors refer to spaces of the form $h(N)$, where h is an injective immersion, as *immersed submanifolds*, and we have adopted this terminology.

However, in general, an immersed submanifold is *not* a submanifold.

One case where this holds is when N is compact, since then, a bijective continuous map is a homeomorphism.