Chapter 18

Metrics, Connections, and Curvature on Lie Groups

18.1 Left (resp. Right) Invariant Metrics

Since a Lie group G is a smooth manifold, we can endow G with a Riemannian metric.

Among all the Riemannian metrics on a Lie groups, those for which the left translations (or the right translations) are isometries are of particular interest because they take the group structure of G into account.

This chapter makes extensive use of results from a beautiful paper of Milnor [34].

Definition 18.1. A metric $\langle -, - \rangle$ on a Lie group G is called *left-invariant* (resp. *right-invariant*) iff

$$\langle u, v\rangle_b = \langle (dL_a)_b u, (dL_a)_b v\rangle_{ab}$$
$$(\text{resp. } \langle u, v\rangle_b = \langle (dR_a)_b u, (dR_a)_b v\rangle_{ba}),$$

for all $a, b \in G$ and all $u, v \in T_bG$.

A Riemannian metric that is both left and right-invariant is called a *bi-invariant metric*.

In the sequel, the identity element of the Lie group, G, will be denoted by e or 1.

Proposition 18.1. *There is a bijective correspondence between left-invariant (resp. right invariant) metrics on a Lie group G, and inner products on the Lie algebra $\mathfrak{g}$ of G.*

Let $\langle -, - \rangle$ be an inner product on $\mathfrak{g}$, and set

$$\langle u, v \rangle_g = \langle (dL_{g^{-1}})_g u, (dL_{g^{-1}})_g v \rangle,$$

for all $u, v \in T_g G$ and all $g \in G$. It is fairly easy to check that the above induces a left-invariant metric on G.

If G has a left-invariant (resp. right-invariant) metric, since left-invariant (resp. right-invariant) translations are isometries and act transitively on G, the space G is called a *homogeneous Riemannian manifold*.

Proposition 18.2. *Every Lie group G equipped with a left-invariant (resp. right-invariant) metric is complete.*

18.2 Bi-Invariant Metrics

Recall that the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ of the Lie group G is the map defined such that $\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ is the linear isomorphism given by

$$\mathrm{Ad}_a = d(\mathbf{Ad}_A)_e = d(R_{a^{-1}} \circ L_a)_e, \quad \text{for every } a \in G.$$

Clearly,

$$\mathrm{Ad}_a = (dR_{a^{-1}})_a \circ (dL_a)_e.$$

Here is the first of four criteria for the existence of a bi-invariant metric on a Lie group.

Proposition 18.3. *There is a bijective correspondence between bi-invariant metrics on a Lie group G and* Ad*-invariant inner products on the Lie algebra $\mathfrak{g}$ of G, namely inner products $\langle -, - \rangle$ on $\mathfrak{g}$ such that* Ad_a *is an isometry of $\mathfrak{g}$ for all $a \in G$; more explicitly,* Ad-*invariant inner inner products satisfy the condition*

$$\langle \mathrm{Ad}_a u, \mathrm{Ad}_a v \rangle = \langle u, v \rangle,$$

for all $a \in G$ and all $u, v \in \mathfrak{g}$.

Proposition 18.3 shows that if a Lie group G possesses a bi-invariant metric, then every linear map Ad_a is an orthogonal transformation of $\mathfrak{g}$.

It follows that $\mathrm{Ad}(G)$ is a subgroup of the orthogonal group of $\mathfrak{g}$, and so its closure $\overline{\mathrm{Ad}(G)}$ is compact.

It turns out that this condition is also sufficient!

To prove the above fact, we make use of an "averaging trick" used in representation theory.

Recall that a *representation* of a Lie group G is a (smooth) homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where V is some finite-dimensional vector space.

For any $g \in G$ and any $u \in V$, we often write $g \cdot u$ for $\rho(g)(u)$.

We say that an inner-product $\langle -, - \rangle$ on V is *invariant under ρ* (or *G-invariant*) iff

$$\langle \rho(g)(u), \rho(g)(v) \rangle = \langle u, v \rangle, \qquad \begin{array}{l} \text{for all } g \in G \\ \text{and all } u, v \in V. \end{array}$$

If G is compact, then the "averaging trick," also called "Weyl's unitarian trick," yields the following important result:

Theorem 18.4. *If G is a compact Lie group, then for every representation $\rho\colon G \to \mathrm{GL}(V)$, there is a G-invariant inner product on V.*

Using Theorem 18.4, we can prove the following result giving a criterion for the existence of a G-invariant inner product for any representation of a Lie group G (see Sternberg [45], Chapter 5, Theorem 5.2).

Theorem 18.5. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a (finite-dim.) representation of a Lie group G. There is a G-invariant inner product on V iff $\overline{\rho(G)}$ is compact. In particular, if G is compact, then there is a G-invariant inner product on V.*

Applying Theorem 18.5 to the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, we get our second criterion for the existence of a bi-invariant metric on a Lie group.

Proposition 18.6. *Given any Lie group G, an inner product $\langle -, - \rangle$ on $\mathfrak{g}$ induces a bi-invariant metric on G iff $\overline{\mathrm{Ad}(G)}$ is compact. In particular, every compact Lie group has a bi-invariant metric.*

Proposition 18.6 can be used to prove that certain Lie groups do not have a bi-invariant metric.

For example, Arsigny, Pennec and Ayache use Proposition 18.6 to give a short and elegant proof of the fact that $\mathbf{SE}(n)$ does not have any bi-invariant metric for all $n \geq 2$.

Recall the adjoint representation of the Lie algebra $\mathfrak{g}$,

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

given by $\mathrm{ad} = d\mathrm{Ad}_1$. Here is our third criterion for the existence of a bi-invariant metric on a connected Lie group.

Proposition 18.7. *If G is a connected Lie group, an inner product $\langle -, - \rangle$ on $\mathfrak{g}$ induces a bi-invariant metric on G iff the linear map $\mathrm{ad}(u)\colon \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint for all $u \in \mathfrak{g}$, which means that*

$$\langle \mathrm{ad}(u)(v), w \rangle = -\langle v, \mathrm{ad}(u)(w) \rangle, \quad \textit{for all } u, v, w \in \mathfrak{g},$$

or equivalently that

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle, \quad \textit{for all } x, y, z \in \mathfrak{g}.$$

It will be convenient to say that an inner product on $\mathfrak{g}$ is *bi-invariant* iff every $\mathrm{ad}(u)$ is skew-adjoint.

The following variant of Proposition 18.7 will also be needed. This is a special case of Lemma 3 in O'Neill [38] (Chapter 11).

Proposition 18.8. *If G is Lie group equipped with an inner product $\langle -, - \rangle$ on $\mathfrak{g}$ that induces a bi-invariant metric on G, then $\mathrm{ad}(X)\colon \mathfrak{g}^L \to \mathfrak{g}^L$ is skew-adjoint for all left-invariant vector fields $X \in \mathfrak{g}^L$, which means that*

$$\langle \mathrm{ad}(X)(Y), Z \rangle = -\langle Y, \mathrm{ad}(X)(Z) \rangle, \quad \textit{for all } X, Y, Z \in \mathfrak{g}^L,$$

or equivalently that

$$\langle [Y, X], Z \rangle = \langle Y, [X, Z] \rangle, \quad \textit{for all } X, Y, Z \in \mathfrak{g}^L.$$

If G is a connected Lie group, then the existence of a bi-invariant metric on G places a heavy restriction on its group structure, as shown by the following result from Milnor's paper [34] (Lemma 7.5):

Theorem 18.9. *A connected Lie group G admits a bi-invariant metric iff it is isomorphic to the cartesian product of a compact group and a vector space ($\mathbb{R}^m$, for some $m \geq 0$).*

A proof of Theorem 18.9 can be found in Milnor [34] (Lemma 7.4 and Lemma 7.5).

The proof uses the universal covering group and it is a bit involved.

One of the steps uses the following proposition.

Proposition 18.10. *Let $\mathfrak{g}$ be a Lie algebra with an inner product such that the linear map* $\mathrm{ad}(u)$ *is skew-adjoint for every $u \in \mathfrak{g}$. Then the orthogonal complement $\mathfrak{a}^\perp$ of any ideal $\mathfrak{a}$ is itself an ideal. Consequently, $\mathfrak{g}$ can be expressed as an orthogonal direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

where each $\mathfrak{g}_i$ is either a simple ideal or a one-dimensional abelian ideal ($\mathfrak{g}_i \cong \mathbb{R}$).

We now investigate connections and curvature on Lie groups with a left-invariant metric.

18.3 Connections and Curvature of Left-Invariant Metrics on Lie Groups

If G is a Lie group equipped with a left-invariant metric, then it is possible to express the Levi-Civita connection and the sectional curvature in terms of quantities defined over the Lie algebra of G, at least for left-invariant vector fields.

When the metric is bi-invariant, much nicer formulae can be obtained.

If $\langle -, - \rangle$ is a left-invariant metric on G, then for any two left-invariant vector fields X, Y, we can show that the function $g \mapsto \langle X, Y \rangle_g$ is constant.

Therefore, for any vector field Z,

$$Z(\langle X, Y \rangle) = 0.$$

If we go back to the Koszul formula (Proposition 12.8)

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &\quad - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle, \end{aligned}$$

we deduce that for all left-invariant vector fields X, Y, Z, we have

$$2\langle \nabla_X Y, Z\rangle = -\langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle,$$

which can be rewritten as

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle. \tag{$\dagger$}$$

The above yields the formula

$$\nabla_X Y = \frac{1}{2}\left([X, Y] - \mathrm{ad}(X)^* Y - \mathrm{ad}(Y)^* X\right), \; X, Y \in \mathfrak{g}^L,$$

where $\mathrm{ad}(X)^*$ denotes the adjoint of $\mathrm{ad}(X)$, where $\mathrm{ad}X$ is defined just after Proposition 16.11.

Remark: Given any two vector $u, v \in \mathfrak{g}$, it is common practice (even though this is quite confusing) to denote by $\nabla_u v$ the result of evaluating the vector field $\nabla_{u^L} v^L$ at e (so, $\nabla_u v = (\nabla_{u^L} v^L)(e)$).

Following Milnor, if we pick an orthonormal basis $(e_1, \ldots, e_n)$ *w.r.t.* our inner product on $\mathfrak{g}$, and if we define the constants α_{ijk} by

$$\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle,$$

we see that

$$(\nabla_{e_i} e_j)(1) = \frac{1}{2} \sum_k (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k. \qquad (*)$$

Now, for orthonormal vectors u, v, the sectional curvature is given by

$$K(u, v) = \langle R(u, v)u, v \rangle,$$

with

$$R(u, v) = \nabla_{[u,v]} - \nabla_u \nabla_v + \nabla_v \nabla_u.$$

If we plug the expressions from equation $(*)$ into the definitions we obtain the following proposition from Milnor [34] (Lemma 1.1):

Proposition 18.11. *Given a Lie group G equipped with a left-invariant metric, for any orthonormal basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$, and with the structure constants $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$, the sectional curvature $K(e_1, e_2)$ is given by*

$$\begin{aligned} K(e_1, e_2) = \sum_k & \frac{1}{2}(\alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) \\ & - \frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) \\ & - \alpha_{k11}\alpha_{k22}). \end{aligned}$$

Although the above formula is not too useful in general, in some cases of interest, a great deal of cancellation takes place so that a more useful formula can be obtained.

An example of this situation is provided by the next proposition (Milnor [34], Lemma 1.2).

Proposition 18.12. *Given a Lie group G equipped with a left-invariant metric, for any $u \in \mathfrak{g}$, if the linear map* $\mathrm{ad}(u)$ *is self-adjoint, then*

$$K(u, v) \geq 0 \quad \textit{for all } v \in \mathfrak{g},$$

where equality holds iff u is orthogonal to $[v, \mathfrak{g}] = \{[v, x] \mid x \in \mathfrak{g}\}$.

For the next proposition we need the following definition.

Definition 18.2. The *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of all elements $u \in \mathfrak{g}$ such that $[u, v] = 0$ for all $v \in \mathfrak{g}$, or equivalently, such that $\mathrm{ad}(u) = 0$.

Proposition 18.13. *Given a Lie group G equipped with a left-invariant metric, for any u in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$,*

$$K(u, v) \geq 0 \quad \textit{for all } v \in \mathfrak{g}.$$

Recall that the Ricci curvature $\mathrm{Ric}(u, v)$ is the trace of the linear map $y \mapsto R(u, y)v$.

With respect to any orthonormal basis $(e_1, \ldots, e_n)$ of $\mathfrak{g}$, we have

$$\mathrm{Ric}(u, v) = \sum_{j=1}^{n} \langle R(u, e_j)v, e_j \rangle = \sum_{j=1}^{n} R(u, e_j, v, e_j).$$

The Ricci curvature is a symmetric form, so it is completely determined by the quadratic form

$$r(u) = \mathrm{Ric}(u, u) = \sum_{j=1}^{n} R(u, e_j, u, e_j).$$

Definition 18.3. If u is a unit vector, $r(u) = \mathrm{Ric}(u, u)$ is called the *Ricci curvature in the direction u*. If we pick an orthonormal basis such that $e_1 = u$, then

$$r(e_1) = \sum_{i=2}^{n} K(e_1, e_i).$$

For computational purposes it may be more convenient to introduce the *Ricci transformation* $\mathrm{Ric}^{\#}$, defined by

$$\mathrm{Ric}^{\#}(x) = \sum_{i=1}^{n} R(e_i, x)e_i.$$

Proposition 18.14. *The Ricci transformation defined by*

$$\mathrm{Ric}^{\#}(x) = \sum_{i=1}^{n} R(e_i, x)e_i$$

is self-adjoint, and it is also the unique map so that

$$r(x) = \mathrm{Ric}(x, x) = \langle \mathrm{Ric}^{\#}(x), x\rangle, \qquad \text{for all } x \in \mathfrak{g}.$$

Definition 18.4. The eigenvalues of $\mathrm{Ric}^{\#}$ are called the *principal Ricci curvatures*.

Proposition 18.15. *Given a Lie group G equipped with a left-invariant metric, if the linear map* $\mathrm{ad}(u)$ *is skew-adjoint, then $r(u) \geq 0$, where equality holds iff u is orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$.*

In particular, if u is in the center of $\mathfrak{g}$, then $r(u) \geq 0$.

As a corollary of Proposition 18.15, we have the following result which is used in the proof of Theorem 18.9:

Proposition 18.16. *If G is a connected Lie group equipped with a bi-invariant metric and if the Lie algebra of G is simple, then there is a constant $c > 0$ so that $r(u) \geq c$ for all unit vector $u \in T_g G$ and for all $g \in G$.*

By Myers' Theorem (Theorem 14.27), the Lie group G is compact and has a finite fundamental group.

The following interesting theorem is proved in Milnor (Milnor [34], Theorem 2.2):

Theorem 18.17. *A connected Lie group G admits a left-invariant metric with $r(u) > 0$ for all unit vectors $u \in \mathfrak{g}$ (all Ricci curvatures are strictly positive) iff G is compact and has a finite fundamental group.*

The following criterion for obtaining a direction of negative curvature is also proved in Milnor (Milnor [34], Lemma 2.3):

Proposition 18.18. *Given a Lie group G equipped with a left-invariant metric, if u is orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, then $r(u) \leq 0$, where equality holds iff $\mathrm{ad}(u)$ is self-adjoint.*

18.4 Connections and Curvature of Bi-Invariant Metrics on Lie Groups

When G possesses a bi-invariant metric, much nicer formulae are obtained.

First of all, since by Proposition 18.8,

$$\langle [Y, Z], X\rangle = \langle Y, [Z, X]\rangle,$$

the last two terms in equation (†), namely

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle,$$

cancel out, and we get

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad \text{for all } X, Y \in \mathfrak{g}^L.$$

Proposition 18.19. *For any Lie group G equipped with a bi-invariant metric, the following properties hold:*

(a) The connection $\nabla_X Y$ is given by

$$\nabla_X Y = \frac{1}{2}\,[X, Y], \qquad \text{for all } X, Y \in \mathfrak{g}^L.$$

(b) The curvature tensor $R(u, v)$ is given by

$$R(u, v) = \frac{1}{4}\,\mathrm{ad}[u, v], \qquad \text{for all } u, v \in \mathfrak{g},$$

or equivalently,

$$R(u, v)w = \frac{1}{4}\,[[u, v], w], \qquad \text{for all } u, v, w \in \mathfrak{g}.$$

(c) The sectional curvature $K(u, v)$ is given by

$$K(u, v) = \frac{1}{4}\,\langle [u, v], [u, v]\rangle,$$

for all pairs of orthonormal vectors $u, v \in \mathfrak{g}$.

(d) The Ricci curvature $\mathrm{Ric}(u, v)$ *is given by*

$$\mathrm{Ric}(u, v) = -\frac{1}{4}\, B(u, v), \qquad \text{for all } u, v \in \mathfrak{g},$$

where B *is the Killing form, with*

$$B(u, v) = \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v)), \qquad \text{for all } u, v \in \mathfrak{g}.$$

Consequently, $K(u, v) \geq 0$*, with equality iff* $[u, v] = 0$ *and* $r(u) \geq 0$*, with equality iff* u *belongs to the center of* $\mathfrak{g}$.

Remark: Proposition 18.19 shows that if a Lie group admits a bi-invariant metric, then its Killing form is negative semi-definite.

What are the geodesics in a Lie group equipped with a bi-invariant metric?

The answer is simple: they are the integral curves of left-invariant vector fields.

Proposition 18.20. *For any Lie group G equipped with a bi-invariant metric, we have:*

(1) The inversion map $\iota \colon g \mapsto g^{-1}$ is an isometry.

(2) For every $a \in G$, if I_a denotes the map given by

$$I_a(b) = ab^{-1}a, \quad \textit{for all } a, b \in G,$$

then I_a is an isometry fixing a which reverses geodesics; that is, for every geodesic γ through a, we have

$$I_a(\gamma)(t) = \gamma(-t).$$

(3) The geodesics through e are the integral curves $t \mapsto \exp(tu)$, where $u \in \mathfrak{g}$; that is, the one-parameter groups. Consequently, the Lie group exponential map $\exp \colon \mathfrak{g} \to G$ coincides with the Riemannian exponential map (at e) from T_eG to G, where G is viewed as a Riemannian manifold.

Remarks:

(1) As $R_g = \iota \circ L_{g^{-1}} \circ \iota$, we deduce that if G has a left-invariant metric, then this metric is also right-invariant iff ι is an isometry.

(2) Property (2) of Proposition 18.20 says that a Lie group with a bi-invariant metric is a *symmetric space*, an important class of Riemannian spaces invented and studied extensively by Elie Cartan. Symmetric spaces are briefly discussed in Section 20.8.

(3) The proof of 18.20 (3) given in O'Neill [38] (Chapter 11, equivalence of (5) and (6) in Proposition 9) appears to be missing the "hard direction," namely, that a geodesic is a one-parameter group.

Many more interesting results about left-invariant metrics on Lie groups can be found in Milnor's paper [34].

We conclude this section by stating the following proposition (Milnor [34], Lemma 7.6):

Proposition 18.21. *If G is any compact, simple, Lie group, then the bi-invariant metric is unique up to a constant. Such a metric necessarily has constant Ricci curvature.*

18.5 Simple and Semisimple Lie Algebras and Lie Groups

In this section, we introduce semisimple Lie algebras.

They play a major role in the structure theory of Lie groups, but we only scratch the surface.

Definition 18.5. A subset $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* iff it is a subspace of $\mathfrak{g}$ (as a vector space) and if it is closed under the bracket operation on $\mathfrak{g}$.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *abelian* iff $[x, y] = 0$ for all $x, y \in \mathfrak{h}$.

An *ideal* in $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h}$ such that

$$[h, g] \in \mathfrak{h}, \qquad \text{for all } h \in \mathfrak{h} \text{ and all } g \in \mathfrak{g}.$$

The *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of all elements $u \in \mathfrak{g}$ such that $[u, v] = 0$ for all $v \in \mathfrak{g}$, or equivalently, such that $\mathrm{ad}(u) = 0$.

Definition 18.6. A Lie algebra $\mathfrak{g}$ is *simple* iff it is nonabelian and if it has **no** ideal other than (0) and $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is *semisimple* iff it has **no abelian ideal other than** (0). A Lie group is *simple* (resp. *semisimple*) iff its Lie algebra is simple (resp. semisimple).

Note that by definition, simple and semisimple Lie algebras are *nonabelian*, and a simple algebra is a semisimple algebra.

Clearly, the trivial subalgebras (0) and $\mathfrak{g}$ itself are ideals, and the center of a Lie algebra is an abelian ideal.

It follows that the center $Z(\mathfrak{g})$ of a semisimple Lie algebra must be the trivial ideal (0).

Definition 18.7. Given two subsets $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, we let $[\mathfrak{a}, \mathfrak{b}]$ be the subspace of $\mathfrak{g}$ consisting of all linear combinations $[a, b]$, with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$, are also ideals (for $[\mathfrak{a}, \mathfrak{b}]$, use the Jacobi identity).

The last fact allows us to make the following definition.

Definition 18.8. Let $\mathfrak{g}$ be a Lie algebra. The ideal $[\mathfrak{g}, \mathfrak{g}]$ is called the *commutator ideal* of $\mathfrak{g}$. The commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ is also denoted by $D^1\mathfrak{g}$ (or $D\mathfrak{g}$).

If $\mathfrak{g}$ is a simple Lie agebra, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, so the simplicity of $\mathfrak{g}$ implies that either $[\mathfrak{g}, \mathfrak{g}] = (0)$ or $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. However, if $[\mathfrak{g}, \mathfrak{g}] = (0)$, then $\mathfrak{g}$ is abelian, a contradiction).

Definition 18.9. The *derived series* (or *commutator series*) $(D^k\mathfrak{g})$ of a Lie algebra (or ideal) $\mathfrak{g}$ is defined as follows:

$$\begin{aligned} D^0\mathfrak{g} &= \mathfrak{g} \\ D^{k+1}\mathfrak{g} &= [D^k\mathfrak{g}, D^k\mathfrak{g}], \quad k \geq 0. \end{aligned}$$

We have a decreasing sequence

$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \cdots .$$

If $\mathfrak{g}$ is an ideal, by induction we see that each $D^k\mathfrak{g}$ is an ideal.

Definition 18.10. We say that a Lie algebra $\mathfrak{g}$ is *solvable* iff $D^k\mathfrak{g} = (0)$ for some $k \geq 0$.

If $\mathfrak{g}$ is abelian, then $[\mathfrak{g}, \mathfrak{g}] = 0$, so $\mathfrak{g}$ is solvable.

Observe that a nonzero solvable Lie algebra has a nonzero abelian ideal, namely, the last nonzero $D^j\mathfrak{g}$.

As a consequence, *a Lie algebra is semisimple iff it has no nonzero solvable ideal*.

It can be shown that every Lie algebra $\mathfrak{g}$ has a largest solvable ideal $\mathfrak{r}$, called the *radical* of $\mathfrak{g}$.

Definition 18.11. The *radical* of a Lie algebra $\mathfrak{g}$ is its largest solvable ideal, and it is denoted $\mathrm{rad}\,\mathfrak{g}$.

Then a Lie algebra is semisimple iff $\mathrm{rad}\,\mathfrak{g} = (0)$.

It can also be shown that for every (finite-dimensional) Lie algebra $\mathfrak{g}$, there is some semisimple Lie algebra $\mathfrak{s}$ such that $\mathfrak{g}$ is a semidirect product

$$\mathfrak{g} = \mathrm{rad}\,\mathfrak{g} \oplus_{\tau} \mathfrak{s}.$$

The above is called a *Levi decomposition*; see Knapp [25] (Appendix B), Serre [44] (Chapter VI, Theorem 4.1 and Corollary 1), and Fulton and Harris [17] (Appendix E).

The Levi decomposition shows the importance of semisimple and solvable Lie algebras: the structure of these algebras determines the structure of arbitrary Lie algebras.

Definition 18.12. The *lower central series* $(C^k\mathfrak{g})$ of a Lie algebra (or ideal) $\mathfrak{g}$ is defined as follows:

$$\begin{aligned} C^0\mathfrak{g} &= \mathfrak{g} \\ C^{k+1}\mathfrak{g} &= [\mathfrak{g}, C^k\mathfrak{g}], \quad k \geq 0. \end{aligned}$$

We have a decreasing sequence

$$\mathfrak{g} = C^0\mathfrak{g} \supseteq C^1\mathfrak{g} \supseteq C^2\mathfrak{g} \supseteq \cdots .$$

Definition 18.13. We say that an ideal $\mathfrak{g}$ is *nilpotent* iff $C^k\mathfrak{g} = (0)$ for some $k \geq 0$.

By induction, it is easy to show that

$$D^k\mathfrak{g} \subseteq C^k\mathfrak{g} \quad k \geq 0.$$

Consequently, we have:

Proposition 18.22. *Every nilpotent Lie algebra is solvable.*

It turns out that a Lie algebra $\mathfrak{g}$ is semisimple iff it can be expressed as a direct sum of ideals $\mathfrak{g}_i$, with each $\mathfrak{g}_i$ a simple algebra (see Knapp [25], Chapter I, Theorem 1.54).

As a conseqence, if $\mathfrak{g}$ is semisimple, then we also have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

If we drop the requirement that a simple Lie algebra be non-abelian, thereby allowing one dimensional Lie algebras to be simple, we run into the trouble that a simple Lie algebra is no longer semisimple, and the above theorem fails for this stupid reason.

Thus, it seems technically advantageous to require that simple Lie algebras be non-abelian.

Nevertheless, in certain situations, it is desirable to drop the requirement that a simple Lie algebra be non-abelian and this is what Milnor does in his paper because it is more convenient for one of his proofs. This is a minor point but it could be confusing for uninitiated readers.

18.6 The Killing Form

The Killing form showed the tip of its nose in Proposition 18.19.

It is an important concept and, in this section, we establish some of its main properties.

Definition 18.14. For any Lie algebra $\mathfrak{g}$ over the field K (where $K = \mathbb{R}$ or $K = \mathbb{C}$), the *Killing form B of $\mathfrak{g}$* is the symmetric K-bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by

$$B(u, v) = \mathrm{tr}(\mathrm{ad}(u) \circ \mathrm{ad}(v)), \qquad \text{for all } u, v \in \mathfrak{g}.$$

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, we also refer to B as the *Killing form of G*.

Remark: According to the experts (see Knapp [25], page 754) the *Killing form* as above, was not defined by Killing, and is closer to a variant due to Elie Cartan.

On the other hand, the notion of "Cartan matrix" is due to Wilhelm Killing!

For example, consider the group $\mathbf{SU}(2)$. Its Lie algebra $\mathfrak{su}(2)$ is the three-dimensional Lie algebra consisting of all skew-Hermitian 2×2 matrices with zero trace; that is, matrices of the form

$$\begin{pmatrix} ai & b+ic \\ -b+ic & -ai \end{pmatrix}, \qquad a, b, c \in \mathbb{R}.$$

By picking a suitable basis of $\mathfrak{su}(2)$, it can be shown that

$$B(X, Y) = 4\mathrm{tr}(XY).$$

Now, if we consider the group $\mathbf{U}(2)$, its Lie algebra $\mathfrak{u}(2)$ is the four-dimensional Lie algebra consisting of all skew-Hermitian 2×2 matrices; that is, matrices of the form

$$\begin{pmatrix} ai & b + ic \\ -b + ic & id \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R},$$

This time, it can be shown that

$$B(X, Y) = 4\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y).$$

For $\mathbf{SO}(3)$, we know that $\mathfrak{so}(3) = \mathfrak{su}(2)$, and we get

$$B(X, Y) = \mathrm{tr}(XY).$$

Actually, the following proposition can be shown.

Proposition 18.23. *The following identities hold:*

$$\begin{aligned}
\mathbf{GL}(n,\mathbb{R}), \mathbf{U}(n)\colon &\quad B(X,Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y)\\
\mathbf{SL}(n,\mathbb{R}), \mathbf{SU}(n)\colon &\quad B(X,Y) = 2n\mathrm{tr}(XY)\\
\mathbf{SO}(n)\colon &\quad B(X,Y) = (n-2)\mathrm{tr}(XY).
\end{aligned}$$

To prove Proposition 18.23, it suffices to compute the quadratic form $B(X,X)$, because $B(X,Y)$ is symmetric bilinear so it can be recovered using the polarization identity

$$B(X,Y) = \frac{1}{2}(B(X+Y,X+Y) - B(X,X) - B(Y,Y)).$$

Furthermore, if $\mathfrak{g}$ is the Lie algebra of a matrix group, since $\mathrm{ad}_X = L_X - R_X$ and L_X and R_X commute, for all $X, Z \in \mathfrak{g}$, we have

$$\begin{aligned}(\mathrm{ad}_X \circ \mathrm{ad}_X)(Z) &= (L_X^2 - 2L_X \circ R_X + R_X^2)(Z)\\ &= X^2 Z - 2XZX + ZX^2.\end{aligned}$$

Therefore, to compute $B(X, X) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_X)$, we can pick a convenient basis of $\mathfrak{g}$ and compute the diagonal entries of the matrix representing the linear map

$$Z \mapsto X^2 Z - 2XZX + ZX^2.$$

Unfortunately, this is usually quite laborious.

Recall that a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map that preserves brackets; that is,

$$\varphi([u, v]) = [\varphi(u), \varphi(v)].$$

Proposition 18.24. *The Killing form B of a Lie algebra $\mathfrak{g}$ has the following properties:*

(1) It is a symmetric bilinear form invariant under all automorphisms of $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then B is Ad_g-invariant, for all $g \in G$.

(2) The linear map $\mathrm{ad}(u)$ is skew-adjoint w.r.t B for all $u \in \mathfrak{g}$; that is,

$$B(\mathrm{ad}(u)(v), w) = -B(v, \mathrm{ad}(u)(w)), \quad \text{for all } u, v, w \in \mathfrak{g},$$

or equivalently,

$$B([u, v], w) = B(u, [v, w]), \quad \text{for all } u, v, w \in \mathfrak{g}.$$

Remarkably, the Killing form yields a simple criterion due to Elie Cartan for testing whether a Lie algebra is semisimple.

Theorem 18.25. *(Cartan's Criterion for Semisimplicity) A lie algebra $\mathfrak{g}$ is semisimple iff its Killing form B is non-degenerate.*

As far as we know, all the known proofs of Cartan's criterion are quite involved.

A fairly easy going proof can be found in Knapp [25] (Chapter 1, Theorem 1.45).

A more concise proof is given in Serre [44] (Chapter VI, Theorem 2.1).

Since a Lie group with trivial Lie algebra is discrete, this implies that the center of a simple Lie group is discrete (because the Lie algebra of the center of a Lie group is the center of its Lie algebra. Prove it!).

We can also characterize which Lie groups have a Killing form which is negative definite.

Theorem 18.26. *A connected Lie group is compact and semisimple iff its Killing form is negative definite.*

Remark: A compact semisimple Lie group equipped with $-B$ as a metric is an Einstein manifold, since Ric is proportional to the metric (see Definition 14.7).

Using Theorem 18.26, since the Killing forms for $\mathbf{U}(n)$, $\mathbf{SU}(n)$ and $\mathbf{S})(n)$ are given by

$$\begin{aligned} \mathbf{GL}(n, \mathbb{R}), \mathbf{U}(n)\colon\ & B(X,Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y) \\ \mathbf{SL}(n, \mathbb{R}), \mathbf{SU}(n)\colon\ & B(X,Y) = 2n\mathrm{tr}(XY) \\ \mathbf{SO}(n)\colon\ & B(X,Y) = (n-2)\mathrm{tr}(XY), \end{aligned}$$

we obtain the following result:

Proposition 18.27. *The Lie group $\mathbf{SU}(n)$ is compact and semisimple for $n \geq 2$, $\mathbf{SO}(n)$ is compact and semisimple for $n \geq 3$, and $\mathbf{SL}(n, \mathbb{R})$ is noncompact and semisimple for $n \geq 2$. However, $\mathbf{U}(n)$, even though it is compact, is not semisimple.*

Another way to determine whether a Lie algebra is semisimple is to consider reductive Lie algebras.

We give a quick exposition without proofs. Details can be found in Knapp [25] (Chapter I, Sections, 7, 8).

Definition 18.15. A Lie algebra $\mathfrak{g}$ is *reductive* iff for every ideal $\mathfrak{a}$ in $\mathfrak{g}$, there is some ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}.$$

The following result is proved in Knapp [25] (Chapter I, Corollary 1.56).

Proposition 18.28. *If $\mathfrak{g}$ is a reductive Lie algebra, then*

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g}),$$

with $[\mathfrak{g}, \mathfrak{g}]$ semisimple and $Z(\mathfrak{g})$ abelian.

Consequently, if $\mathfrak{g}$ is reductive, then it is semisimple iff its center $Z(\mathfrak{g})$ is trivial.

For Lie algebras of matrices, a simple condition implies that a Lie algera is reductive.

The following result is proved in Knapp [25] (Chapter I, Proposition 1.59).

Proposition 18.29. *If $\mathfrak{g}$ is a real Lie algebra of matrices over $\mathbb{R}$ or $\mathbb{C}$, and if $\mathfrak{g}$ is closed under conjugate transpose (that is, if $A \in \mathfrak{g}$, then $A^* \in \mathfrak{g}$), then $\mathfrak{g}$ is reductive.*

The familiar Lie algebras $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n)$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$, $\mathfrak{so}(p, q)$, $\mathfrak{u}(p, q)$, $\mathfrak{su}(p, q)$ are all closed under conjugate transpose.

Among those, by computing their center, we find that $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{C})$ are semisimple for $n \geq 2$, $\mathfrak{so}(n)$, $\mathfrak{so}(n, \mathbb{C})$ are semisimple for $n \geq 3$, $\mathfrak{su}(n)$ is semisimple for $n \geq 2$, $\mathfrak{so}(p, q)$ is semisimple for $p + q \geq 3$, and $\mathfrak{su}(p, q)$ is semisimple for $p + q \geq 2$.

Semisimple Lie algebras and semisimple Lie groups have been investigated extensively, starting with the complete classification of the complex semisimple Lie algebras by Killing (1888) and corrected by Elie Cartan in his thesis (1894).

One should read the Notes, especially on Chapter II, at the end of Knapp's book [25] for a fascinating account of the history of the theory of semisimple Lie algebras.

The theories and the body of results that emerged from these investigations play a very important role, not only in mathematics, but also in physics, and constitute one of the most beautiful chapters of mathematics.

18.7 Left-Invariant Connections and Cartan Connections

Unfortunately, if a Lie group G does not admit a bi-invariant metric, under the Levi-Civita connection, geodesics are generally not given by the exponential map $\exp\colon \mathfrak{g} \to G$.

If we are willing to consider connections not induced by a metric, then it turns out that there is a fairly natural connection for which the geodesics coincide with integral curves of left-invariant vector fields.

These connections are called *Cartan connections*.

Such connections are torsion-free (symmetric), but *the price that we pay is that in general they are not compatible with the chosen metric*.

As a consequence, even though geodesics exist for all $t \in \mathbb{R}$, it is generally false that any two points can connected by a geodesic.

This has to do with the failure of the exponential to be surjective.

This section is heavily inspired by Postnikov [40] (Chapter 6, Sections 3–6); see also Kobayashi and Nomizu [26] (Chapter X, Section 2).

Recall that a vector field X on a Lie group G is left-invariant if the following diagram commutes for all $a \in G$:

$$\begin{tikzcd}
TG \arrow[r, "d(L_a)"] & TG \\
G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u, "X"']
\end{tikzcd}$$

In this section, we use freely the fact that there is a bijection between the Lie algebra $\mathfrak{g}$ and the Lie algebra $\mathfrak{g}^L$ of left-invariant vector fields on G.

For every $X \in \mathfrak{g}$, we denote by $X^L \in \mathfrak{g}^L$ the unique left-invariant vector field such that $X^L_1 = X$.

Definition 18.16. A connection ∇ on a Lie group G is *left-invariant* if for any two left-invariant vector fields X^L, Y^L with $X, Y \in \mathfrak{g}$, the vector field $\nabla_{X^L} Y^L$ is also left-invariant.

By analogy with left-invariant metrics, there is a version of Proposition 18.1 stating that there is a one-to-one correspondence between left-invariant connections and bilinear forms $\alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proposition 18.30. *There is a one-to-one correspondence between left-invariant connections on G and bilinear forms on $\mathfrak{g}$.*

Given a left-invariant connection ∇ on G, we get the map $\alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by

$$\alpha(X, Y) = (\nabla_{X^L} Y^L)_1, \quad X, Y \in \mathfrak{g}.$$

We can also show that every bilinear map $\alpha\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defines a unique left-invariant connection (we use a basis of $\mathfrak{g}$).

Every bilinear form α can be written as the sum of a symmetric bilinear form

$$\alpha_H(X,Y) = \frac{\alpha(X,Y) + \alpha(Y,X)}{2}$$

and a skew-symmetric bilinear form

$$\alpha_S(X,Y) = \frac{\alpha(X,Y) - \alpha(Y,X)}{2}.$$

Proposition 18.31. *The left-invariant connection ∇ induced by a bilinear map α on $\mathfrak{g}$ has the property that, for every $X \in \mathfrak{g}$, the curve $t \mapsto \exp_{\mathrm{gr}}(tX) = e^{tX}$ is a geodesic iff α is skew-symmetric.*

Definition 18.17. A left-invariant connection satisfying the property that for every $X \in \mathfrak{g}$, the curve $t \mapsto e^{tX}$ is a geodesic, is called a *Cartan connection*.

It is easy to find out when the Cartan connection ∇ associated with a bilinear map α on $\mathfrak{g}$ is torsion-free (symmetric).

Proposition 18.32. *The Cartan connection ∇ associated with a bilinear map α on $\mathfrak{g}$ is torsion-free (symmetric) iff*

$$\alpha_S(X, Y) = \frac{1}{2}[X, Y], \quad \textit{for all } X, Y \in \mathfrak{g},$$

This implies the following result.

Proposition 18.33. *Given any Lie group G, there is a unique (torsion-free) symmetric Cartan connection ∇ given by*

$$\nabla_{X^L} Y^L = \frac{1}{2}[X, Y]^L, \quad \textit{for all } X, Y \in \mathfrak{g}.$$

Then, the same calculation that we used in the case of a bi-invariant metric on a Lie group shows that the curvature tensor is given by

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z], \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

Proposition 18.34. *For any* $X \in \mathfrak{g}$ *and any point* $a \in G$*, the unique geodesic* $\gamma_{a,X}$ *such that* $\gamma_{a,X}(0) = a$ *and* $\gamma'_{a,X}(0) = X$*, is given by*

$$\gamma_{a,X}(t) = e^{td(R_{a^{-1}})_a X} a;$$

that is,

$$\gamma_{a,X} = R_a \circ \gamma_{d(R_{a^{-1}})_a X},$$

where $\gamma_{d(R_{a^{-1}})_a X}(t) = e^{td(R_{a^{-1}})_a X}$.

Remark: Observe that the bilinear forms given by

$$\alpha(X, Y) = \lambda[X, Y] \quad \text{for some } \lambda \in \mathbb{R}$$

are skew-symmetric, and thus induce Cartan connections.

Easy computations show that the torsion is given by

$$T(X,Y) = (2\lambda - 1)[X,Y],$$

and the curvature by

$$R(X,Y)Z = \lambda(1-\lambda)[[X,Y],Z].$$

It follows that for $\lambda = 0$ and $\lambda = 1$, we get connections where the curvature vanishes.

However, these connections have torsion. Again, we see that $\lambda = 1/2$ is the only value for which the Cartan connection is symmetric.

In the case of a bi-invariant metric, the Levi-Civita connection coincides with the Cartan connection.