Chapter 3

Adjoint Representations and the Derivative of exp

3.1 The Adjoint Representations Ad and ad

Given any two vector spaces E and F, recall that the vector space of all linear maps from E to F is denoted by $\mathrm{Hom}(E, F)$.

The vector space of all invertible linear maps from E to itself is a group denoted $\mathbf{GL}(E)$.

When $E = \mathbb{R}^n$, we often denote $\mathbf{GL}(\mathbb{R}^n)$ by $\mathbf{GL}(n, \mathbb{R})$ (and if $E = \mathbb{C}^n$, we often denote $\mathbf{GL}(\mathbb{C}^n)$ by $\mathbf{GL}(n, \mathbb{C})$).

The vector space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ matrices is also denoted by $\mathfrak{gl}(n, \mathbb{R})$ (and $\mathrm{M}_n(\mathbb{C})$ by $\mathfrak{gl}(n, \mathbb{C})$).

Then, $\mathbf{GL}(\mathfrak{gl}(n,\mathbb{R}))$ is the vector space of all invertible linear maps from $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{M}_n(\mathbb{R})$ to itself.

For any matrix $A \in \mathrm{M}_A(\mathbb{R})$ (or $A \in \mathrm{M}_A(\mathbb{C})$), define the maps $L_A\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ and $R_A\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ by

$$L_A(B) = AB, \quad R_A(B) = BA, \quad \text{for all } B \in \mathrm{M}_n(\mathbb{R}).$$

Observe that $L_A \circ R_B = R_B \circ L_A$ for all $A, B \in \mathrm{M}_n(\mathbb{R})$.

For any matrix $A \in \mathbf{GL}(n,\mathbb{R})$, let

$$\mathbf{Ad}_A\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R}) \quad \text{(conjugation by } A)$$

be given by

$$\mathbf{Ad}_A(B) = ABA^{-1} \quad \text{for all } B \in \mathrm{M}_n(\mathbb{R}).$$

Observe that $\mathbf{Ad}_A = L_A \circ R_{A^{-1}}$ and that $\mathbf{Ad}_A$ is an invertible linear map with inverse $\mathbf{Ad}_{A^{-1}}$.

The restriction of $\mathbf{Ad}_A$ to invertible matrices $B \in \mathbf{GL}(n, \mathbb{R})$ yields the map

$$\mathbf{Ad}_A \colon \mathbf{GL}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$$

also given by

$$\mathbf{Ad}_A(B) = ABA^{-1} \quad \text{for all } B \in \mathbf{GL}(n, \mathbb{R}).$$

This time, observe that $\mathbf{Ad}_A$ is a group homomorphism (with respect to multiplication), since

$$\begin{aligned} &\mathbf{Ad}_A(BC) = ABCA^{-1} \\ &= ABA^{-1}ACA^{-1} = \mathbf{Ad}_A(B)\mathbf{Ad}_A(C). \end{aligned}$$

In fact, $\mathbf{Ad}_A$ is a group isomorphism (because its inverse is $\mathbf{Ad}_{A^{-1}}$).

Beware that $\mathbf{Ad}_A$ is **not** a linear map on $\mathbf{GL}(n, \mathbb{R})$ because $\mathbf{GL}(n, \mathbb{R})$ is not a vector space!

However, $\mathbf{GL}(n, \mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, because it is the complement of the set of singular matrices

$$\{A \in \mathrm{M}_n(\mathbb{R}) \mid \det(A) = 0\},$$

a closed set, since it is the inverse image of the closed set $\{0\}$ by the determinant function, which is continuous.

Since $\mathbf{GL}(n, \mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, for every $B \in \mathbf{GL}(n, \mathbb{R})$, there is an open ball $B(B, \eta) \subseteq \mathbf{GL}(n, \mathbb{R})$ such that $B + X \in B(B, \eta)$ for all $X \in \mathrm{M}_n(\mathbb{R})$ with $\|X\| < \eta$, so $\mathbf{Ad}_A(B + X)$ is well defined and

$$\begin{aligned}&\mathbf{Ad}_A(B + X) - \mathbf{Ad}_A(B)\\&= A(B + X)A^{-1} - ABA^{-1} = AXA^{-1},\end{aligned}$$

which shows that $d(\mathbf{Ad}_A)_B$ exists and is given by

$$d(\mathbf{Ad}_A)_B(X) = AXA^{-1}, \quad \text{for all } X \in \mathrm{M}_n(\mathbb{R}).$$

In particular, for $B = I$, we see that the derivative $d(\mathbf{Ad}_A)_I$ of $\mathbf{Ad}_A$ at I is a linear map of $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}_n(\mathbb{R})$ denoted by $\mathrm{Ad}(A)$ or Ad_A (or $\mathrm{Ad}\, A$), and given by

$$\mathrm{Ad}_A(X) = AXA^{-1} \quad \text{for all } X \in \mathfrak{gl}(n, \mathbb{R}).$$

The inverse of Ad_A is $\mathrm{Ad}_{A^{-1}}$, so $\mathrm{Ad}_A \in \mathbf{GL}(\mathfrak{gl}(n, \mathbb{R}))$.

Note that

$$\mathrm{Ad}_{AB} = \mathrm{Ad}_A \circ \mathrm{Ad}_B,$$

so the map $A \mapsto \mathrm{Ad}_A$ is a group homomorphism denoted

$$\mathrm{Ad}\colon \mathbf{GL}(n, \mathbb{R}) \to \mathbf{GL}(\mathfrak{gl}(n, \mathbb{R})).$$

The homomorphism Ad is called the *adjoint representation* of $\mathbf{GL}(n, \mathbb{R})$.

We also would like to compute the derivative $d(\mathrm{Ad})_I$ of Ad at I.

For all $X, Y \in \mathrm{M}_n(\mathbb{R})$, with $\|X\|$ small enough we have $I + X \in \mathbf{GL}(n, \mathbb{R})$, and

$$\begin{aligned}&\mathrm{Ad}_{I+X}(Y) - \mathrm{Ad}_I(Y) - (XY - YX)\\&= (YX^2 - XYX)(I + X)^{-1}.\end{aligned}$$

Then, if we let

$$\epsilon(X, Y) = \frac{(YX^2 - XYX)(I + X)^{-1}}{\|X\|},$$

we proved that for $\|X\|$ small enough

$$\mathrm{Ad}_{I+X}(Y) - \mathrm{Ad}_I(Y) = (XY - YX) + \epsilon(X, Y) \|X\|,$$

with $\|\epsilon(X, Y)\| \leq 2 \|X\| \|Y\| \left\|(I + X)^{-1}\right\|$, and with $\epsilon(X, Y)$ linear in Y.

Let $\mathrm{ad}_X \colon \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ be the linear map given by

$$\mathrm{ad}_X(Y) = XY - YX = [X, Y],$$

and ad be the linear map

$$\mathrm{ad} \colon \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{Hom}(\mathfrak{gl}(n, \mathbb{R}), \mathfrak{gl}(n, \mathbb{R}))$$

given by

$$\mathrm{ad}(X) = \mathrm{ad}_X.$$

We also define $\epsilon_X \colon \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ as the linear map given by

$$\epsilon_X(Y) = \epsilon(X, Y).$$

If $\|\epsilon_X\|$ is the operator norm of ϵ_X, we have

$$\|\epsilon_X\| = \max_{\|Y\|=1} \|\epsilon(X, Y)\| \leq 2 \|X\| \left\|(I + X)^{-1}\right\|.$$

Then, the equation

$$\mathrm{Ad}_{I+X}(Y) - \mathrm{Ad}_I(Y) = (XY - YX) + \epsilon(X,Y)\,\|X\|,$$

which holds for all Y, yields

$$\mathrm{Ad}_{I+X} - \mathrm{Ad}_I = \mathrm{ad}_X + \epsilon_X\,\|X\|,$$

and because $\|\epsilon_X\| \leq 2\,\|X\|\,\left\|(I+X)^{-1}\right\|$, we have $\lim_{X\mapsto 0}\epsilon_X = 0$, which shows that $d(\mathrm{Ad})_I(X) = \mathrm{ad}_X$; that is,

$$d(\mathrm{Ad})_I = \mathrm{ad}.$$

The notation $\mathrm{ad}(X)$ (or $\mathrm{ad}\,X$) is also used instead ad_X.

The map ad is a linear map

$$\mathrm{ad}\colon \mathfrak{gl}(n,\mathbb{R}) \to \mathrm{Hom}(\mathfrak{gl}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}))$$

called the *adjoint representation* of $\mathfrak{gl}(n,\mathbb{R})$.

One will check that

$$\begin{aligned}\mathrm{ad}([X,Y]) &= \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X)\\ &= [\mathrm{ad}(X), \mathrm{ad}(Y)],\end{aligned}$$

the Lie bracket on linear maps on $\mathfrak{gl}(n,\mathbb{R})$.

This means that ad is a Lie algebra homomorphism. It can be checked that this property is equivalent to the following identity known as the *Jacobi identity*:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,$$

for all $X, Y, Z \in \mathfrak{gl}(n, \mathbb{R})$.

Note that

$$\mathrm{ad}_X = L_X - R_X.$$

Finally, we prove a formula relating Ad and ad through the exponential.

Proposition 3.1. *For any $X \in \mathrm{M}_n(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$, we have*

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X} = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}_X)^k}{k!};$$

that is,

$$\begin{aligned} e^X Y e^{-X} &= e^{\mathrm{ad}_X} Y \\ &= Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] \\ &\quad + \cdots \end{aligned}$$

for all $X, Y \in \mathrm{M}_n(\mathbb{R})$

3.2 The Derivative of exp

It is also possible to find a formula for the derivative $d\exp_A$ of the exponential map at A, but this is a bit tricky.

It can be shown that

$$d(\exp)_A = e^A \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (\mathrm{ad}_A)^k,$$

so

$$\begin{aligned} d(\exp)_A(B) = e^A \biggl(B &- \frac{1}{2!}[A, B] + \frac{1}{3!}[A, [A, B]] \\ &- \frac{1}{4!}[A, [A, [A, B]]] + \cdots \biggr). \end{aligned}$$

It is customary to write

$$\frac{\mathrm{id} - e^{-\mathrm{ad}_A}}{\mathrm{ad}_A}$$

for the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(\mathrm{ad}_A)^k,$$

and the formula for the derivative of exp is usually stated as

$$d(\exp)_A = e^A \left(\frac{\mathrm{id} - e^{-\mathrm{ad}_A}}{\mathrm{ad}_A} \right).$$

The formula for the exponential tells us when the derivative $d(\exp)_A$ is invertible.

Indeed, it is easy to see that if the eigenvalues of the matrix X are $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of the matrix

$$\frac{\mathrm{id} - e^{-X}}{X} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} X^k$$

are

$$\frac{1 - e^{-\lambda_j}}{\lambda_j} \quad \text{if } \lambda_j \neq 0, \text{ and } 1 \text{ if } \lambda_j = 0.$$

It follows that the matrix $\frac{\mathrm{id} - e^{-X}}{X}$ is invertible iff no λ_j if of the form $k2\pi i$ for some $k \in \mathbb{Z} - \{0\}$, so $d(\exp)_A$ is invertible iff no eigenvalue of ad_A is of the form $k2\pi i$ for some $k \in \mathbb{Z} - \{0\}$.

However, it can also be shown that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of ad_A are the $\lambda_i - \lambda_j$, with $1 \leq i, j \leq n$.

In conclusion, $d(\exp)_A$ is invertible iff for all i, j we have

$$\lambda_i - \lambda_j \neq k2\pi i, \quad k \in \mathbb{Z} - \{0\}. \tag{$*$}$$

This suggests defining the following subset $\mathcal{E}(n)$ of $\mathrm{M}_n(\mathbb{R})$.

The set $\mathcal{E}(n)$ consists of all matrices $A \in \mathrm{M}_n(\mathbb{R})$ whose eigenvalue $\lambda + i\mu$ of A ($\lambda, \mu \in \mathbb{R}$) lie in the horizontal strip determined by the condition $-\pi < \mu < \pi$.

Then, it is clear that the matrices in $\mathcal{E}(n)$ satisfy the condition $(*)$, so $d(\exp)_A$ is invertible for all $A \in \mathcal{E}(n)$.

By the inverse function theorem, the exponential map is a local diffeomorphism between $\mathcal{E}(n)$ and $\exp(\mathcal{E}(n))$.

Remarkably, more is true: the exponential map is diffeomorphism between $\mathcal{E}(n)$ and $\exp(\mathcal{E}(n))$ (in particular, it is a bijection).

This takes quite a bit of work to be proved. For example, see Mnemné and Testard [40]. We have the following result.

Theorem 3.2. *The restriction of the exponential map to $\mathcal{E}(n)$ is a diffeomorphism of $\mathcal{E}(n)$ onto its image $\exp(\mathcal{E}(n))$. Furthermore, $\exp(\mathcal{E}(n))$ consists of all invertible matrices that have no real negative eigenvalues; it is an open subset of $\mathbf{GL}(n, \mathbb{R})$; it contains the open ball $B(I, 1) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \|A - I\| < 1\}$, for every matrix norm $\| \; \|$ on $n \times n$ matrices.*

Theorem 3.2 has some practical applications because there are algorithms for finding a real log of a matrix with no real negative eigenvalues; for more on applications of Theorem 3.2 to medical imaging, see Chapter **??**.