Chapter 5

Review of Groups and Group Actions

5.1 Basic Concepts of Groups

Definition 5.1. A *group* is a set G equipped with a binary operation $\cdot\colon G \times G \to G$ that associates an element $a \cdot b \in G$ to every pair of elements $a, b \in G$, and having the following properties: $\cdot$ is *associative*, has an *identity element*, $e \in G$, and every element in G is *invertible* (w.r.t. $\cdot$). More explicitly, this means that the following equations hold for all $a, b, c \in G$:

(G1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$ (associativity);

(G2) $a \cdot e = e \cdot a = a.$ (identity);

(G3) For every $a \in G$, there is some $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (inverse).

A group G is *abelian* (or *commutative*) if

$$a \cdot b = b \cdot a$$

for all $a, b \in G$.

Observe that a group is never empty, since $e \in G$.

Given a group, G, for any two subsets $R, S \subseteq G$, we let

$$RS = \{r \cdot s \mid r \in R,\ s \in S\}.$$

In particular, for any $g \in G$, if $R = \{g\}$, we write

$$gS = \{g \cdot s \mid s \in S\}$$

and similarly, if $S = \{g\}$, we write

$$Rg = \{r \cdot g \mid r \in R\}.$$

From now on, we will drop the multiplication sign and write $g_1 g_2$ for $g_1 \cdot g_2$.

Definition 5.2. Given a group, G, a subset, H, of G is a *subgroup of G* iff

(1) The identity element, e, of G also belongs to H ($e \in H$);

(2) For all $h_1, h_2 \in H$, we have $h_1 h_2 \in H$;

(3) For all $h \in H$, we have $h^{-1} \in H$.

It is easily checked that a subset, $H \subseteq G$, is a subgroup of G iff H is nonempty and whenever $h_1, h_2 \in H$, then $h_1 h_2^{-1} \in H$.

If H is a subgroup of G and $g \in G$ is any element, the sets of the form gH are called *left cosets of H in G* and the sets of the form Hg are called *right cosets of H in G*.

The left cosets (resp. right cosets) of H induce an equivalence relation, $\sim$, defined as follows: For all $g_1, g_2 \in G$,

$$g_1 \sim g_2 \quad \text{iff} \quad g_1 H = g_2 H$$

(resp. $g_1 \sim g_2$ iff $Hg_1 = Hg_2$).

Obviously, $\sim$ is an equivalence relation. Now, it is easy to see that $g_1 H = g_2 H$ iff $g_2^{-1} g_1 \in H$, so the equivalence class of an element $g \in G$ is the coset gH (resp. Hg).

The set of left cosets of H in G (which, in general, is **not** a group) is denoted G/H. The "points" of G/H are obtained by "collapsing" all the elements in a coset into a single element.

The set of right cosets is denoted by $H \backslash G$.

It is tempting to define a multiplication operation on left cosets (or right cosets) by setting

$$(g_1H)(g_2H) = (g_1g_2)H,$$

but this operation is not well defined in general, unless the subgroup H possesses a special property.

This property is typical of the kernels of group homomorphisms.

Definition 5.3. Given any two groups, G, G', a function $\varphi\colon G \to G'$ is a *homomorphism* iff

$$\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2), \quad \text{for all } g_1, g_2 \in G.$$

Taking $g_1 = g_2 = e$ (in G), we see that

$$\varphi(e) = e',$$

and taking $g_1 = g$ and $g_2 = g^{-1}$, we see that

$$\varphi(g^{-1}) = \varphi(g)^{-1}.$$

If $\varphi\colon G \to G'$ and $\psi\colon G' \to G''$ are group homomorphisms, then $\psi \circ \varphi\colon G \to G''$ is also a homomorphism.

If $\varphi\colon G \to G'$ is a homomorphism of groups and $H \subseteq G$ and $H' \subseteq G'$ are two subgroups, then it is easily checked that

$$\mathrm{Im}\ H = \varphi(H) = \{\varphi(g) \mid g \in H\}$$

is a subgroup of G' called the *image of H by φ*, and

$$\varphi^{-1}(H') = \{g \in G \mid \varphi(g) \in H'\}$$

is a subgroup of G. In particular, when $H' = \{e'\}$, we obtain the *kernel*, $\mathrm{Ker}\ \varphi$, of φ. Thus,

$$\mathrm{Ker}\ \varphi = \{g \in G \mid \varphi(g) = e'\}.$$

It is immediately verified that $\varphi\colon G \to G'$ is injective iff $\mathrm{Ker}\ \varphi = \{e\}$. (We also write $\mathrm{Ker}\ \varphi = (0)$.)

We say that φ is an *isomorphism* if there is a homomorphism, $\psi\colon G' \to G$, so that

$$\psi \circ \varphi = \mathrm{id}_G \quad \text{and} \quad \varphi \circ \psi = \mathrm{id}_{G'}.$$

In this case, ψ is unique and it is denoted φ^{-1}.

When φ is an isomorphism we say the the groups G and G' are *isomorphic*. When $G' = G$, a group isomorphism is called an *automorphism*.

We claim that $H = \mathrm{Ker}\,\varphi$ satisfies the following property:

$$gH = Hg, \quad \text{for all } g \in G. \tag{$*$}$$

First, note that $(*)$ is equivalent to

$$gHg^{-1} = H, \quad \text{for all } g \in G,$$

and the above is equivalent to

$$gHg^{-1} \subseteq H, \quad \text{for all } g \in G. \tag{$**$}$$

Definition 5.4. For any group, G, a subgroup, $N \subseteq G$, is a *normal subgroup* of G iff

$$gNg^{-1} = N, \quad \text{for all } g \in G.$$

This is denoted by $N \lhd G$.

If N is a normal subgroup of G, the equivalence relation induced by left cosets is the same as the equivalence induced by right cosets.

Furthermore, this equivalence relation, $\sim$, is a *congruence*, which means that: For all $g_1, g_2, g_1', g_2' \in G$,

(1) If $g_1N = g_1'N$ and $g_2N = g_2'N$, then $g_1g_2N = g_1'g_2'N$, and

(2) If $g_1N = g_2N$, then $g_1^{-1}N = g_2^{-1}N$.

As a consequence, we can define a group structure on the set $G/\sim$ of equivalence classes modulo $\sim$, by setting

$$(g_1N)(g_2N) = (g_1g_2)N.$$

This group is denoted G/N. The equivalence class, gN, of an element $g \in G$ is also denoted $\overline{g}$. The map $\pi\colon G \to G/N$, given by

$$\pi(g) = \overline{g} = gN,$$

is clearly a group homomorphism called the *canonical projection*.

Given a homomorphism of groups, $\varphi\colon G \to G'$, we easily check that the groups $G/\mathrm{Ker}\ \varphi$ and $\mathrm{Im}\ \varphi = \varphi(G)$ are isomorphic.

5.2 Group Actions: Part I, Definitions and Examples

If X is a set (usually, some kind of geometric space, for example, the sphere in $\mathbb{R}^3$, the upper half-plane, etc.), the "symmetries" of X are often captured by the action of a group, G, on X.

In fact, if G is a Lie group and the action satisfies some simple properties, the set X can be given a manifold structure which makes it a projection (quotient) of G, a so-called "homogeneous space."

Definition 5.5. Given a set, X, and a group, G, a *left action of G on X* (for short, an *action of G on X*) is a function, $\varphi\colon G \times X \to X$, such that

(1) For all $g, h \in G$ and all $x \in X$,
$$\varphi(g, \varphi(h, x)) = \varphi(gh, x),$$

(2) For all $x \in X$,
$$\varphi(1, x) = x,$$
where $1 \in G$ is the identity element of G.

To alleviate the notation, we usually write $g \cdot x$ or even gx for $\varphi(g, x)$, in which case, the above axioms read:

(1) For all $g, h \in G$ and all $x \in X$,
$$g \cdot (h \cdot x) = gh \cdot x,$$

(2) For all $x \in X$,
$$1 \cdot x = x.$$

The set X is called a *(left) G-set*.

The action φ is *faithful* or *effective* iff for every g, if $g \cdot x = x$ for all $x \in X$, then $g = 1$;

the action φ is *transitive* iff for any two elements $x, y \in X$, there is some $g \in G$ so that $g \cdot x = y$.

Given an action, $\varphi\colon G \times X \to X$, for every $g \in G$, we have a function, $\varphi_g\colon X \to X$, defined by

$$\varphi_g(x) = g \cdot x, \quad \text{for all } x \in X.$$

Observe that φ_g has $\varphi_{g^{-1}}$ as inverse.

Therefore, φ_g is a bijection of X, i.e., a permutation of X.

Moreover, we check immediately that

$$\varphi_g \circ \varphi_h = \varphi_{gh},$$

so, the map $g \mapsto \varphi_g$ is a group homomorphism from G to $\mathfrak{S}_X$, the group of permutations of X.

With a slight abuse of notation, this group homomorphism $G \longrightarrow \mathfrak{S}_X$ is also denoted φ.

Conversely, it is easy to see that any group homomorphism, $\varphi\colon G \to \mathfrak{S}_X$, yields a group action $\cdot\colon G \times X \longrightarrow X$, by setting

$$g \cdot x = \varphi(g)(x).$$

Observe that an action, φ, is faithful iff the group homomorphism, $\varphi\colon G \to \mathfrak{S}_X$, is injective.

Also, we have $g \cdot x = y$ iff $g^{-1} \cdot y = x$.

Definition 5.6. Given two G-sets, X and Y, a function, $f\colon X \to Y$, is said to be *equivariant*, or a *G-map* iff for all $x \in X$ and all $g \in G$, we have

$$f(g \cdot x) = g \cdot f(x).$$

Equivalently, if the G-actions are denoted by $\varphi\colon G \times X \to X$ and $\psi\colon G \times Y \to Y$, we have the following commutative diagram for all $g \in G$:

$$\begin{tikzcd} X \arrow[r, "\varphi_g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\psi_g"'] & Y. \end{tikzcd}$$

Remark: We can also define a *right action*, $\cdot\colon X \times G \to X$, of a group G on a set X, as a map satisfying the conditions

(1) For all $g, h \in G$ and all $x \in X$,

$$(x \cdot g) \cdot h = x \cdot gh,$$

(2) For all $x \in X$,

$$x \cdot 1 = x.$$

However, one change is necessary. For every $g \in G$, the map $\varphi_g \colon X \to X$ must be defined as

$$\varphi_g(x) = x \cdot g^{-1},$$

in order for the map $g \mapsto \varphi_g$ from G to $\mathfrak{S}_X$ to be a homomorphism ($\varphi_g \circ \varphi_h = \varphi_{gh}$).

Conversely, given a homomorphism $\varphi \colon G \to \mathfrak{S}_X$, we get a right action $\cdot \colon X \times G \longrightarrow X$ by setting

$$x \cdot g = \varphi(g^{-1})(x).$$

Every notion defined for left actions is also defined for right actions, in the obvious way.

Here are some examples of (left) group actions.

Example 5.1. The unit sphere S^2 (more generally, S^{n-1}).

Recall that for any $n \geq 1$, the *(real) unit sphere*, S^{n-1}, is the set of points in $\mathbb{R}^n$ given by

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

In particular, S^2 is the usual sphere in $\mathbb{R}^3$.

Since the group $\mathbf{SO}(3) = \mathbf{SO}(3, \mathbb{R})$ consists of (orientation preserving) linear isometries, i.e., *linear* maps that are distance preserving (and of determinant $+1$), and every linear map leaves the origin fixed, we see that any rotation maps S^2 into itself.

Beware that this would be false if we considered the group of *affine* isometries, $\mathbf{SE}(3)$, of $\mathbb{E}^3$. For example, a screw motion does *not* map S^2 into itself, even though it is distance preserving, because the origin is translated.

Thus, we have an action, $\cdot \colon \mathbf{SO}(3) \times S^2 \to S^2$, given by

$$R \cdot x = Rx.$$

The verification that the above is indeed an action is trivial. This action is transitive; see Figure 5.1.

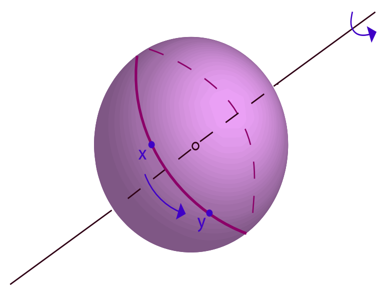

Figure 5.1: The rotation which maps x to y.

Similarly, for any $n \geq 1$, we get an action,

$$\cdot \colon \mathbf{SO}(n) \times S^{n-1} \to S^{n-1}.$$

It is easy to show that this action is transitive.

Analogously, we can define the *(complex) unit sphere*, Σ^{n-1}, as the set of points in $\mathbb{C}^n$ given by

$$\Sigma^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1\overline{z}_1 + \cdots + z_n\overline{z}_n = 1\}.$$

If we write $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$, then

$$\Sigma^{n-1} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \\ x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 = 1\}.$$

Therefore, we can view the complex sphere, Σ^{n-1} (in $\mathbb{C}^n$), as the real sphere, S^{2n-1} (in $\mathbb{R}^{2n}$).

By analogy with the real case, we can define an action,

$$\cdot\colon \mathbf{SU}(n) \times \Sigma^{n-1} \to \Sigma^{n-1},$$

of the group, $\mathbf{SU}(n)$, of *linear* maps of $\mathbb{C}^n$ preserving the hermitian inner product (and the origin, as all linear maps do) and this action is transitive.

One should not confuse the unit sphere, Σ^{n-1}, with the hypersurface, $S^{n-1}_{\mathbb{C}}$, given by

$$S^{n-1}_{\mathbb{C}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 = 1\}.$$

For instance, one should check that a line, L, through the origin intersects Σ^{n-1} in a circle, whereas it intersects $S^{n-1}_{\mathbb{C}}$ in exactly two points!

Example 5.2. The upper half-plane.

The *upper half-plane*, H, is the open subset of $\mathbb{R}^2$ consisting of all points, $(x, y) \in \mathbb{R}^2$, with $y > 0$.

It is convenient to identify H with the set of complex numbers, $z \in \mathbb{C}$, such that $\Im z > 0$. Then, we can define an action,

$$\cdot \colon \mathbf{SL}(2, \mathbb{R}) \times H \to H,$$

as follows: For any $z \in H$, for any $A \in \mathbf{SL}(2, \mathbb{R})$,

$$A \cdot z = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$.

It is easily verified that $A \cdot z$ is indeed always well defined and in H when $z \in H$. This action is transitive (check this).

Maps of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $z \in \mathbb{C}$ and $ad - bc = 1$, are called *Möbius transformations*.

Here, $a, b, c, d \in \mathbb{R}$, but in general, we allow $a, b, c, d \in \mathbb{C}$. Actually, these transformations are not necessarily defined everywhere on $\mathbb{C}$, for example, for $z = -d/c$ if $c \neq 0$.

To fix this problem, we add a "point at infinity", ∞, to $\mathbb{C}$ and define Möbius transformations as functions $\mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$.

If $c = 0$, the Möbius transformation sends ∞ to itself, otherwise, $-d/c \mapsto \infty$ and $\infty \mapsto a/c$.

The space $\mathbb{C} \cup \{\infty\}$ can be viewed as the plane, $\mathbb{R}^2$, extended with a point at infinity.

Using a stereographic projection from the sphere S^2 to the plane, (say from the north pole to the equatorial plane), we see that there is a bijection between the sphere, S^2, and $\mathbb{C} \cup \{\infty\}$.

More precisely, the *stereographic projection* of the sphere S^2 from the north pole, $N = (0, 0, 1)$, to the plane $z = 0$ (extended with the point at infinity, ∞) is given by

$$(x, y, z) \in S^2 - \{(0, 0, 1)\} \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = \frac{x+iy}{1-z},$$

with $(0, 0, 1) \mapsto \infty$.

The inverse stereographic projection is given by

$$(x, y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right),$$

with $\infty \mapsto (0, 0, 1)$.

Intuitively, the inverse stereographic projection "wraps" the equatorial plane around the sphere. The space $\mathbb{C} \cup \{\infty\}$ is known as the *Riemann sphere*.

We will see shortly that $\mathbb{C} \cup \{\infty\} \cong S^2$ is also the complex projective line, $\mathbb{CP}^1$.

In summary, Möbius transformations are bijections of the Riemann sphere. It is easy to check that these transformations form a group under composition for all $a, b, c, d \in \mathbb{C}$, with $ad - bc = 1$.

This is the *Möbius group*, denoted $\mathbf{M\ddot{o}b}^+$.

The Möbius transformations corresponding to the case $a, b, c, d \in \mathbb{R}$, with $ad - bc = 1$ form a subgroup of $\mathbf{M\ddot{o}b}^+$ denoted $\mathbf{M\ddot{o}b}^+_{\mathbb{R}}$.

The map from $\mathbf{SL}(2, \mathbb{C})$ to $\mathbf{M\ddot{o}b}^+$ that sends $A \in \mathbf{SL}(2, \mathbb{C})$ to the corresponding Möbius transformation is a surjective group homomorphism and one checks easily that its kernel is $\{-I, I\}$ (where I is the 2×2 identity matrix).

Therefore, the Möbius group $\mathbf{M\ddot{o}b}^+$ is isomorphic to the quotient group $\mathbf{SL}(2, \mathbb{C})/\{-I, I\}$, denoted $\mathbf{PSL}(2, \mathbb{C})$.

This latter group turns out to be the group of projective transformations of the projective space $\mathbb{CP}^1$.

The same reasoning shows that the subgroup $\mathbf{M\ddot{o}b}^+_{\mathbb{R}}$ is isomorphic to $\mathbf{SL}(2, \mathbb{R})/\{-I, I\}$, denoted $\mathbf{PSL}(2, \mathbb{R})$.

Example 5.3. The Riemann sphere $\mathbb{C} \cup \{\infty\}$.

The group $\mathbf{SL}(2, \mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\} \cong S^2$ the same way that $\mathbf{SL}(2, \mathbb{R})$ acts on H, namely: For any $A \in \mathbf{SL}(2, \mathbb{C})$, for any $z \in \mathbb{C} \cup \{\infty\}$,

$$A \cdot z = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1.$$

This action is clearly transitive.

Example 5.4. The unit disk.

One may recall from complex analysis that the (complex) Möbius transformation

$$z \mapsto \frac{z-i}{z+i}$$

is a biholomorphic isomorphism between the upper half plane, H, and the open unit disk,

$$D = \{z \in \mathbb{C} \mid |z| < 1\}.$$

As a consequence, it is possible to define a transitive action of $\mathbf{SL}(2, \mathbb{R})$ on D.

This can be done in a more direct fashion, using a group isomorphic to $\mathbf{SL}(2, \mathbb{R})$, namely, $\mathbf{SU}(1, 1)$ (a group of complex matrices), but we don't want to do this right now.

Example 5.5. The unit Riemann sphere revisited.

Another interesting action is the action of $\mathbf{SU}(2)$ on the extended plane $\mathbb{C} \cup \{\infty\}$.

Recall that the group $\mathbf{SU}(2)$ consists of all complex matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}, \qquad \alpha\overline{\alpha} + \beta\overline{\beta} = 1,$$

Let $X = \mathbb{C} \cup \{\infty\}$ and $G = \mathbf{SU}(2)$. The action $\cdot \colon \mathbf{SU}(2) \times (\mathbb{C} \cup \{\infty\}) \to \mathbb{C} \cup \{\infty\}$ is given by

$$A \cdot w = \frac{\alpha w + \beta}{-\overline{\beta} w + \overline{\alpha}}, \quad w \in \mathbb{C} \cup \{\infty\}.$$

This action is transitive, but the proof of this fact relies on the surjectivity of the group homomorphism

$$\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$$

defined below, and the stereographic projection σ_N from S^2 onto $\mathbb{C} \cup \{\infty\}$.

In particular, take $z, w \in \mathbb{C} \cup \{\infty\}$, use the inverse stereographic projection to obtain two points on S^2, namely $\sigma_N^{-1}(z)$ and $\sigma_N^{-1}(w)$.

Then apply the appropriate rotation $R \in \mathbf{SO}(3)$ to map $\sigma_N^{-1}(z)$ onto $\sigma_N^{-1}(w)$.

Such a rotation exists by the argument presented in Example 5.1.

Since $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is surjective (see below), we know there must exist $A \in \mathbf{SU}(2)$ such that $\rho(A) = R$ and $A \cdot z = w$.

Using the stereographic projection σ_N from S^2 onto $\mathbb{C} \cup \{\infty\}$ and its inverse σ_N^{-1}, we can define an action of $\mathbf{SU}(2)$ on S^2 by

$$A \cdot (x, y, z) = \sigma_N^{-1}(A \cdot \sigma_N(x, y, z)), \quad (x, y, z) \in S^2.$$

Although this is not immediately obvious, it turns out that $\mathbf{SU}(2)$ acts on S^2 by maps that are restrictions of linear maps to S^2, and since these linear maps preserve S^2, they are orthogonal transformations.

Thus, we obtain a continuous (in fact, smooth) group homomorphism

$$\rho \colon \mathbf{SU}(2) \to \mathbf{O}(3).$$

Since $\mathbf{SU}(2)$ is connected and ρ is continuous, the image of $\mathbf{SU}(2)$ is contained in the connected component of I in $\mathbf{O}(3)$, namely $\mathbf{SO}(3)$, so ρ is a homomorphism

$$\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3).$$

We will see that this homomorphism is surjective and that its kernel is $\{I, -I\}$. The upshot is that we have an isomorphism

$$\mathbf{SO}(3) \cong \mathbf{SU}(2)/\{I, -I\}.$$

The homomorphism ρ is a way of describing how a unit quaternion (any element of $\mathbf{SU}(2)$) induces a rotation, *via* the stereographic projection and its inverse.

If we write $\alpha = a + ib$ and $\beta = c + id$, a rather tedious computation yields

$$\rho(A) = \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & -2ab - 2cd & -2ac + 2bd \\ 2ab - 2cd & a^2 - b^2 + c^2 - d^2 & -2ad - 2bc \\ 2ac + 2bd & 2ad - 2bc & a^2 + b^2 - c^2 - d^2 \end{pmatrix}.$$

One can check that $\rho(A)$ is indeed a rotation matrix which represents the rotation whose axis is the line determined by the vector $(d, -c, b)$ and whose angle $\theta \in [-\pi, \pi]$ is determined by

$$\cos\frac{\theta}{2} = |a|.$$

We can also compute the derivative $d\rho_I \colon \mathfrak{su}(2) \to \mathfrak{so}(3)$ of ρ at I as follows.

Recall that $\mathfrak{su}(2)$ consists of all complex matrices of the form

$$\begin{pmatrix} ib & c+id \\ -c+id & -ib \end{pmatrix}, \quad b, c, d \in \mathbb{R},$$

so pick the following basis for $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and define the curves in $\mathbf{SU}(2)$ through I given by

$$\begin{aligned} c_1(t) &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \\ c_2(t) &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \\ c_3(t) &= \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}. \end{aligned}$$

It is easy to check that $c_i'(0) = X_i$ for $i = 1, 2, 3$, and that

$$d\rho_I(X_1) = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$d\rho_I(X_2) = 2\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$d\rho_I(X_3) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus we have

$$d\rho_I(X_1) = 2E_3, \quad d\rho_I(X_2) = -2E_2, \quad d\rho_I(X_3) = 2E_1,$$

where (E_1, E_2, E_3) is the basis of $\mathfrak{so}(3)$ given in Section 4, which means that $d\rho_I$ is an isomorphism between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

Recall from Proposition 4.12 that we have the commutative diagram

$$\begin{tikzcd} \mathbf{SU}(2) \arrow[r, "\rho"] & \mathbf{SO}(3) \\ \mathfrak{su}(2) \arrow[u, "\exp"] \arrow[r, "d\rho_I"'] & \mathfrak{so}(3) \arrow[u, "\exp"'] \end{tikzcd}.$$

Since $d\rho_I$ is surjective and the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective, we conclude that ρ is surjective.

(We also know from Section 4 that $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ is surjective.)

Observe that $\rho(-A) = \rho(A)$, and it is easy to check that $\mathrm{Ker}\,\rho = \{I, -I\}$.

Example 5.6. The set of $n \times n$ symmetric, positive, definite matrices, $\mathbf{SPD}(n)$.

The group $\mathbf{GL}(n) = \mathbf{GL}(n, \mathbb{R})$ acts on $\mathbf{SPD}(n)$ as follows: For all $A \in \mathbf{GL}(n)$ and all $S \in \mathbf{SPD}(n)$,

$$A \cdot S = ASA^{\top}.$$

It is easily checked that $ASA^{\top}$ is in $\mathbf{SPD}(n)$ if S is in $\mathbf{SPD}(n)$.

This action is transitive because every SPD matrix, S, can be written as $S = AA^{\top}$, for some invertible matrix, A (prove this as an exercise).

Example 5.7. The projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

The *(real) projective space*, $\mathbb{RP}^n$, is the set of all lines through the origin in $\mathbb{R}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{R}^{n+1}$ (where $n \geq 0$).

Since a one-dimensional subspace, $L \subseteq \mathbb{R}^{n+1}$, is spanned by any nonzero vector, $u \in L$, we can view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

In terms of this definition, there is a projection,

$$pr\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n,$$

given by $pr(u) = [u]_\sim$, the equivalence class of u modulo $\sim$.

Write $[u]$ for the line defined by the nonzero vector, u.

Since every line, L, in $\mathbb{R}^{n+1}$ intersects the sphere S^n in two antipodal points, we can view $\mathbb{RP}^n$ as the quotient of the sphere S^n by identification of antipodal points. See Figures 5.2 and 5.3.

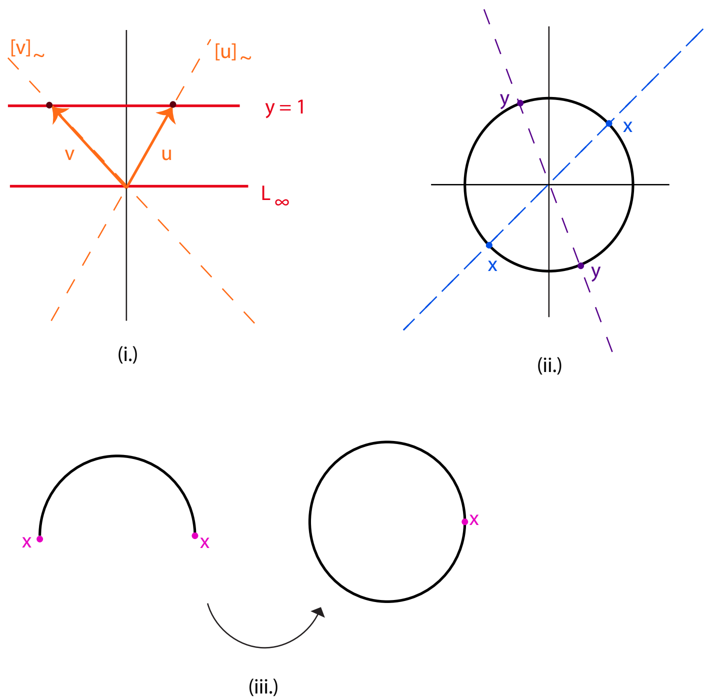

Figure 5.2: Three constructions for $\mathbb{RP}^1 \cong S^1$. Illustration (i.) applies the equivalence relation. Since any line through the origin, excluding the x-axis, intersects the line $y = 1$, its equivalence class is represented by its point of intersection on $y = 1$. Hence, $\mathbb{RP}^n$ is the disjoint union of the line $y = 1$ and the point of infinity given by the x-axis. Illustration (ii.) represents $\mathbb{RP}^1$ as the quotient of the circle S^1 by identification of antipodal points. Illustration (iii.) is a variation which glues the equatorial points of the upper semicircle.

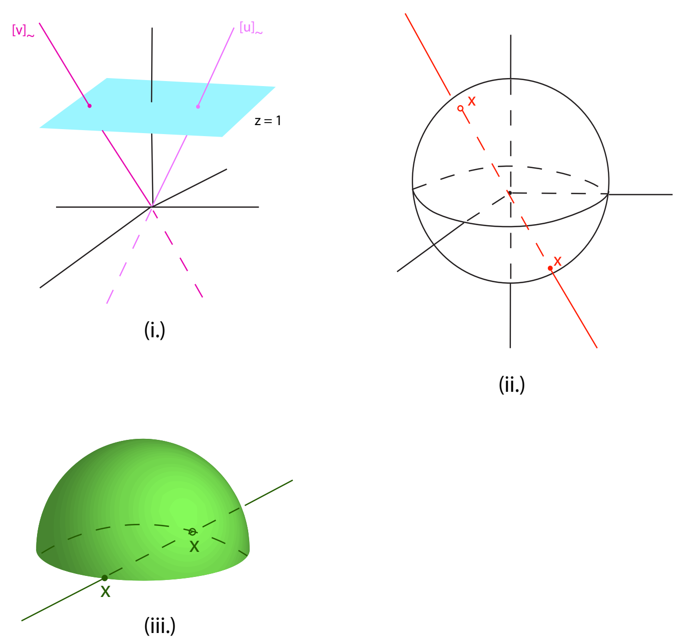

Figure 5.3: Three constructions for $\mathbb{RP}^2$. Illustration (i.) applies the equivalence relation. Since any line through the origin which is not contained in the xy-plane intersects the plane $z = 1$, its equivalence class is represented by its point of intersection on $z = 1$. Hence, $\mathbb{RP}^2$ is the disjoint union of the plane $z = 1$ and the copy of $\mathbb{RP}^1$ provided by the xy-plane. Illustration (ii.) represents $\mathbb{RP}^2$ as the quotient of the sphere S^2 by identification of antipodal points. Illustration (iii.) is a variation which glues the antipodal points on boundary of the unit disk, which is represented as as the upper hemisphere.

We define an action of $\mathbf{SO}(n+1)$ on $\mathbb{RP}^n$ as follows: For any line, $L = [u]$, for any $R \in \mathbf{SO}(n+1)$,

$$R \cdot L = [Ru].$$

Since R is linear, the line $[Ru]$ is well defined, i.e., does not depend on the choice of $u \in L$. It is clear that this action is transitive.

The *(complex) projective space*, $\mathbb{CP}^n$, is defined analogously as the set of all lines through the origin in $\mathbb{C}^{n+1}$, i.e., the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$ (where $n \geq 0$).

This time, we can view $\mathbb{CP}^n$ as the set of equivalence classes of vectors in $\mathbb{C}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{C}.$$

We have the projection,

$$pr\colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n,$$

given by $pr(u) = [u]_\sim$, the equivalence class of u modulo $\sim$.

Again, write $[u]$ for the line defined by the nonzero vector, u.

We define an action of $\mathbf{SU}(n+1)$ on $\mathbb{CP}^n$ as follows: For any line, $L = [u]$, for any $R \in \mathbf{SU}(n+1)$,

$$R \cdot L = [Ru].$$

Again, this action is well defined and it is transitive.

Recall that $\Sigma^n \subseteq \mathbb{C}^{n+1}$, the unit sphere in $\mathbb{C}^{n+1}$, is defined by

$$\Sigma^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid z_1\overline{z}_1 + \cdots + z_{n+1}\overline{z}_{n+1} = 1\}.$$

For any line, $L = [u]$, where $u \in \mathbb{C}^{n+1}$ is a nonzero vector, writing $u = (u_1, \ldots, u_{n+1})$, a point $z \in \mathbb{C}^{n+1}$ belongs to L iff $z = \lambda(u_1, \ldots, u_{n+1})$, for some $\lambda \in \mathbb{C}$.

Therefore, the intersection, $L \cap \Sigma^n$, of the line L and the sphere Σ^n is given by

$$\begin{aligned} L \cap \Sigma^n = \{\lambda(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} \mid \\ \lambda \in \mathbb{C},\ \lambda\overline{\lambda}(u_1\overline{u}_1 + \cdots + u_{n+1}\overline{u}_{n+1}) = 1\}, \end{aligned}$$

i.e.,

$$\begin{aligned} L \cap \Sigma^n = \left\{\lambda(u_1, \ldots, u_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \lambda \in \mathbb{C}, \right. \\ \left. |\lambda| = \frac{1}{\sqrt{|u_1|^2 + \cdots + |u_{n+1}|^2}} \right\}. \end{aligned}$$

Thus, we see that there is a bijection between $L \cap \Sigma^n$ and the circle, S^1, i.e., geometrically, $L \cap \Sigma^n$ is a circle.

Moreover, since any line, L, through the origin is determined by just one other point, we see that for any two lines L_1 and L_2 through the origin,

$$L_1 \neq L_2 \quad \text{iff} \quad (L_1 \cap \Sigma^n) \cap (L_2 \cap \Sigma^n) = \emptyset.$$

However, Σ^n is the sphere S^{2n+1} in $\mathbb{R}^{2n+2}$.

It follows that $\mathbb{CP}^n$ is the quotient of S^{2n+1} by the equivalence relation, $\sim$, defined such that

$$y \sim z \quad \text{iff} \quad y, z \in L \cap \Sigma^n,$$

for some line, L, through the origin.

Therefore, we can write

$$S^{2n+1}/S^1 \cong \mathbb{CP}^n.$$

The case $n = 1$ is particularly interesting, as it turns out that

$$S^3/S^1 \cong S^2.$$

This is the famous *Hopf fibration*. To show this, proceed as follows: As

$$S^3 \cong \Sigma^1 = \{(z, z') \in \mathbb{C}^2 \mid |z|^2 + |z'|^2 = 1\},$$

define a map, $\mathrm{HF}\colon S^3 \to S^2$, by

$$\mathrm{HF}((z, z')) = (2z\overline{z'}, |z|^2 - |z'|^2).$$

We leave as a homework exercise to prove that this map has range S^2 and that

$$\mathrm{HF}((z_1, z_1')) = \mathrm{HF}((z_2, z_2'))$$

iff

$$(z_1, z_1') = \lambda(z_2, z_2'), \quad \text{for some } \lambda \text{ with } |\lambda| = 1.$$

In other words, for any point, $p \in S^2$, the inverse image, $\mathrm{HF}^{-1}(p)$ (also called *fibre* over p), is a circle on S^3.

Consequently, S^3 can be viewed as the union of a family of disjoint circles. This is the *Hopf fibration*.

It is possible to visualize the Hopf fibration using the stereographic projection from S^3 onto $\mathbb{R}^3$. This is a beautiful and puzzling picture. For example, see Berger [4].

Therefore, HF induces a bijection from $\mathbb{CP}^1$ to S^2, and it is a homeomorphism.

Example 5.8. Affine spaces.

If E is any (real) vector space and X is any set, a transitive and faithful action, $\cdot \colon E \times X \to X$, of the additive group of E on X makes X into an *affine space*.

The intuition is that the members of E are translations.

Those familiar with affine spaces as in Gallier [21] (Chapter 2) or Berger [4] will point out that if X is an affine space, then, not only is the action of E on X transitive, but more is true:

For any two points, $a, b \in E$, there is a *unique* vector, $u \in E$, such that $u \cdot a = b$.

By the way, the action of E on X is usually considered to be a right action and is written additively, so $u \cdot a$ is written $a + u$ (the result of translating a by u).

Thus, it would seem that we have to require more of our action.

However, this is not necessary because E (under addition) is *abelian*.

Proposition 5.1. *If G is an abelian group acting on a set X and the action $\cdot\colon G \times X \to X$ is transitive and faithful, then for any two elements $x, y \in X$, there is a unique $g \in G$ so that $g \cdot x = y$ (the action is simply transitive).*

More examples will be considered later.

5.3 Group Actions: Part II, Stabilizers and Homogeneous Spaces

The subset of group elements that leave some given element $x \in X$ fixed plays an important role.

Definition 5.7. Given an action, $\cdot\colon G \times X \to X$, of a group G on a set X, for any $x \in X$, the group G_x (also denoted $\mathrm{Stab}_G(x)$), called the *stabilizer* of x or *isotropy group at x* is given by

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

It is easy to verify that G_x is indeed a subgroup of G.

In general, G_x is **not** a normal subgroup.

Observe that

$$G_{g\cdot x} = gG_x g^{-1},$$

for all $g \in G$ and all $x \in X$.

Therefore, the stabilizers of x and $g \cdot x$ are conjugate of each other.

When the action of G on X is transitive, for any fixed $x \in G$, the set X is a quotient (as a set, not as group) of G by G_x.

Proposition 5.2. *If $\cdot\colon G \times X \to X$ is a transitive action of a group G on a set X, for every fixed $x \in X$, the surjection, $\pi\colon G \to X$, given by*

$$\pi(g) = g \cdot x$$

induces a bijection

$$\overline{\pi}\colon G/G_x \to X,$$

where G_x is the stabilizer of x. See Figure 5.4.

The map $\pi\colon G \to X$ (corresponding to a fixed $x \in X$) is sometimes called a *projection* of G onto X.

Proposition 5.2 shows that for every $y \in X$, the subset $\pi^{-1}(y)$, (called the *fibre above y*) is equal to some coset gG_x of G, and thus is in bijection with the group G_x itself.

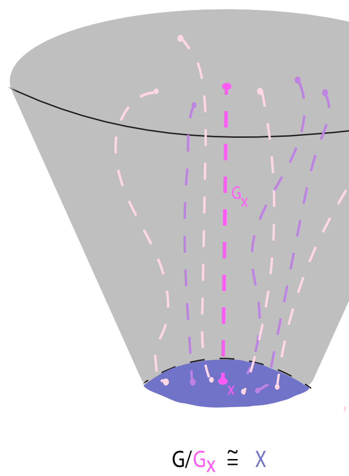

Figure 5.4: A schematic representation of $G/G_x \cong X$, where G is the gray solid, X is its purple circular base, and G_x is the pink vertical strand. The dotted strands are the fibres gG_x.

We can think of G as a moving family of fibres, G_x, parametrized by X.

This point of view of viewing a space as a moving family of simpler spaces is typical in (algebraic) geometry, and underlies the notion of (principal) fibre bundle.

Note that if the action $\cdot\colon G \times X \to X$ is transitive, then the stabilizers G_x and G_y of any two elements $x, y \in X$ are isomorphic, as they as conjugates.

Thus, in this case, it is enough to compute one of these stabilizers for a "convenient" x.

Definition 5.8. A set, X, is said to be a *homogeneous space* if there is a transitive action, $\cdot\colon G \times X \to X$, of some group, G, on X.

We see that all the spaces of Example 5.1–5.8, are homogeneous spaces.

Another example that will play an important role when we deal with Lie groups is the situation where we have a group, G, a subgroup, H, of G (not necessarily normal) and where $X = G/H$, the set of left cosets of G modulo H.

The group G acts on G/H by left multiplication:

$$a \cdot (gH) = (ag)H,$$

where $a, g \in G$. This action is clearly transitive and one checks that the stabilizer of gH is gHg^{-1}.

If G is a topological group and H is a closed subgroup of G (see later for an explanation), it turns out that G/H is *Hausdorff*.

If G is a Lie group, we obtain a manifold.

Even if G and X are topological spaces and the action, $\cdot\colon G \times X \to X$, is continuous, the space G/G_x under the quotient topology is, in general, **not** homeomorphic to X.

We will give later sufficient conditions that insure that X is indeed a topological space or even a manifold.

In particular, X will be a manifold when G is a Lie group.

In general, an action $\cdot\colon G \times X \to X$ is not transitive on X, but for every $x \in X$, it is transitive on the set

$$O(x) = G \cdot x = \{g \cdot x \mid g \in G\}.$$

Such a set is called the *orbit* of x. The orbits are the equivalence classes of the following equivalence relation:

Definition 5.9. Given an action, $\cdot \colon G \times X \to X$, of some group, G, on X, the equivalence relation, $\sim$, on X is defined so that, for all $x, y \in X$,

$$x \sim y \quad \text{iff} \quad y = g \cdot x, \quad \text{for some } g \in G.$$

For every $x \in X$, the equivalence class of x is the *orbit of x*, denoted $O(x)$ or $G \cdot x$, with

$$O(x) = G \cdot x = \{g \cdot x \mid g \in G\}.$$

The set of orbits is denoted X/G.

We warn the reader that some authors use the notation $G \backslash X$ for the the set of orbits $G \cdot x$, because these orbits can be considered as right orbits, by analogy with right cosets Hg of a subgroup H of G.

The orbit space, X/G, is obtained from X by an identification (or merging) process: For every orbit, all points in that orbit are merged into a single point.

For example, if $X = S^2$ and G is the group consisting of the restrictions of the two linear maps I and $-I$ of $\mathbb{R}^3$ to S^2 (where $(-I)(x) = -x$), then

$$X/G = S^2/\{I, -I\} \cong \mathbb{RP}^2.$$

More generally, if S^n is the n-sphere in $\mathbb{R}^{n+1}$, then we have a bijection between the orbit space $S^n/\{I, -I\}$ and $\mathbb{RP}^n$:

$$S^n/\{I, -I\} \cong \mathbb{RP}^n.$$

Many manifolds can be obtained in this fashion, including the torus, the Klein bottle, the Möbius band, etc.

Since the action of G is transitive on $O(x)$, by Proposition 5.2, we see that for every $x \in X$, we have a bijection

$$O(x) \cong G/G_x.$$

As a corollary, if both X and G are finite, for any set $A \subseteq X$ of representatives from every orbit, we have the *orbit formula*:

$$|X| = \sum_{a \in A} [G \colon G_a] = \sum_{a \in A} |G|/|G_a|.$$

Even if a group action, $\cdot \colon G \times X \to X$, is not transitive, when X is a manifold, we can consider the set of orbits, X/G, and if the action of G on X satisfies certain conditions, X/G is actually a manifold.

Spaces arising in this fashion are often called *orbifolds*.

In summary, we see that manifolds arise in at least two ways from a group action:

(1) As homogeneous spaces, G/G_x, if the action is transitive.

(2) As orbifolds, X/G.

Of course, in both cases, the action must satisfy some additional properties.

Let us now determine some stabilizers for the actions of Examples Examples 5.1–5.8 and for more examples of homogeneous spaces.

(a) Consider the action

$$\cdot \colon \mathbf{SO}(n) \times S^{n-1} \to S^{n-1},$$

of $\mathbf{SO}(n)$ on the sphere S^{n-1} ($n \geq 1$) defined in Example 5.1. Since this action is transitive, we can determine the stabilizer of any convenient element of S^{n-1}, say $e_1 = (1, 0, \ldots, 0)$.

In order for any $R \in \mathbf{SO}(n)$ to leave e_1 fixed, the first column of R must be e_1, so R is an orthogonal matrix of the form

$$R = \begin{pmatrix} 1 & U \\ 0 & S \end{pmatrix}, \quad \text{with} \quad \det(S) = 1.$$

As the rows of R must be unit vector, we see that $U = 0$ and $S \in \mathbf{SO}(n-1)$.

Therefore, the stabilizer of e_1 is isomorphic to $\mathbf{SO}(n-1)$, and we deduce the bijection

$$\mathbf{SO}(n)/\mathbf{SO}(n-1) \cong S^{n-1}.$$

Strictly speaking, $\mathbf{SO}(n-1)$ is not a subgroup of $\mathbf{SO}(n)$ and in all rigor, we should consider the subgroup, $\widetilde{\mathbf{SO}}(n-1)$, of $\mathbf{SO}(n)$ consisting of all matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad \det(S) = 1$$

and write

$$\mathbf{SO}(n)/\widetilde{\mathbf{SO}}(n-1) \cong S^{n-1}.$$

However, it is common practice to identify $\mathbf{SO}(n-1)$ with $\widetilde{\mathbf{SO}}(n-1)$.

When $n = 2$, as $\mathbf{SO}(1) = \{1\}$, we find that $\mathbf{SO}(2) \cong S^1$, a circle, a fact that we already knew.

When $n = 3$, we find that $\mathbf{SO}(3)/\mathbf{SO}(2) \cong S^2$.

This says that $\mathbf{SO}(3)$ is somehow the result of gluing circles to the surface of a sphere (in $\mathbb{R}^3$), in such a way that these circles do not intersect. This is hard to visualize!

A similar argument for the complex unit sphere, Σ^{n-1}, shows that

$$\mathbf{SU}(n)/\mathbf{SU}(n-1) \cong \Sigma^{n-1} \cong S^{2n-1}.$$

Again, we identify $\mathbf{SU}(n-1)$ with a subgroup of $\mathbf{SU}(n)$, as in the real case. In particular, when $n = 2$, as $\mathbf{SU}(1) = \{1\}$, we find that

$$\mathbf{SU}(2) \cong S^3,$$

i.e., the group $\mathbf{SU}(2)$ is topologically the sphere S^3!

Actually, this is not surprising if we remember that $\mathbf{SU}(2)$ is in fact the group of unit quaternions.

(b) We saw in Example 5.2 that the action

$$\cdot\colon \mathbf{SL}(2, \mathbb{R}) \times H \to H$$

of the group $\mathbf{SL}(2, \mathbb{R})$ on the upper half plane is transitive. Let us find out what the stabilizer of $z = i$ is.

We should have

$$\frac{ai + b}{ci + d} = i,$$

that is, $ai + b = -c + di$, i.e.,

$$(d - a)i = b + c.$$

Since a, b, c, d are real, we must have $d = a$ and $b = -c$. Moreover, $ad - bc = 1$, so we get $a^2 + b^2 = 1$.

We conclude that a matrix in $\mathbf{SL}(2, \mathbb{R})$ fixes i iff it is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with} \quad a^2 + b^2 = 1.$$

Clearly, these are the rotation matrices in $\mathbf{SO}(2)$ and so, the stabilizer of i is $\mathbf{SO}(2)$. We conclude that

$$\mathbf{SL}(2,\mathbb{R})/\mathbf{SO}(2) \cong H.$$

This time, we can view $\mathbf{SL}(2,\mathbb{R})$ as the result of gluing circles to the upper half plane. This is not so easy to visualize.

There is a better way to visualize the topology of $\mathbf{SL}(2,\mathbb{R})$ by making it act on the open disk, D. We will return to this action in a little while.

(c) Now, consider the action of $\mathbf{SL}(2, \mathbb{C})$ on $\mathbb{C} \cup \{\infty\} \cong S^2$. As it is transitive, let us find the stabilizer of $z = 0$. We must have

$$\frac{b}{d} = 0,$$

and as $ad - bc = 1$, we must have $b = 0$ and $ad = 1$.

Thus, the stabilizer of 0 is the subgroup, $\mathbf{SL}(2, \mathbb{C})_0$, of $\mathbf{SL}(2, \mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad \text{where} \quad a \in \mathbb{C} - \{0\} \quad \text{and} \quad c \in \mathbb{C}.$$

We get

$$\mathbf{SL}(2, \mathbb{C})/\mathbf{SL}(2, \mathbb{C})_0 \cong \mathbb{C} \cup \{\infty\} \cong S^2,$$

but this is not very illuminating.

(d) In Example 5.6, we considered the action

$$\cdot \colon \mathbf{GL}(n) \times \mathbf{SPD}(n) \to \mathbf{SPD}(n)$$

of $\mathbf{GL}(n)$ on $\mathbf{SPD}(n)$, the set of symmetric positive definite matrices.

As this action is transitive, let us find the stabilizer of I. For any $A \in \mathbf{GL}(n)$, the matrix A stabilizes I iff

$$AIA^\top = AA^\top = I.$$

Therefore, the stabilizer of I is $\mathbf{O}(n)$ and we find that

$$\mathbf{GL}(n)/\mathbf{O}(n) = \mathbf{SPD}(n).$$

Observe that if $\mathbf{GL}^+(n)$ denotes the subgroup of $\mathbf{GL}(n)$ consisting of all matrices with a strictly positive determinant, then we have an action

$$\cdot\colon \mathbf{GL}^+(n) \times \mathbf{SPD}(n) \to \mathbf{SPD}(n).$$

This action is transitive and we find that the stabilizer of I is $\mathbf{SO}(n)$; consequently, we get

$$\mathbf{GL}^+(n)/\mathbf{SO}(n) = \mathbf{SPD}(n).$$

(e) In Example 5.7, we considered the action

$$\cdot\colon \mathbf{SO}(n+1) \times \mathbb{RP}^n \to \mathbb{RP}^n$$

of $\mathbf{SO}(n+1)$ on the (real) projective space, $\mathbb{RP}^n$. As this action is transitive, let us find the stabilizer of the line, $L = [e_1]$, where $e_1 = (1, 0, \ldots, 0)$.

We find that the stabilizer of $L = [e_1]$ is isomorphic to the group $\mathbf{O}(n)$ and so,

$$\mathbf{SO}(n+1)/\mathbf{O}(n) \cong \mathbb{RP}^n.$$

Strictly speaking, $\mathbf{O}(n)$ is not a subgroup of $\mathbf{SO}(n+1)$, so the above equation does not make sense. We should write

$$\mathbf{SO}(n+1)/\widetilde{\mathbf{O}}(n) \cong \mathbb{RP}^n,$$

where $\widetilde{\mathbf{O}}(n)$ is the subgroup of $\mathbf{SO}(n+1)$ consisting of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad S \in \mathbf{O}(n),\ \ \alpha = \pm 1, \quad \det(S) = \alpha.$$

However, the common practice is to write $\mathbf{O}(n)$ instead of $\widetilde{\mathbf{O}}(n)$.

We should mention that $\mathbb{RP}^3$ and $\mathbf{SO}(3)$ are homeomorphic spaces. This is shown using the quaternions, for example, see Gallier [21], Chapter 8.

A similar argument applies to the action

$$\cdot \colon \mathbf{SU}(n+1) \times \mathbb{CP}^n \to \mathbb{CP}^n$$

of $\mathbf{SU}(n+1)$ on the (complex) projective space, $\mathbb{CP}^n$. We find that

$$\mathbf{SU}(n+1)/\mathbf{U}(n) \cong \mathbb{CP}^n.$$

Again, the above is a bit sloppy as $\mathbf{U}(n)$ is not a subgroup of $\mathbf{SU}(n+1)$. To be rigorous, we should use the subgroup, $\widetilde{\mathbf{U}}(n)$, consisting of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & S \end{pmatrix}, \quad \text{with} \quad S \in \mathbf{U}(n), \ |\alpha| = 1, \quad \det(S) = \overline{\alpha}.$$

The common practice is to write $\mathbf{U}(n)$ instead of $\widetilde{\mathbf{U}}(n)$.

In particular, when $n = 1$, we find that

$$\mathbf{SU}(2)/\mathbf{U}(1) \cong \mathbb{CP}^1.$$

But, we know that $\mathbf{SU}(2) \cong S^3$ and, clearly, $\mathbf{U}(1) \cong S^1$.

So, again, we find that $S^3/S^1 \cong \mathbb{CP}^1$ (but we know, more, namely, $S^3/S^1 \cong S^2 \cong \mathbb{CP}^1$.)

We now return to case (b) to give a better picture of $\mathbf{SL}(2, \mathbb{R})$. Instead of having $\mathbf{SL}(2, \mathbb{R})$ act on the upper half plane we define an action of $\mathbf{SL}(2, \mathbb{R})$ on the open unit disk, D.

Technically, it is easier to consider the group, $\mathbf{SU}(1, 1)$, which is isomorphic to $\mathbf{SL}(2, \mathbb{R})$, and to make $\mathbf{SU}(1, 1)$ act on D.

The group $\mathbf{SU}(1, 1)$ is the group of 2×2 complex matrices of the form

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{with} \quad a\overline{a} - b\overline{b} = 1.$$

The reader should check that if we let

$$g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

then the map from $\mathbf{SL}(2, \mathbb{R})$ to $\mathbf{SU}(1, 1)$ given by

$$A \mapsto gAg^{-1}$$

is an isomorphism.

Observe that the Möbius transformation associated with g is

$$z \mapsto \frac{z - i}{z + 1},$$

which is the holomorphic isomorphism mapping H to D mentionned earlier!

Now, we can define a bijection between $\mathbf{SU}(1, 1)$ and $S^1 \times D$ given by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto (a/|a|, b/a).$$

We conclude that $\mathbf{SL}(2, \mathbb{R}) \cong \mathbf{SU}(1, 1)$ is topologically an open solid torus (i.e., with the surface of the torus removed).

It is possible to further classify the elements of $\mathbf{SL}(2, \mathbb{R})$ into three categories and to have geometric interpretations of these as certain regions of the torus.

For details, the reader should consult Carter, Segal and Macdonald [13] or Duistermatt and Kolk [18] (Chapter 1, Section 1.2).

The group $\mathbf{SU}(1, 1)$ acts on D by interpreting any matrix in $\mathbf{SU}(1, 1)$ as a Möbius tranformation, i.e.,

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto \left(z \mapsto \frac{az + b}{\overline{b}z + \overline{a}} \right).$$

The reader should check that these transformations preserve D.

Both the upper half-plane and the open disk are models of Lobachevsky's non-Euclidean geometry (where the parallel postulate fails).

They are also models of hyperbolic spaces (Riemannian manifolds with constant negative curvature, see Gallot, Hulin and Lafontaine [22], Chapter III).

According to Dubrovin, Fomenko, and Novikov [17] (Chapter 2, Section 13.2), the open disk model is due to Poincaré and the upper half-plane model to Klein, although Poincaré was the first to realize that the upper half-plane is a hyperbolic space.

5.4 The Grassmann and Stiefel Manifolds

We now consider a generalization of projective spaces (real and complex). First, consider the real case.

Definition 5.10. Given any $n \geq 1$, for any k with $0 \leq k \leq n$, the set $G(k, n)$ of all linear k-dimensional subspaces of $\mathbb{R}^n$ (also called *k-planes*) is called a *Grassmannian* (or *Grassmann manifold*).

Any k-dimensional subspace, U, of $\mathbb{R}$ is spanned by k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; write $U = \mathrm{span}(u_1, \ldots, u_k)$.

We can define an action,

$$\cdot \colon \mathbf{O}(n) \times G(k, n) \to G(k, n)$$

as follows: For any $R \in \mathbf{O}(n)$, for any
$U = \mathrm{span}(u_1, \ldots, u_k)$, let

$$R \cdot U = \mathrm{span}(Ru_1, \ldots, Ru_k).$$

We have to check that the above is well defined but this is not hard.

It is also easy to see that this action is transitive.

Thus, it is enough to find the stabilizer of any k-plane.

We can show that the stabilizer of U is isomorphic to $\mathbf{O}(k) \times \mathbf{O}(n-k)$ and we find that

$$\mathbf{O}(n)/(\mathbf{O}(k) \times \mathbf{O}(n-k)) \cong G(k, n).$$

It turns out that this makes $G(k, n)$ into a smooth manifold of dimension $k(n-k)$ called a *Grassmannian*.

The restriction of the action of $\mathbf{O}(n)$ on $G(k, n)$ to $\mathbf{SO}(n)$ yields an action

$$\cdot \colon \mathbf{SO}(n) \times G(k, n) \to G(k, n)$$

of $\mathbf{SO}(n)$ on $G(k, n)$.

Then, it is easy to see that the stabilizer of the subspace U is isomorphic to the subgroup $S(\mathbf{O}(k) \times \mathbf{O}(n-k))$ of $\mathbf{SO}(n)$ consisting of the rotations of the form

$$R = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix},$$

with $S \in \mathbf{O}(k)$, $T \in \mathbf{O}(n-k)$ and $\det(S)\det(T) = 1$.

Thus, we also have

$$\mathbf{SO}(n)/S(\mathbf{O}(k) \times \mathbf{O}(n-k)) \cong G(k, n).$$

If we recall the projection $pr\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$, by definition, a *$k$-plane* in $\mathbb{RP}^n$ is the image under pr of any $(k+1)$-plane in $\mathbb{R}^{n+1}$.

So, for example, a line in $\mathbb{RP}^n$ is the image of a 2-plane in $\mathbb{R}^{n+1}$, and a hyperplane in $\mathbb{RP}^n$ is the image of a hyperplane in $\mathbb{R}^{n+1}$.

The advantage of this point of view is that the k-planes in $\mathbb{RP}^n$ are arbitrary, i.e., they do not have to go through “the origin” (which does not make sense, anyway!).

Then, we see that we can interpret the Grassmannian $G(k+1, n+1)$ as a space of “parameters” for the k-planes in $\mathbb{RP}^n$. For example, $G(2, n+1)$ parametrizes the lines in $\mathbb{RP}^n$.

In this viewpoint, $G(k+1, n+1)$ is usually denoted $\mathbb{G}(k, n)$.

It can be proved (using some exterior algebra) that $G(k, n)$ can be embedded in $\mathbb{RP}^{\binom{n}{k}-1}$.

Much more is true. For example, $G(k, n)$ is a projective variety, which means that it can be defined as a subset of $\mathbb{RP}^{\binom{n}{k}-1}$ equal to the zero locus of a set of homogeneous equations.

There is even a set of quadratic equations, known as the *Plücker equations*, defining $G(k, n)$.

In particular, when $n = 4$ and $k = 2$, we have $G(2, 4) \subseteq \mathbb{RP}^5$, and $G(2, 4)$ is defined by a single equation of degree 2.

The Grassmannian $G(2, 4) = \mathbb{G}(1, 3)$ is known as the *Klein quadric*. This hypersurface in $\mathbb{RP}^5$ parametrizes the lines in $\mathbb{RP}^3$. It play an important role in computer vision.

Complex Grassmannians are defined in a similar way, by replacing $\mathbb{R}$ by $\mathbb{C}$ throughout.

The complex Grassmannian, $G_{\mathbb{C}}(k, n)$, is a complex manifold as well as a real manifold and we have

$$\mathbf{U}(n)/(\mathbf{U}(k) \times \mathbf{U}(n-k)) \cong G_{\mathbb{C}}(k, n).$$

As in the case of the real Grassmannians, the action of $\mathbf{U}(n)$ on $G_{\mathbb{C}}(k,n)$ yields an action of $\mathbf{SU}(n)$ on $G_{\mathbb{C}}(k,n)$, and we get

$$\mathbf{SU}(n)/S(\mathbf{U}(k)\times\mathbf{U}(n-k))\cong G_{\mathbb{C}}(k,n),$$

where $S(\mathbf{U}(k)\times\mathbf{U}(n-k))$ is the subgroup of $\mathbf{SU}(n)$ consisting of all matrices $R\in\mathbf{SU}(n)$ of the form

$$R=\begin{pmatrix} S & 0\\ 0 & T\end{pmatrix},$$

with $S\in\mathbf{U}(k)$, $T\in\mathbf{U}(n-k)$ and $\det(S)\det(T)=1$.

Closely related to Grassmannians are the *Stiefel manifolds*.

We begin with the real case.

Definition 5.11. For any $n \geq 1$ and any k with $1 \leq k \leq n$, the set $S(k, n)$ of all orthonormal *k-frames*, that is, of k-tuples of orthonormal vectors $(u_1, \ldots, u_k)$ with $u_i \in \mathbb{R}^n$, is called a *Stiefel manifold*.

Obviously, $S(1, n) = S^{n-1}$ and $S(n, n) = \mathbf{O}(n)$, so assume $k \leq n - 1$.

There is a natural action

$$\cdot \colon \mathbf{SO}(n) \times S(k, n) \to S(k, n)$$

of $\mathbf{SO}(n)$ on $S(k, n)$ given by

$$R \cdot (u_1, \ldots, u_k) = (Ru_1, \ldots, Ru_k).$$

This action is transitive.

Let us find the stabilizer of the orthonormal k-frame $(e_1, \ldots, e_k)$ consisting of the first canonical k-basis vectors of $\mathbb{R}^n$.

A matrix $R \in \mathbf{SO}(n)$ stabilizes $(e_1, \ldots, e_k)$ iff it is of the form

$$R = \begin{pmatrix} I_k & 0 \\ 0 & S \end{pmatrix}$$

where $S \in \mathbf{SO}(n-k)$.

Therefore,

$$\mathbf{SO}(n)/\mathbf{SO}(n-k) \cong S(k,n).$$

This makes $S(k,n)$ a smooth manifold of dimension

$$nk - \frac{k(k+1)}{2} = k(n-k) + \frac{k(k-1)}{2}.$$

Remark: It should be noted that we can define another type of Stiefel manifolds, denoted by $V(k, n)$, using linearly independent k-tuples $(u_1, \ldots, u_k)$ that do not necessarily form an orthonormal system.

In this case, there is an action

$$\cdot\colon \mathbf{GL}(n, \mathbb{R}) \times V(k, n) \to V(k, n)$$

and the stabilizer H of the first k canonical basis vectors $(e_1, \ldots, e_k)$ is a closed subgroup of $\mathbf{GL}(n, \mathbb{R})$, but it doesn't have a simple description,

We get an isomorphism

$$V(k, n) \cong \mathrm{GL}(n, \mathbb{R})/H.$$

The version of the Stiefel manifold $S(k, n)$ using orthonormal frames is sometimes denoted by $V^0(k, n)$ (Milnor and Stasheff [39] use the notation $V_k^0(\mathbb{R}^n)$).

Beware that the notation is not standardized. Certain authors use $V(k, n)$ for what we denote by $S(k, n)$!

Complex Stiefel manifolds are defined in a similar way by replacing $\mathbb{R}$ by $\mathbb{C}$ and $\mathbf{O}(n)$ by $\mathbf{U}(n)$.

For $1 \leq k \leq n-1$, the complex Stiefel manifold $S_{\mathbb{C}}(k, n)$ is isomorphic to the quotient

$$\mathbf{SU}(n)/\mathbf{SU}(n-k) \cong S_{\mathbb{C}}(k, n).$$

If $k = 1$, we have $S_{\mathbb{C}}(1, n) = S^{2n-1}$, and if $k = n$, we have $S_{\mathbb{C}}(n, n) = \mathbf{U}(n)$.

The Grassmannians can also be viewed as quotient spaces of the Stiefel manifolds.

Every k-frame $(u_1, \ldots, u_k)$ can be represented by an $n \times k$ matrix Y over the canonical basis of $\mathbb{R}^n$, and such a matrix Y satisfies the equation

$$Y^\top Y = I.$$

We have a right action

$$\cdot\colon S(k,n) \times \mathbf{O}(k) \to S(k,n)$$

given by

$$Y \cdot R = YR,$$

for any $R \in \mathbf{O}(k)$.

However, this action is not transitive (unless $k = 1$), but the orbit space $S(k,n)/\mathbf{O}(k)$ is isomorphic to the Grassmannian $G(k,n)$, so we can write

$$G(k,n) \cong S(k,n)/\mathbf{O}(k).$$

Similarly, the complex Grassmannian is isomorphic to the orbit space $S_{\mathbb{C}}(k,n)/\mathbf{U}(k)$:

$$G_{\mathbb{C}}(k,n) \cong S_{\mathbb{C}}(k,n)/\mathbf{U}(k).$$

5.5 Topological Groups

Since Lie groups are topological groups (and manifolds), it is useful to gather a few basic facts about topological groups.

Definition 5.12. A set, G, is a *topological group* iff

(a) G is a Hausdorff topological space;

(b) G is a group (with identity 1);

(c) Multiplication, $\cdot\colon G \times G \to G$, and the inverse operation, $G \longrightarrow G\colon g \mapsto g^{-1}$, are continuous, where $G \times G$ has the product topology.

It is easy to see that the two requirements of condition (c) are equivalent to

(c′) The map $G \times G \longrightarrow G\colon (g, h) \mapsto gh^{-1}$ is continuous.

Given a topological group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Observe that $L_{a^{-1}}$ is the inverse of L_a and similarly, $R_{a^{-1}}$ is the inverse of R_a. As multiplication is continuous, we see that L_a and R_a are continuous.

Moreover, since they have a continuous inverse, they are homeomorphisms.

As a consequence, if U is an open subset of G, then so is $gU = L_g(U)$ (resp. $Ug = R_gU$), for all $g \in G$.

Therefore, the topology of a topological group (i.e., its family of open sets) is *determined* by the knowledge of the open subsets containing the identity, 1.

Given any subset, $S \subseteq G$, let $S^{-1} = \{s^{-1} \mid s \in S\}$; let $S^0 = \{1\}$ and $S^{n+1} = S^n S$, for all $n \geq 0$.

Property (c) of Definition 5.12 has the following useful consequences:

Proposition 5.3. *If G is a topological group and U is any open subset containing 1, then there is some open subset, $V \subseteq U$, with $1 \in V$, so that $V = V^{-1}$ and $V^2 \subseteq U$. Furthermore, $\overline{V} \subseteq U$.*

A subset, U, containing 1 such that $U = U^{-1}$, is called *symmetric*.

Using Proposition 5.3, we can give a very convenient characterization of the Hausdorff separation property in a topological group.

Proposition 5.4. *If G is a topological group, then the following properties are equivalent:*

(1) G is Hausdorff;

(2) The set $\{1\}$ is closed;

(3) The set $\{g\}$ is closed, for every $g \in G$.

If H is a subgroup of G (not necessarily normal), we can form the set of left cosets, G/H and we have the projection, $p\colon G \to G/H$, where $p(g) = gH = \overline{g}$.

If G is a topological group, then G/H can be given the *quotient topology*, where a subset $U \subseteq G/H$ is open iff $p^{-1}(U)$ is open in G.

With this topology, p is continuous.

The trouble is that G/H is not necessarily Hausdorff. However, we can neatly characterize when this happens.

Proposition 5.5. *If G is a topological group and H is a subgroup of G then the following properties hold:*

(1) The map $p\colon G \to G/H$ is an open map, which means that $p(V)$ is open in G/H whenever V is open in G.

(2) The space G/H is Hausdorff iff H is closed in G.

(3) If H is open, then H is closed and G/H has the discrete topology (every subset is open).

(4) The subgroup H is open iff $1 \in \overset{\circ}{H}$ (i.e., there is some open subset, U, so that $1 \in U \subseteq H$).

Proposition 5.6. *If G is a connected topological group, then G is generated by any symmetric neighborhood, V, of 1. In fact,*

$$G = \bigcup_{n \geq 1} V^n.$$

A subgroup, H, of a topological group G is *discrete* iff the induced topology on H is discrete, i.e., for every $h \in H$, there is some open subset, U, of G so that $U \cap H = \{h\}$.

Proposition 5.7. *If G is a topological group and H is discrete subgroup of G, then H is closed.*

Proposition 5.8. *If G is a topological group and H is any subgroup of G, then the closure, $\overline{H}$, of H is a subgroup of G.*

Proposition 5.9. *Let G be a topological group and H be any subgroup of G. If H and G/H are connected, then G is connected.*

Proposition 5.10. *Let G be a topological group and let V be any connected symmetric open subset containing* 1*. Then, if G_0 is the connected component of the identity, we have*

$$G_0 = \bigcup_{n \geq 1} V^n$$

and G_0 is a normal subgroup of G. Moreover, the group G/G_0 is discrete.

A topological space, X is *locally compact* iff for every point $p \in X$, there is a compact neighborhood, C of p, i.e., there is a compact, C, and an open, U, with $p \in U \subseteq C$.

For example, manifolds are locally compact.

Proposition 5.11. *Let G be a topological group and assume that G is connected and locally compact. Then, G is countable at infinity, which means that G is the union of a countable family of compact subsets. In fact, if V is any symmetric compact neighborhood of* 1*, then*

$$G = \bigcup_{n \geq 1} V^n.$$

Definition 5.13. Let G be a topological group and let X be a topological space. An action $\varphi \colon G \times X \to X$ is *continuous* (and *G acts continuously on X*) if the map φ is continuous.

If an action $\varphi \colon G \times X \to X$ is continuous, then each map $\varphi_g \colon X \to X$ is a homeomorphism of X (recall that $\varphi_g(x) = g \cdot x$, for all $x \in X$).

Under some mild assumptions on G and X, the quotient space G/G_x is homeomorphic to X. For example, this happens if X is a Baire space.

Recall that a *Baire space* X is a topological space with the property that if $\{F\}_{i\geq 1}$ is any countable family of closed sets F_i such that each F_i has empty interior, then $\bigcup_{i\geq 1} F_i$ also has empty interior.

By complementation, this is equivalent to the fact that for every countable family of open sets U_i such that each U_i is dense in X (i.e., $\overline{U}_i = X$), then $\bigcap_{i\geq 1} U_i$ is also dense in X.

Remark: A subset $A \subseteq X$ is *rare* if its closure $\overline{A}$ has empty interior. A subset $Y \subseteq X$ is *meager* if it is a countable union of rare sets.

Then, it is immediately verified that a space X is a Baire space iff every nonempty open subset of X is not meager.

The following theorem shows that there are plenty of Baire spaces:

Theorem 5.12. *(Baire) (1) Every locally compact topological space is a Baire space.*

(2) Every complete metric space is a Baire space.

Theorem 5.13. *Let G be a topological group which is locally compact and countable at infinity, X a Hausdorff topological space which is a Baire space, and assume that G acts transitively and continuously on X. Then, for any $x \in X$, the map $\varphi\colon G/G_x \to X$ is a homeomorphism.*

By Theorem 5.12, we get the following important corollary:

Theorem 5.14. *Let G be a topological group which is locally compact and countable at infinity, X a locally compact Hausdorff topological space and assume that G acts transitively and continuously on X. Then, for any $x \in X$, the map $\varphi\colon G/G_x \to X$ is a homeomorphism.*

Remark: If a topological group acts continuously and transitively on a Hausdorff topological space, then for every $x \in X$, the stabilizer, G_x, is a closed subgroup of G.

This is because, as the action is continuous, the projection $\pi\colon G \longrightarrow X\colon g \mapsto g \cdot x$ is continuous, and $G_x = \pi^{-1}(\{x\})$, with $\{x\}$ closed.