Chapter 6

The Lorentz Groups

6.1 The Lorentz Groups $\mathbf{O}(n,1)$, $\mathbf{SO}(n,1)$ and $\mathbf{SO}_0(n,1)$

The Lorentz group $\mathbf{SO}(3,1)$ shows up in an interesting way in computer vision.

Denote the $p\times p$-identity matrix by I_p, for $p, q, \geq 1$, and define

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

If $n = p+q$, the matrix $I_{p,q}$ is associated with the non-degenerate symmetric bilinear form

$$\varphi_{p,q}((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j$$

with associated quadratic form

$$\Phi_{p,q}((x_1, \ldots, x_n)) = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{n} x_j^2.$$

In particular, when $p = 1$ and $q = 3$, we have the *Lorentz metric*

$$x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

In physics, x_1 is interpreted as time and written t and x_2, x_3, x_4 as coordinates in $\mathbb{R}^3$ and written x, y, z. Thus, the Lozentz metric is usually written a

$$t^2 - x^2 - y^2 - z^2.$$

The space $\mathbb{R}^4$ with the Lorentz metric is called *Minkowski space*. It plays an important role in Einstein's theory of special relativity.

The group $\mathbf{O}(p, q)$ is the set of all $n \times n$-matrices

$$\mathbf{O}(p, q) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q}\}.$$

This is the group of all invertible linear maps of $\mathbb{R}^n$ that preserve the quadratic form, $\Phi_{p,q}$, i.e., the group of isometries of $\Phi_{p,q}$.

Clearly, $I_{p,q}^2 = I$, so the condition $A^\top I_{p,q} A = I_{p,q}$ implies that

$$A^{-1} = I_{p,q} A^\top I_{p,q}.$$

Thus, $A I_{p,q} A^\top = I_{p,q}$ also holds, which shows that $\mathbf{O}(p,q)$ is closed under transposition (i.e., if $A \in \mathbf{O}(p,q)$, then $A^\top \in \mathbf{O}(p,q)$).

We have the subgroup

$$\mathbf{SO}(p,q) = \{A \in \mathbf{O}(p,q) \mid \det(A) = 1\}$$

consisting of the isometries of $(\mathbb{R}^n, \Phi_{p,q})$ with determinant $+1$.

It is clear that $\mathbf{SO}(p,q)$ is also closed under transposition.

The condition $A^\top I_{p,q} A = I_{p,q}$ has an interpretation in terms of the inner product $\varphi_{p,q}$ and the columns (and rows) of A.

Indeed, if we denote the jth column of A by A_j, then

$$A^\top I_{p,q} A = (\varphi_{p,q}(A_i, A_j)),$$

so $A \in \mathbf{O}(p,q)$ iff the columns of A form an "orthonormal basis" w.r.t. $\varphi_{p,q}$, i.e.,

$$\varphi_{p,q}(A_i, A_j) = \begin{cases} \delta_{ij} & \text{if } 1 \leq i, j \leq p; \\ -\delta_{ij} & \text{if } p+1 \leq i, j \leq p+q. \end{cases}$$

The difference with the usual orthogonal matrices is that $\varphi_{p,q}(A_i, A_i) = -1$, if $p+1 \leq i \leq p+q$. As $\mathbf{O}(p,q)$ is closed under transposition, the rows of A also form an orthonormal basis w.r.t. $\varphi_{p,q}$.

It turns out that $\mathbf{SO}(p,q)$ has two connected components and the component containing the identity is a subgroup of $\mathbf{SO}(p,q)$ denoted $\mathbf{SO}_0(p,q)$.

The group $\mathbf{SO}_0(p,q)$ turns out to be homeomorphic to $\mathbf{SO}(p) \times \mathbf{SO}(q) \times \mathbb{R}^{pq}$, but this is not easy to prove. (One way to prove it is to use results on pseudo-algebraic subgroups of $\mathbf{GL}(n,\mathbb{C})$, see Knapp [28] or Gallier's notes on Clifford algebras (on the web)).

We will now determine the polar decomposition and the SVD decomposition of matrices in the Lorentz groups $\mathbf{O}(n,1)$ and $\mathbf{SO}(n,1)$.

Write $J = I_{n,1}$ and, given any $A \in \mathbf{O}(n,1)$, write

$$A = \begin{pmatrix} B & u \\ v^\top & c \end{pmatrix},$$

where B is an $n \times n$ matrix, u, v are (column) vectors in $\mathbb{R}^n$ and $c \in \mathbb{R}$.

Proposition 6.1. *Every matrix $A \in \mathbf{O}(n,1)$ has a polar decomposition of the form*

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

or

$$A = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $Q \in \mathbf{O}(n)$ and $c = \sqrt{\|v\|^2 + 1}$.

Thus, we see that $\mathbf{O}(n,1)$ has four components corresponding to the cases:

(1) $Q \in \mathbf{O}(n)$; $\det(Q) < 0$; $+1$ as the lower right entry of the orthogonal matrix;

(2) $Q \in \mathbf{SO}(n)$; -1 as the lower right entry of the orthogonal matrix;

(3) $Q \in \mathbf{O}(n)$; $\det(Q) < 0$; -1 as the lower right entry of the orthogonal matrix;

(4) $Q \in \mathbf{SO}(n)$; $+1$ as the lower right entry of the orthogonal matrix.

Observe that $\det(A) = -1$ in cases (1) and (2) and that $\det(A) = +1$ in cases (3) and (4).

Thus, (3) and (4) correspond to the group $\mathbf{SO}(n, 1)$, in which case the polar decomposition is of the form

$$A = \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $Q \in \mathbf{O}(n)$, with $\det(Q) = -1$ and $c = \sqrt{\|v\|^2 + 1}$, or

$$A = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

where $Q \in \mathbf{SO}(n)$ and $c = \sqrt{\|v\|^2 + 1}$.

The components in (1) and (2) are not groups. We will show later that all four components are connected and that case (4) corresponds to a group (Proposition 6.6).

This last group is the connected component of the identity and it is denoted $\mathbf{SO}_0(n, 1)$ (see Corollary 6.9).

For the time being, note that $A \in \mathbf{SO}_0(n,1)$ iff $A \in \mathbf{SO}(n,1)$ and $a_{n+1\,n+1}$ $(= c) > 0$ (here, $A = (a_{i\,j})$.)

In fact, we proved above that if $a_{n+1\,n+1} > 0$, then $a_{n+1\,n+1} \geq 1$.

Remark: If we let

$$\Lambda_P = \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda_T = I_{n,1},$$

where

$$I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix},$$

then we have the disjoint union

$$\begin{aligned} \mathbf{O}(n,1) = \mathbf{SO}_0(n,1) &\cup \Lambda_P \mathbf{SO}_0(n,1) \\ &\cup \Lambda_T \mathbf{SO}_0(n,1) \cup \Lambda_P \Lambda_T \mathbf{SO}_0(n,1). \end{aligned}$$

The positive definite symmetric matrix

$$S = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix}$$

involved in Proposition 6.1 is called a *Lorentz boost*. Observe that if $v = 0$, then $c = 1$ and $S = I_{n+1}$.

Proposition 6.2. *Assume $v \neq 0$. The eigenvalues of the symmetric positive definite matrix*

$$S = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

where $c = \sqrt{\|v\|^2 + 1}$, are 1 with multiplicity $n - 1$, and e^α and $e^{-\alpha}$ each with multiplicity 1 (for some $\alpha \geq 0$). An orthonormal basis of eigenvectors of S consists of vectors of the form

$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{v}{\sqrt{2}\|v\|} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{v}{\sqrt{2}\|v\|} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

where the $u_i \in \mathbb{R}^n$ are all orthogonal to v and pairwise orthogonal.

Corollary 6.3. *The singular values of any matrix* $A \in \mathbf{O}(n,1)$ *are* 1 *with multiplicity* $n-1$, e^{α}, *and* $e^{-\alpha}$, *for some* $\alpha \geq 0$.

Note that the case $\alpha = 0$ is possible, in which case, A is an orthogonal matrix of the form

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix},$$

with $Q \in \mathbf{O}(n)$. The two singular values e^{α} and $e^{-\alpha}$ tell us how much A deviates from being orthogonal.

We can now determine a convenient form for the SVD of matrices in $\mathbf{O}(n,1)$.

Theorem 6.4. *Every matrix $A \in \mathbf{O}(n,1)$ can be written as*

$$A = \begin{pmatrix} P & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} Q^\top & 0 \\ 0 & 1 \end{pmatrix}$$

with $\epsilon = \pm 1$, $P \in \mathbf{O}(n)$ and $Q \in \mathbf{SO}(n)$. When $A \in \mathbf{SO}(n,1)$, we have $\det(P)\epsilon = +1$, and when $A \in \mathbf{SO}_0(n,1)$, we have $\epsilon = +1$ and $P \in \mathbf{SO}(n)$, that is,

$$A = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} Q^\top & 0 \\ 0 & 1 \end{pmatrix}$$

with $P \in \mathbf{SO}(n)$ and $Q \in \mathbf{SO}(n)$.

Remark: We warn our readers about Chapter 6 of Baker's book [3]. Indeed, this chapter is seriously flawed.

The main two Theorems (Theorem 6.9 and Theorem 6.10) are false and as consequence, the proof of Theorem 6.11 is wrong too. Theorem 6.11 states that the exponential map $\exp\colon \mathfrak{so}(n,1) \to \mathbf{SO}_0(n,1)$ is surjective, which is correct, but known proofs are nontrivial and quite lengthy (see Section 11.8).

The proof of Theorem 6.12 is also false, although the theorem itself is correct (this is our Theorem 11.31, see Section 11.8).

For a thorough analysis of the eigenvalues of Lorentz isometries (and much more), one should consult Riesz [46] (Chapter III).

Clearly, a result similar to Theorem 6.4 also holds for the matrices in the groups $\mathbf{O}(1, n)$, $\mathbf{SO}(1, n)$ and $\mathbf{SO}_0(1, n)$.

For example, every matrix $A \in \mathbf{SO}_0(1, n)$ can be written as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & \cdots & 0 \\ \sinh \alpha & \cosh \alpha & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q^\top \end{pmatrix},$$

where $P, Q \in \mathbf{SO}(n)$.

In the case $n = 3$, we obtain the *proper orthochronous Lorentz group*, $\mathbf{SO}_0(1, 3)$, also denoted $\mathbf{Lor}(1, 3)$.

By the way, $\mathbf{O}(1, 3)$ is called the *(full) Lorentz group* and $\mathbf{SO}(1, 3)$ is the *special Lorentz group*.

Theorem 6.4 (really, the version for $\mathbf{SO}_0(1,n)$) shows that the Lorentz group $\mathbf{SO}_0(1,3)$ is generated by the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \quad \text{with } P \in \mathbf{SO}(3)$$

and the matrices of the form

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This fact will be useful when we prove that the homomorphism $\varphi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$ is surjective.

Remark: Unfortunately, unlike orthogonal matrices which can always be diagonalized over $\mathbb{C}$, **not** every matrix in $\mathbf{SO}(1,n)$ can be diagonalized for $n \geq 2$.

This has to do with the fact that the Lie algebra $\mathfrak{so}(1,n)$ has non-zero idempotents (see Section 11.8).

It turns out that the group $\mathbf{SO}_0(1,3)$ admits another interesting characterization involving the hypersurface

$$\mathcal{H} = \{(t, x, y, z) \in \mathbb{R}^4 \mid t^2 - x^2 - y^2 - z^2 = 1\}.$$

This surface has two sheets and it is not hard to show that $\mathbf{SO}_0(1,3)$ is the subgroup of $\mathbf{SO}(1,3)$ that preserves these two sheets (does not swap them).

Actually, we will prove this fact for any n.

Let us switch back to $\mathbf{SO}(n,1)$.

First, as a matter of notation, we write every $u \in \mathbb{R}^{n+1}$ as $u = (\mathbf{u}, t)$, where $\mathbf{u} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, so that the Lorentz inner product can be expressed as

$$\langle u, v\rangle = \langle (\mathbf{u}, t), (\mathbf{v}, s)\rangle = \mathbf{u} \cdot \mathbf{v} - ts,$$

where $\mathbf{u} \cdot \mathbf{v}$ is the standard Euclidean inner product (the Euclidean norm of x is denoted $\|x\|$).

Definition 6.1. A nonzero vector, $u = (\mathbf{u}, t) \in \mathbb{R}^{n+1}$ is called

(a) *spacelike* iff $\langle u, u \rangle > 0$, i.e., iff $\|\mathbf{u}\|^2 > t^2$;

(b) *timelike* iff $\langle u, u \rangle < 0$, i.e., iff $\|\mathbf{u}\|^2 < t^2$;

(c) *lightlike* or *isotropic* iff $\langle u, u \rangle = 0$, i.e., iff $\|\mathbf{u}\|^2 = t^2$.

A spacelike (resp. timelike, resp. lightlike) vector is said to be *positive* iff $t > 0$ and *negative* iff $t < 0$.

The set of all isotropic vectors

$$\mathcal{H}_n(0) = \{u = (\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 = t^2\}$$

is called the *light cone*.

For every $r > 0$, let

$$\mathcal{H}_n(r) = \{u = (\mathbf{u}, t) \in \mathbb{R}^{n+1} \mid \|\mathbf{u}\|^2 - t^2 = -r\},$$

a hyperboloid of two sheets.

The space $\mathcal{H}_n(r)$ has two connected components.:

$\mathcal{H}_n^+(r)$ is the sheet containing $(0, \ldots, 0, \sqrt{r})$

$\mathcal{H}_n^-(r)$ is the sheet containing $(0, \ldots, 0, -\sqrt{r})$.

Since every Lorentz isometry, $A \in \mathbf{SO}(n, 1)$, preserves the Lorentz inner product, we conclude that A globally preserves every hyperboloid, $\mathcal{H}_n(r)$, for $r > 0$.

We claim that every $A \in \mathbf{SO}_0(n, 1)$ preserves both $\mathcal{H}_n^+(r)$ and $\mathcal{H}_n^-(r)$.

Proposition 6.5. *If $a_{n+1\,n+1} > 0$, then every isometry, $A \in \mathbf{SO}(n, 1)$, preserves all positive (resp. negative) timelike vectors and all positive (resp. negative) lightlike vectors. Moreover, if $A \in \mathbf{SO}(n, 1)$ preserves all positive timelike vectors, then $a_{n+1\,n+1} > 0$.*

Let $\mathbf{O}^+(n,1)$ denote the subset of $\mathbf{O}(n,1)$ consisting of all matrices, $A = (a_{i\,j})$, such that $a_{n+1\,n+1} > 0$.

Recall that

$$\mathbf{SO}_0(n,1) = \{A \in \mathbf{SO}(n,1) \mid a_{n+1\,n+1} > 0\}.$$

Note that $\mathbf{SO}_0(n,1) = \mathbf{O}^+(n,1) \cap \mathbf{SO}(n,1)$.

Proposition 6.6. *The set $\mathbf{O}^+(n,1)$ is a subgroup of $\mathbf{O}(n,1)$ and the set $\mathbf{SO}_0(n,1)$ is a subgroup of $\mathbf{SO}(n,1)$.*

Next, we wish to prove that the action $\mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ is transitive.

Proposition 6.7. *Let $u = (\mathbf{u}, t)$ and $v = (\mathbf{v}, s)$ be nonzero vectors in $\mathbb{R}^{n+1}$ with $\langle u, v\rangle = 0$. If u is timelike, then v is spacelike (i.e., $\langle v, v\rangle > 0$).*

Proposition 6.8. *The action* $\mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ *is transitive.*

Let us find the stabilizer of $e_{n+1} = (0, \ldots, 0, 1)$.

We must have $Ae_{n+1} = e_{n+1}$, and the polar form implies that

$$A = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with} \quad P \in \mathbf{SO}(n).$$

Therefore, the stabilizer of e_{n+1} is isomorphic to $\mathbf{SO}(n)$ and we conclude that $\mathcal{H}_n^+(1)$, as a homogeneous space, is

$$\mathcal{H}_n^+(1) \cong \mathbf{SO}_0(n,1)/\mathbf{SO}(n).$$

As an application of Theorem 5.14 and Proposition 5.9, we show that the Lorentz group $\mathbf{SO}_0(n,1)$ is connected.

Firstly, it is easy to check that $\mathbf{SO}_0(n,1)$ and $\mathcal{H}_n^+(1)$ satisfy the assumptions of Theorem 5.14 because they are both manifolds.

Also, we saw at the end of Section 6.1 that the action $\cdot\colon \mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ of $\mathbf{SO}_0(n,1)$ on $\mathcal{H}_n^+(1)$ is transitive, so that, as topological spaces

$$\mathbf{SO}_0(n,1)/\mathbf{SO}(n) \cong \mathcal{H}_n^+(1).$$

Now, we already showed that $\mathcal{H}_n^+(1)$ is connected so, by Proposition 5.9, the connectivity of $\mathbf{SO}_0(n,1)$ follows from the connectivity of $\mathbf{SO}(n)$ for $n \geq 1$.

The connectivity of $\mathbf{SO}(n)$ is a consequence of the surjectivity of the exponential map (see Theorem 1.11) but we can also give a quick proof using Proposition 5.9.

Indeed, $\mathbf{SO}(n+1)$ and S^n are both manifolds and we saw in Section 5.2 that

$$\mathbf{SO}(n+1)/\mathbf{SO}(n) \cong S^n.$$

Now, S^n is connected for $n \geq 1$ and $\mathbf{SO}(1) \cong S^1$ is connected. We finish the proof by induction on n.

Corollary 6.9. *The Lorentz group* $\mathbf{SO}_0(n,1)$ *is connected; it is the component of the identity in* $\mathbf{O}(n,1)$.