Chapter 16

Lie Groups, Lie Algebras and the Exponential Map

16.1 Lie Groups and Lie Algebras

Now that we have the general concept of a manifold, we can define Lie groups in more generality.

If every Lie group was a linear group (a group of matrices), then there would be no need for a more general definition.

However, there are Lie groups that are not matrix groups, although it is not a trivial task to exhibit such groups and to prove that they are not matrix groups.

An example of a Lie group which is not a matrix group is the quotient group $G = H/N$, where H (the Heisenberg group) is the group of 3×3 upper triangular matrices given by

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\},$$

and N is the discrete group

$$N = \left\{ \begin{pmatrix} 1 & 0 & k2\pi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

Both groups H and N are matrix groups, N is closed and normal, yet $G = H/N$ is a Lie group and it can be shown using some representation theory that G is not a matrix group

Another example of a Lie group that is not a matrix group is obtained by considering the universal cover $\widetilde{\mathbf{SL}(n,\mathbb{R})}$ of $\mathbf{SL}(n,\mathbb{R})$ for $n \geq 2$.

The group $\mathbf{SL}(n,\mathbb{R})$ is a matrix group which is not simply connected for $n \geq 2$, and its universal cover $\widetilde{\mathbf{SL}(n,\mathbb{R})}$ is a Lie group which is not a matrix group.

Definition 16.1. A *Lie group* is a nonempty subset, G, satisfying the following conditions:

(a) G is a group (with identity element denoted e or 1).

(b) G is a smooth manifold.

(c) G is a topological group. In particular, the group operation, $\cdot : G \times G \to G$, and the inverse map, ${}^{-1}\colon G \to G$, are smooth.

Remark: The smoothness of inversion follows automatically from the smoothness of multiplication. This can be shown by applying the inverse function theorem to the map $(g, h) \mapsto (g, gh)$, from $G \times G$ to $G \times G$.

We have already met a number of Lie groups: $\mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$, $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{SL}(n, \mathbb{C})$, $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{SE}(n, \mathbb{R})$.

Also, every linear Lie group (i.e., a closed subgroup of $\mathbf{GL}(n, \mathbb{C})$) is a Lie group.

We saw in the case of linear Lie groups that the tangent space to G at the identity, $\mathfrak{g} = T_1 G$, plays a very important role. This is again true in this more general setting.

Definition 16.2. A *(real) Lie algebra*, $\mathcal{A}$, is a real vector space together with a bilinear map, $[\cdot, \cdot]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the *Lie bracket* on $\mathcal{A}$ such that the following two identities hold for all $a, b, c \in \mathcal{A}$:

$$[a,\, a] = 0,$$

and the so-called *Jacobi identity*

$$[a,\, [b,\, c]] + [c,\, [a,\, b]] + [b,\, [c,\, a]] = 0.$$

It is immediately verified that $[b, a] = -[a, b]$.

For every $a \in \mathcal{A}$, it is customary to define the linear map $\mathrm{ad}(a)\colon \mathcal{A} \to \mathcal{A}$ by

$$\mathrm{ad}(a)(b) = [a, b], \quad b \in \mathcal{A}.$$

The map $\mathrm{ad}(a)$ is also denoted ad_a or $\mathrm{ad}\, a$.

Let us also recall the definition of homomorphisms of Lie groups and Lie algebras.

Definition 16.3. Given two Lie groups G_1 and G_2, a *homomorphism (or map) of Lie groups* is a function, $f\colon G_1 \to G_2$, that is a homomorphism of groups and a smooth map (between the manifolds G_1 and G_2). Given two Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *homomorphism (or map) of Lie algebras* is a function, $f\colon \mathcal{A}_1 \to \mathcal{A}_2$, that is a linear map between the vector spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ and that preserves Lie brackets, i.e.,

$$f([A, B]) = [f(A), f(B)]$$

for all $A, B \in \mathcal{A}_1$.

An *isomorphism of Lie groups* is a bijective function f such that both f and f^{-1} are maps of Lie groups, and an *isomorphism of Lie algebras* is a bijective function f such that both f and f^{-1} are maps of Lie algebras.

The Lie bracket operation on $\mathfrak{g}$ can be defined in terms of the so-called adjoint representation.

Given a Lie group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Because multiplication and the inverse maps are smooth, the maps L_a and R_a are diffeomorphisms, and their derivatives play an important role.

The inner automorphisms $R_{a^{-1}} \circ L_a$ (also written $R_{a^{-1}}L_a$ or $\mathbf{Ad}_a$) also play an important role. Note that

$$\mathbf{Ad}_a(b) = R_{a^{-1}}L_a(b) = aba^{-1}.$$

The derivative

$$d(\mathbf{Ad}_a)_1 \colon T_1G \to T_1G$$

of $\mathbf{Ad}_a \colon G \to G$ at 1 is an isomorphism of Lie algebras, denoted by $\mathrm{Ad}_a \colon \mathfrak{g} \to \mathfrak{g}$.

The map $\mathrm{Ad} \colon G \to \mathbf{GL}(\mathfrak{g})$ given by $a \mapsto \mathrm{Ad}_a$ is a group homomorphism from G to $\mathbf{GL}(\mathfrak{g})$. Furthermore, this map is smooth.

Proposition 16.1. *The map* $\mathrm{Ad} \colon G \to \mathbf{GL}(\mathfrak{g})$ *is smooth. Thus it is a Lie group homomorphism.*

Definition 16.4. The map $a \mapsto \mathrm{Ad}_a$ is a map of Lie groups

$$\mathrm{Ad} \colon G \to \mathbf{GL}(\mathfrak{g}),$$

called the *adjoint representation of G* (where $\mathbf{GL}(\mathfrak{g})$ denotes the Lie group of all bijective linear maps on $\mathfrak{g}$).

In the case of a Lie linear group, we have verified earlier that

$$\mathrm{Ad}(a)(X) = \mathrm{Ad}_a(X) = aXa^{-1}$$

for all $a \in G$ and all $X \in \mathfrak{g}$.

Since $\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$ is smooth, its derivative $d\mathrm{Ad}_1\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ exists.

Definition 16.5. The derivative

$$d\mathrm{Ad}_1\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

of $\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$ at 1 is map of Lie algebras, denoted by

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

called the *adjoint representation of $\mathfrak{g}$*.

In the case of a linear group, we showed that

$$\mathrm{ad}(A)(B) = [A,\ B]$$

for all $A, B \in \mathfrak{g}$.

One can also check (in general) that the Jacobi identity on $\mathfrak{g}$ is equivalent to the fact that ad preserves Lie brackets, i.e., ad is a map of Lie algebras:

$$\mathrm{ad}([u,\, v]) = [\mathrm{ad}(u),\, \mathrm{ad}(v)],$$

for all $u, v \in \mathfrak{g}$ (where on the right, the Lie bracket is the commutator of linear maps on $\mathfrak{g}$).

In the case of an abstract Lie group G, since ad is defined, we would like to define the Lie bracket of $\mathfrak{g}$ in terms of ad.

This is the key to the definition of the Lie bracket in the case of a general Lie group (not just a linear Lie group).

Definition 16.6. Given a Lie group, G, the tangent space, $\mathfrak{g} = T_1 G$, at the identity with the Lie bracket defined by

$$[u, v] = \mathrm{ad}(u)(v), \quad \text{for all } u, v \in \mathfrak{g},$$

is the *Lie algebra of the Lie group G*.

Actually, we have to justify why $\mathfrak{g}$ really is a Lie algebra. For this, we have

Proposition 16.2. *Given a Lie group, G, the Lie bracket, $[u, v] = \mathrm{ad}(u)(v)$, of Definition 16.6 satisfies the axioms of a Lie algebra (given in Definition 16.2). Therefore, $\mathfrak{g}$ with this bracket is a Lie algebra.*

Remark: After proving that $\mathfrak{g}$ is isomorphic to the vector space of left-invariant vector fields on G, we get another proof of Proposition 16.2.

16.2 Left and Right Invariant Vector Fields, the Exponential Map

A fairly convenient way to define the exponential map is to use left-invariant vector fields.

Definition 16.7. If G is a Lie group, a vector field, X, on G is *left-invariant* (resp. *right-invariant*) iff

$$d(L_a)_b(X(b)) = X(L_a(b)) = X(ab), \quad \text{for all } a, b \in G.$$

(resp.

$$d(R_a)_b(X(b)) = X(R_a(b)) = X(ba), \quad \text{for all } a, b \in G.)$$

Equivalently, a vector field, X, is left-invariant iff the following diagram commutes (and similarly for a right-invariant vector field):

$$\begin{tikzcd}
TG \arrow[r, "d(L_a)"] & TG \\
G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u, "X"']
\end{tikzcd}$$

If X is a left-invariant vector field, setting $b = 1$, we see that

$$X(a) = d(L_a)_1(X(1)),$$

which shows that X is determined by its value, $X(1) \in \mathfrak{g}$, at the identity (and similarly for right-invariant vector fields).

Conversely, given any $v \in \mathfrak{g}$, we can define the vector field, v^L, by

$$v^L(a) = d(L_a)_1(v), \quad \text{for all } a \in G.$$

We claim that v^L is left-invariant. This follows by an easy application of the chain rule:

$$\begin{aligned} v^L(ab) &= d(L_{ab})_1(v) \\ &= d(L_a \circ L_b)_1(v) \\ &= d(L_a)_b(d(L_b)_1(v)) \\ &= d(L_a)_b(v^L(b)). \end{aligned}$$

Furthermore, $v^L(1) = v$.

In summary, we proved the following result.

Proposition 16.3. *Given a Lie group G, the map $X \mapsto X(1)$ establishes an isomorphism between the space of left-invariant vector fields on G and $\mathfrak{g}$. In fact, the map $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^L(a)$ is an isomorphism between $G \times \mathfrak{g}$ and the tangent bundle TG.*

Definition 16.8. The vector space of left-invariant vector fields on a Lie group G is denoted by $\mathfrak{g}^L$.

Because the derivative of any Lie group homomorphism is a Lie algebra homomorphism, $(dL_a)_b$ is a Lie algebra homomorphism, so $\mathfrak{g}^L$ is a Lie algebra.

Given any $v \in \mathfrak{g}$, we can also define the vector field, v^R, by

$$v^R(a) = d(R_a)_1(v), \quad \text{for all } a \in G.$$

It is easily shown that v^R is right-invariant and we also have an isomorphism $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^R(a)$.

Definition 16.9. The vector space of right-invariant vector fields on a Lie group G is denoted by $\mathfrak{g}^R$.

The vector space $\mathfrak{g}^R$ is also a Lie algebra.

Another reason left-invariant (resp. right-invariant) vector fields on a Lie group are important is that they are complete, i.e., they define a flow whose domain is $\mathbb{R} \times G$. To prove this, we begin with the following easy proposition:

Proposition 16.4. *Given a Lie group, G, if X is a left-invariant (resp. right-invariant) vector field and Φ is its flow, then*

$$\Phi(t, g) = g\Phi(t, 1) \quad (\textit{resp.} \quad \Phi(t, g) = \Phi(t, 1)g),$$

for all $(t, g) \in \mathcal{D}(X)$.

Proposition 16.5. *Given a Lie group, G, for every $v \in \mathfrak{g}$, there is a unique smooth homomorphism, $h_v \colon (\mathbb{R}, +) \to G$, such that $\dot{h}_v(0) = v$. Furthermore, $h_v(t)$ is the maximal integral curve of both v^L and v^R with initial condition 1 and the flows of v^L and v^R are defined for all $t \in \mathbb{R}$.*

Since $h_v \colon (\mathbb{R}, +) \to G$ is a homomorphism, the following terminology is often used.

Definition 16.10. The integral curve $h_v \colon (\mathbb{R}, +) \to G$ of Proposition 16.5 is often referred to as a *one-parameter group*.

Proposition 16.5 yields the definition of the exponential map in terms of maximal integral curves.

Definition 16.11. Given a Lie group, G, the *exponential map*, $\exp \colon \mathfrak{g} \to G$, is given by

$$\exp(v) = h_v(1) = \Phi_1^v(1), \quad \text{for all } v \in \mathfrak{g},$$

where Φ_t^v denotes the flow of v^L.

It is not difficult to prove that exp is smooth.

Observe that for any fixed $t \in \mathbb{R}$, the map

$$s \mapsto h_v(st)$$

is a smooth homomorphism, h, such that $\dot{h}(0) = tv$.

By uniqueness, we have

$$h_v(st) = h_{tv}(s).$$

Setting $s = 1$, we find that

$$h_v(t) = \exp(tv), \quad \text{for all } v \in \mathfrak{g} \text{ and all } t \in \mathbb{R}.$$

Then, differentiating with respect to t at $t = 0$, we get

$$v = d\exp_0(v),$$

i.e., $d\exp_0 = \mathrm{id}_{\mathfrak{g}}$.

By the inverse function theorem, exp is a local diffeomorphism at 0. This means that there is some open subset, $U \subseteq \mathfrak{g}$, containing 0, such that the restriction of exp to U is a diffeomorphism onto $\exp(U) \subseteq G$, with $1 \in \exp(U)$.

In fact, by left-translation, the map $v \mapsto g\exp(v)$ is a local diffeomorphism between some open subset, $U \subseteq \mathfrak{g}$, containing 0 and the open subset, $\exp(U)$, containing g.

Proposition 16.6. *Given a Lie group G, the exponential map* $\exp\colon \mathfrak{g} \to G$ *is smooth and is a local diffeomorphism at* 0.

Remark: Given any Lie group G, we have a notion of exponential map $\exp\colon \mathfrak{g} \to G$ given by the maximal integral curves of left-invariant vector fields on G (see Proposition 16.5 and Definition 16.11).

This exponential does not require any connection or any metric in order to be defined; let us call it the *group exponential*.

If G is endowed with a connection or a Riemannian metric (the Levi-Civita connection if G has a Riemannnian metric), then we also have the notion of exponential induced by geodesics (see Definition 13.4); let us call this exponential the *geodesic exponential*.

To avoid ambiguities when both kinds of exponentials arise, we propose to denote the group exponential by $\exp_{\mathrm{gr}}$ and the geodesic exponential by exp, as before.

Even if the geodesic exponential is defined on the whole of $\mathfrak{g}$ (which may not be the case), these two notions of exponential differ in general.

The exponential map is also natural in the following sense:

Proposition 16.7. *Given any two Lie groups, G and H, for every Lie group homomorphism, $f\colon G \to H$, the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "df_1"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

As useful corollary of Proposition 16.7 is:

Proposition 16.8. *Let G be a connected Lie group and H be any Lie group. For any two homomorphisms, $\phi_1\colon G \to H$ and $\phi_2\colon G \to H$, if $d(\phi_1)_1 = d(\phi_2)_1$, then $\phi_1 = \phi_2$.*

Corollary 16.9. *If G is a connected Lie group, then a Lie group homomorphism $\phi\colon G \to H$ is uniquely determined by the Lie algebra homomorphism $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$.*

We obtain another useful corollary of Proposition 16.7 when we apply it to the adjoint representation of G,

$$\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$$

and to the conjugation map,

$$\mathbf{Ad}_a\colon G \to G,$$

where $\mathbf{Ad}_a(b) = aba^{-1}$.

In the first case, $d\mathrm{Ad}_1 = \mathrm{ad}$, with $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and in the second case, $d(\mathbf{Ad}_a)_1 = \mathrm{Ad}_a$.

Proposition 16.10. *Given any Lie group, G, the following properties hold:*

(1)
$$\mathrm{Ad}(\exp(u)) = e^{\mathrm{ad}(u)}, \qquad \text{for all } u \in \mathfrak{g},$$

where $\exp\colon \mathfrak{g} \to G$ *is the exponential of the Lie group, G, and $f \mapsto e^f$ is the exponential map given by*

$$e^f = \sum_{k=0}^{\infty} \frac{f^k}{k!},$$

for any linear map (matrix), $f \in \mathfrak{gl}(\mathfrak{g})$. Equivalently, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathbf{GL}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathfrak{gl}(\mathfrak{g}). \arrow[u, "f \mapsto e^f"'] \end{tikzcd}$$

(2)

$$\exp(t\mathrm{Ad}_g(u)) = g\exp(tu)g^{-1},$$

for all $u \in \mathfrak{g}$*, all* $g \in G$ *and all* $t \in \mathbb{R}$*. Equivalently, the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\mathbf{Ad}_g"] & G \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{Ad}_g"'] & \mathfrak{g}. \arrow[u, "\exp"'] \end{tikzcd}$$

Since the Lie algebra $\mathfrak{g} = T_1G$ is isomorphic to the vector space of left-invariant vector fields on G and since the Lie bracket of vector fields makes sense (see Definition 8.3), it is natural to ask if there is any relationship between, $[u, v]$, where $[u, v] = \mathrm{ad}(u)(v)$, and the Lie bracket, $[u^L, v^L]$, of the left-invariant vector fields associated with $u, v \in \mathfrak{g}$.

The answer is: Yes, they coincide (*via* the correspondence $u \mapsto u^L$).

Proposition 16.11. *Given a Lie group, G, we have*

$$[u^L, v^L](1) = \mathrm{ad}(u)(v), \quad \textit{for all } u, v \in \mathfrak{g}.$$

Proposition 16.11 shows that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^L$ are isomorphic (where $\mathfrak{g}^L$ is the Lie algebra of left-invariant vector fields on G).

In view of this isomorphism, if X and Y are any two left-invariant vector fields on G, we define $\mathrm{ad}(X)(Y)$ by

$$\mathrm{ad}(X)(Y) = [X, Y],$$

where the Lie bracket on the right-hand side is the Lie bracket on vector fields.

Proposition 16.12. *Given a Lie group G, the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^L$ are isomorphic, and the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^R$ are anti-isomorphic.*

If G is a Lie group, let G_0 be the connected component of the identity. We know G_0 is a topological normal subgroup of G and it is a submanifold in an obvious way, so it is a Lie group.

Proposition 16.13. *If G is a Lie group and G_0 is the connected component of 1, then G_0 is generated by $\exp(\mathfrak{g})$. Moreover, G_0 is countable at infinity.*

16.3 Homomorphisms of Lie Groups and Lie Algebras, Lie Subgroups

If G and H are two Lie groups and $\phi\colon G \to H$ is a homomorphism of Lie groups, then $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H.

In fact, it is a Lie algebra homomorphism.

Proposition 16.14. *If G and H are two Lie groups and $\phi\colon G \to H$ is a homomorphism of Lie groups, then*

$$d\phi_1 \circ \mathrm{Ad}_g = \mathrm{Ad}_{\phi(g)} \circ d\phi_1, \quad \text{for all } g \in G,$$

that is, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "d\phi_1"] \arrow[d, "\mathrm{Ad}_g"'] & \mathfrak{h} \arrow[d, "\mathrm{Ad}_{\phi(g)}"] \\
\mathfrak{g} \arrow[r, "d\phi_1"'] & \mathfrak{h}
\end{tikzcd}$$

and $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Remark: If we identify the Lie algebra, $\mathfrak{g}$, of G with the space of left-invariant vector fields on G, the map $d\phi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is viewed as the map such that, for every left-invariant vector field, X, on G, the vector field $d\phi_1(X)$ is the unique left-invariant vector field on H such that

$$d\phi_1(X)(1) = d\phi_1(X(1)),$$

i.e., $d\phi_1(X) = d\phi_1(X(1))^L$. Then, we can give another proof of the fact that $d\phi_1$ is a Lie algebra homomorphism.

Proposition 16.15. *If G and H are two Lie groups and $\phi \colon G \to H$ is a homomorphism of Lie groups, if we identify $\mathfrak{g}$ (resp. $\mathfrak{h}$) with the space of left-invariant vector fields on G (resp. left-invariant vector fields on H), then,*

(a) X and $d\phi_1(X)$ are ϕ-related, for every left-invariant vector field, X, on G;

(b) $d\phi_1 \colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

We now consider Lie subgroups. The following proposition shows that an injective Lie group homomorphism is an immersion.

Proposition 16.16. *If $\phi\colon G \to H$ is an injective Lie group homomorphism, then the map $d\phi_g\colon T_gG \to T_{\phi(g)}H$ is injective for all $g \in G$.*

Therefore, if $\phi\colon G \to H$ is injective, it is automatically an immersion.

Definition 16.12. Let G be a Lie group. A set, H, is an *immersed (Lie) subgroup* of G iff

(a) H is a Lie group;

(b) There is an injective Lie group homomorphism, $\phi\colon H \to G$ (and thus, ϕ is an immersion, as noted above).

We say that H is a *Lie subgroup* (or *closed Lie subgroup*) of G iff H is a Lie group that is a subgroup of G and also a submanifold of G.

Observe that an immersed Lie subgroup, H, is an immersed submanifold, since ϕ is an injective immersion.

However, $\phi(H)$ may *not* have the subspace topology inherited from G and $\phi(H)$ may not be closed.

An example of this situation is provided by the 2-torus, $T^2 \cong \mathbf{SO}(2) \times \mathbf{SO}(2)$, which can be identified with the group of 2×2 complex diagonal matrices of the form

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}$$

where $\theta_1, \theta_2 \in \mathbb{R}$.

For any $c \in \mathbb{R}$, let S_c be the subgroup of T^2 consisting of all matrices of the form

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{ict} \end{pmatrix}, \quad t \in \mathbb{R}.$$

It is easily checked that S_c is an immersed Lie subgroup of T^2 iff c is irrational.

However, when c is irrational, one can show that S_c is dense in T^2 but not closed.

As we will see below, *a Lie subgroup is always closed*.

We borrowed the terminology "immersed subgroup" from Fulton and Harris [17] (Chapter 7), but we warn the reader that most books call such subgroups "Lie subgroups" and refer to the second kind of subgroups (that are submanifolds) as "closed subgroups."

Theorem 16.17. *Let G be a Lie group and let (H, ϕ) be an immersed Lie subgroup of G. Then, ϕ is an embedding iff $\phi(H)$ is closed in G. As as consequence, any Lie subgroup of G is closed.*

We can prove easily that a Lie subgroup, H, of G is closed.

If G is a Lie group, say that a subset, $H \subseteq G$, is an *abstract subgroup* iff it is just a subgroup of the underlying group of G (i.e., we forget the topology and the manifold structure).

Theorem 16.18. *Let G be a Lie group. An abstract subgroup, H, of G is a submanifold (i.e., a Lie subgroup) of G iff H is closed (i.e, H with the induced topology is closed in G).*

16.4 The Correspondence Lie Groups–Lie Algebras

Historically, Lie was the first to understand that a lot of the structure of a Lie group is captured by its Lie algebra, a simpler object (since it is a vector space).

In this short section, we state without proof some of the "Lie theorems," although not in their original form.

Definition 16.13. If $\mathfrak{g}$ is a Lie algebra, a *subalgebra*, $\mathfrak{h}$, of $\mathfrak{g}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$, for all $u, v \in \mathfrak{h}$. If $\mathfrak{h}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$ for all $u \in \mathfrak{h}$ and all $v \in \mathfrak{g}$, we say that $\mathfrak{h}$ is an *ideal* in $\mathfrak{g}$.

For a proof of the theorem below, see Warner [47] (Chapter 3) or Duistermaat and Kolk [15] (Chapter 1, Section 10).

Theorem 16.19. *Let G be a Lie group with Lie algebra, $\mathfrak{g}$, and let (H, ϕ) be an immersed Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $d\phi_1\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*

Conversely, for each subalgebra, $\widetilde{\mathfrak{h}}$, of $\mathfrak{g}$, there is a unique connected immersed subgroup, (H, ϕ), of G so that $d\phi_1\mathfrak{h} = \widetilde{\mathfrak{h}}$. In fact, as a group, $\phi(H)$ is the subgroup of G generated by $\exp(\widetilde{\mathfrak{h}})$.

Furthermore, normal subgroups correspond to ideals.

Theorem 16.19 shows that *there is a one-to-one correspondence between connected immersed subgroups of a Lie group and subalgebras of its Lie algebra*.

Theorem 16.20. *Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For every homomorphism, $\psi\colon \mathfrak{g} \to \mathfrak{h}$, there is a unique Lie group homomorphism, $\phi\colon G \to H$, so that $d\phi_1 = \psi$.*

Again a proof of the theorem above is given in Warner [47] (Chapter 3) or Duistermaat and Kolk [15] (Chapter 1, Section 10).

Corollary 16.21. *If G and H are connected and simply connected Lie groups, then G and H are isomorphic iff $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

It can also be shown that for every finite-dimensional Lie algebra $\mathfrak{g}$, there is a connected and simply connected Lie group G such that $\mathfrak{g}$ is the Lie algebra of G.

This result is known as *Lie's third theorem*.

Lie's third theorem was first proven by Élie Cartan; see Serre [44].

It is also a consequence of deep theorem known as *Ado's theorem*.

Ado's theorem states that every finite-dimensional Lie algebra has a faithful representation in $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}_n(\mathbb{R})$ for some n.

The proof is quite involved; see Knapp [25] (Appendix C) Fulton and Harris [17] (Appendix E), or Bourbaki [8] (Chapter 1, Section §7).

As a corollary of Lie's third theorem, there is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply-connected Lie groups, given by associating each simply connnected Lie group with its Lie algebra.; see Lee [31] (Theorem 20.20) and Warner [47] (Theorem 3.28).

In summary, following Fulton and Harris, we have the following two principles of the Lie group/Lie algebra correspondence:

First Principle: If G and H are Lie groups, with G connected, then a homomorphism of Lie groups, $\phi\colon G \to H$, is uniquely determined by the Lie algebra homomorphism, $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$.

Second Principle: Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras.

A linear map, $\psi\colon \mathfrak{g} \to \mathfrak{h}$, is a Lie algebra map iff there is a unique Lie group homomorphism, $\phi\colon G \to H$, so that $d\phi_1 = \psi$.

16.5 Semidirect Products of Lie Algebras and Lie Groups

If $\mathfrak{a}$ and $\mathfrak{b}$ are two Lie algebras, recall that the direct sum $\mathfrak{a} \oplus \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is $\mathfrak{a} \times \mathfrak{b}$ with the product vector space structure where

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

for all $a_1, a_2 \in \mathfrak{a}$ and all $b_1, b_2 \in \mathfrak{b}$, and

$$\lambda(a, b) = (\lambda a, \lambda b)$$

for all $\lambda \in \mathbb{R}$, all $a \in \mathfrak{a}$, and all $b \in \mathfrak{b}$.

The map $a \mapsto (a, 0)$ is an isomorphism of $\mathfrak{a}$ with the subspace $\{(a, 0) \mid a \in \mathfrak{a}\}$ of $\mathfrak{a} \oplus \mathfrak{b}$ and the map $b \mapsto (0, b)$ is an isomorphism of $\mathfrak{b}$ with the subspace $\{(0, b) \mid b \in \mathfrak{b}\}$ of $\mathfrak{a} \oplus \mathfrak{b}$.

These isomorphisms allow us to identify $\mathfrak{a}$ with the subspace $\{(a, 0) \mid a \in \mathfrak{a}\}$ and $\mathfrak{b}$ with the subspace $\{(0, b) \mid b \in \mathfrak{b}\}$.

We can make the direct sum $\mathfrak{a} \oplus \mathfrak{b}$ into a Lie algebra by defining the Lie bracket $[-, -]$ such that $[a_1, a_2]$ agrees with the Lie bracket on $\mathfrak{a}$ for all $a_1, a_2, \in \mathfrak{a}$, $[b_1, b_2]$ agrees with the Lie bracket on $\mathfrak{b}$ for all $b_1, b_2, \in \mathfrak{b}$, and $[a, b] = [b, a] = 0$ for all $a \in \mathfrak{a}$ and all $b \in \mathfrak{b}$.

Definition 16.14. If $\mathfrak{a}$ and $\mathfrak{b}$ are two Lie algebras, the direct sum $\mathfrak{a} \oplus \mathfrak{b}$ with the bracket defined by

$$[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2]_{\mathfrak{a}}, [b_1, b_2]_{\mathfrak{b}})$$

for all $a_1, a_2, \in \mathfrak{a}$ and all $b_1, b_2, \in \mathfrak{b}$ is a Lie algebra is called the *Lie algebra direct sum* of $\mathfrak{a}$ and $\mathfrak{b}$.

Observe that with this Lie algebra structure, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals.

The above construction is sometimes called an "external direct sum" because it does not assume that the constituent Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of some given Lie algebra $\mathfrak{g}$.

Definition 16.15. If $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of a given Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum as a vector space and if both $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then for all $a \in \mathfrak{a}$ and all $b \in \mathfrak{b}$, we have $[a, b] \in \mathfrak{a} \cap \mathfrak{b} = (0)$, so $\mathfrak{a} \oplus \mathfrak{b}$ is the Lie algebra direct sum of $\mathfrak{a}$ and $\mathfrak{b}$. This Lie algeba is called an *internal direct sum*.

We now would like to generalize this construction to the situation where the Lie bracket $[a, b]$ of some $a \in \mathfrak{a}$ and some $b \in \mathfrak{b}$ is given in terms of a map from $\mathfrak{b}$ to $\mathrm{Hom}(\mathfrak{a}, \mathfrak{a})$. For this to work, we need to consider derivations.

Definition 16.16. Given a Lie algebra $\mathfrak{g}$, a *derivation* is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ satisfying the following condition:

$$D([X,Y]) = [D(X),Y] + [X,D(Y)], \quad \text{for all } X,Y \in \mathfrak{g}.$$

The vector space of all derivations on $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$.

The first thing to observe is that the Jacobi identity can be expressed as

$$[Z,[X,Y]] = [[Z,X],Y] + [X,[Z,Y]],$$

which holds iff

$$(\mathrm{ad}\, Z)[X,Y] = [(\mathrm{ad}\, Z)X, Y] + [X, (\mathrm{ad}\, Z)Y],$$

and the above equation means that $\mathrm{ad}(Z)$ is a derivation.

In fact, it is easy to check that the Jacobi identity holds iff $\mathrm{ad}\, Z$ is a derivation for every $Z \in \mathfrak{g}$.

It tuns out that the vector space of derivations $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra under the commutator bracket.

Proposition 16.22. *For any Lie algebra $\mathfrak{g}$, the vector space $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra under the commutator bracket. Furthermore, the map $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism.*

Proposition 16.23. *For any Lie algebra $\mathfrak{g}$ If $D \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, then*

$$[D, \mathrm{ad}\, X] = \mathrm{ad}\,(DX).$$

Proposition 16.24. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras, and suppose τ is a Lie algebra homomorphism $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$. Then there is a unique Lie algebra structure on the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ whose Lie bracket agrees with the Lie bracket on $\mathfrak{a}$ and the Lie bracket on $\mathfrak{b}$, and such that*

$$[(0, B), (A, 0)]_{\mathfrak{g}} = \tau(B)(A) \quad \text{for all } A \in \mathfrak{a} \text{ and all } B \in \mathfrak{b}.$$

The Lie bracket on $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is given by

$$\begin{aligned}&[(A, B), (A', B')]_{\mathfrak{g}} \\ &\quad = ([A, A']_{\mathfrak{a}} + \tau(B)(A') - \tau(B')(A),\ [B, B']_{\mathfrak{b}}),\end{aligned}$$

for all $A, A' \in \mathfrak{a}$ and all $B, B' \in \mathfrak{b}$. In particular,

$$[(0, B), (A', 0)]_{\mathfrak{g}} = \tau(B)(A') \in \mathfrak{a}.$$

With this Lie algebra structure, $\mathfrak{a}$ is an ideal and $\mathfrak{b}$ is a subalgebra.

Definition 16.17. The Lie algebra obtained in Proposition 16.24 is denoted by

$$\mathfrak{a} \oplus_\tau \mathfrak{b} \quad \text{or} \quad \mathfrak{a} \rtimes_\tau \mathfrak{b}$$

and is called the *semidirect product of $\mathfrak{b}$ by $\mathfrak{a}$ with respect to $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$*.

When τ is the zero map, we get back the Lie algebra direct sum.

Remark: A sequence of Lie algebra maps

$$\mathfrak{a} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\psi} \mathfrak{b}$$

with φ injective, ψ surjective, and with $\mathrm{Im}\,\varphi = \mathrm{Ker}\,\psi = \mathfrak{n}$, is called an *extension of $\mathfrak{b}$ by $\mathfrak{a}$ with kernel $\mathfrak{n}$*.

If there is a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$, then we say that this extension is *inessential*.

Given a semidirect product $\mathfrak{g} = \mathfrak{a} \rtimes_\tau \mathfrak{b}$ of $\mathfrak{b}$ by $\mathfrak{a}$, if $\varphi\colon \mathfrak{a} \to \mathfrak{g}$ is the map given $\varphi(a) = (a, 0)$ and ψ is the map $\psi\colon \mathfrak{g} \to \mathfrak{b}$ given by $\psi(a, b) = b$, then $\mathfrak{g}$ is an inessential extension of $\mathfrak{b}$ by $\mathfrak{a}$.

Conversely, it is easy to see that every inessential extension of of $\mathfrak{b}$ by $\mathfrak{a}$ is a semidirect product of of $\mathfrak{b}$ by $\mathfrak{a}$.

Proposition 16.24 is an external construction. The notion of semidirect product has a corresponding internal construction.

If $\mathfrak{g}$ is a Lie algebra and if $\mathfrak{a}$ and $\mathfrak{b}$ are subspaces of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b},$$

$\mathfrak{a}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, then for every $B \in \mathfrak{b}$, because $\mathfrak{a}$ is an ideal, the restriction of $\mathrm{ad}\, B$ to $\mathfrak{a}$ leaves $\mathfrak{a}$ invariant, so by Proposition 16.22, the map $B \mapsto \mathrm{ad}\, B \restriction \mathfrak{a}$ is a Lie algebra homomorphism $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$.

Observe that $[B, A] = \tau(B)(A)$, for all $A \in \mathfrak{a}$ and all $B \in \mathfrak{b}$, so the Lie bracket on $\mathfrak{g}$ is completely determined by the Lie brackets on $\mathfrak{a}$ and $\mathfrak{b}$ and the homomorphism τ.

We say that $\mathfrak{g}$ is the *semidirect product* of $\mathfrak{b}$ and $\mathfrak{a}$ and we write

$$\mathfrak{g} = \mathfrak{a} \oplus_\tau \mathfrak{b}.$$

Let $\mathfrak{g}$ be any Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}_n(\mathbb{R})$, let $\mathfrak{a} = \mathbb{R}^n$ with the zero bracket making $\mathbb{R}^n$ into an abelian Lie algebra.

Then, $\mathrm{Der}(\mathfrak{a}) = \mathfrak{gl}(n, \mathbb{R})$, and we let $\tau \colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ be the inclusion map.

The resulting semidirect product $\mathbb{R}^n \rtimes \mathfrak{g}$ is the affine Lie algebra associated with $\mathfrak{g}$. Its Lie bracket is defined by

$$[(u, A), (v, B)] = (Av - Bu, [A, B]).$$

In particular, if $\mathfrak{g} = \mathfrak{so}(n)$, the Lie algebra of $\mathbf{SO}(n)$, then $\mathbb{R}^n \rtimes \mathfrak{so}(n) = \mathfrak{se}(n)$, the Lie algebra of $\mathbf{SE}(n)$.

Before turning our attention to semidirect products of Lie groups, let us consider the group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra isomorphisms of a Lie algebra $\mathfrak{g}$.

The group $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of the groups $\mathbf{GL}(\mathfrak{g})$ of linear automorphisms of $\mathfrak{g}$, and it is easy to see that it is closed, so it is a Lie group.

Proposition 16.25. *For any (real) Lie algebra $\mathfrak{g}$, the Lie algebra $\mathrm{L}(\mathrm{Aut}(\mathfrak{g}))$ of the group $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$.*

We know that Ad is a Lie group homomorphism

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}),$$

and Proposition 16.25 implies that ad is a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}).$$

We now define semidirect products of Lie groups and show how their algebras are semidirect products of Lie algebras.

Proposition 16.26. *Let H and K be two groups and let $\tau\colon K \to \mathrm{Aut}(H)$ be a homomorphism of K into the automorphism group of H. Let $G = H \times K$ with multiplication defined as follows:*

$$(h_1, k_1)(h_2, k_2) = (h_1\tau(k_1)(h_2), k_1k_2),$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K$. Then, the following properties hold:

(1) This multiplication makes G into a group with identity $(1, 1)$ and with inverse given by

$$(h, k)^{-1} = (\tau(k^{-1})(h^{-1}), k^{-1}).$$

(2) The maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms from H to the subgroup $\{(h, 1) \mid h \in H\}$ of G and from K to the subgroup $\{(1, k) \mid k \in K\}$ of G.

(3) Using the isomorphisms from (2), the group H is a normal subgroup of G.

(4) Using the isomorphisms from (2), $H \cap K = (1)$.

(5) For all $h \in H$ an all $k \in K$, we have

$$(1, k)(h, 1)(1, k)^{-1} = (\tau(k)(h), 1).$$

In view of Proposition 16.26, we make the following definition.

Definition 16.18. Let H and K be two groups and let $\tau\colon K \to \mathrm{Aut}(H)$ be a homomorphism of K into the automorphism group of H. The group defined in Proposition 16.26 is called the *semidirect product of K by H with respect to τ*, and it is denoted $H \rtimes_\tau K$ (or even $H \rtimes K$).

Note that $\tau\colon K \to \mathrm{Aut}(H)$ can be viewed as a left action $\cdot\colon K \times H \to H$ of K on H "acting by automorphisms," which means that for every $k \in K$, the map $h \mapsto \tau(k, h)$ is an automorphism of H.

Note that when τ is the trivial homomorphism (that is, $\tau(k) = \mathrm{id}$ for all $k \in K$), the semidirect product is just the direct product $H \times K$ of the groups H and K, and K is also a normal subgroup of G.

Let $H = \mathbb{R}^n$ under addition, let $K = \mathbf{SO}(n)$, and let τ be the inclusion map of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$.

In other words, τ is the action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$ given by $R \cdot u = Ru$.

Then, the semidirect product $\mathbb{R}^n \rtimes \mathbf{SO}(n)$ is isomorphic to the group $\mathbf{SE}(n)$ of direct affine rigid motions of $\mathbb{R}^n$ (translations and rotations), since the multiplication is given by

$$(u, R)(v, S) = (Rv + u, RS).$$

We obtain other affine groups by letting K be $\mathbf{SL}(n)$, $\mathbf{GL}(n)$, *etc.*

As in the case of Lie algebras, a sequence of groups homomorphisms

$$H \xrightarrow{\varphi} G \xrightarrow{\psi} K$$

with φ injective, ψ surjective, and with $\operatorname{Im} \varphi = \operatorname{Ker} \psi = N$, is called an *extension of K by H with kernel N*.

If $H \rtimes_\tau K$ is a semidirect product, we have the homomorphisms $\varphi\colon H \to G$ and $\psi\colon G \to K$ given by

$$\varphi(h) = (h, 1), \qquad \psi(h, k) = k,$$

and it is clear that we have an extension of K by H with kernel $N = \{(h, 1) \mid h \in H\}$. Note that we have a homomorphism $\gamma\colon K \to G$ (a section of ψ) given by

$$\gamma(k) = (1, k),$$

and that

$$\psi \circ \gamma = \mathrm{id}.$$

Conversely, it can be shown that if an extension of K by H has a section $\gamma \colon K \to G$, then G is isomorphic to a semidirect product of K by H with respect to a certain homomorphism τ; find it!

I claim that if H and K are two Lie groups and if the map from $H \times K$ to H given by $(h, k) \mapsto \tau(k)(h)$ is smooth, then the semidirect product $H \rtimes_\tau K$ is a Lie group (see Varadarajan [46] (Section 3.15), Bourbaki [8], (Chapter 3, Section 1.4)). This is because

$$
\begin{aligned}
(h_1, k_1)(h_2, k_2)^{-1} &= (h_1, k_1)(\tau(k_2^{-1})(h_2^{-1}), k_2^{-1}) \\
&= (h_1 \tau(k_1)(\tau(k_2^{-1})(h_2^{-1})), k_1 k_2^{-1}) \\
&= (h_1 \tau(k_1 k_2^{-1})(h_2^{-1}), k_1 k_2^{-1}),
\end{aligned}
$$

which shows that multiplication and inversion in $H \rtimes_\tau K$ are smooth.

For every $k \in K$, the derivative of $d(\tau(k))_1$ of $\tau(k)$ at 1 is a Lie algebra isomorphism of $\mathfrak{h}$, and just like Ad, it can be shown that the map $\widetilde{\tau}\colon K \to \mathrm{Aut}(\mathfrak{h})$ given by

$$\widetilde{\tau}(k) = d(\tau(k))_1 \quad k \in K$$

is a smooth homomorphism from K into $\mathrm{Aut}(\mathfrak{h})$.

It follows by Proposition 16.25 that its derivative $d\widetilde{\tau}_1\colon \mathfrak{k} \to \mathrm{Der}(\mathfrak{h})$ at 1 is a homomorphism of $\mathfrak{k}$ into $\mathrm{Der}(\mathfrak{h})$.

Proposition 16.27. *Using the notations just introduced, the Lie algebra of the semidirect product $H \rtimes_\tau K$ of K by H with respect to τ is the semidirect product $\mathfrak{h} \rtimes_{d\widetilde{\tau}_1} \mathfrak{k}$ of $\mathfrak{k}$ by $\mathfrak{h}$ with respect to $d\widetilde{\tau}_1$.*

Proposition 16.27 applied to the semidirect product $\mathbb{R}^n \rtimes_\tau \mathbf{SO}(n) \cong \mathbf{SE}(n)$ where τ is the inclusion map of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$ confirms that $\mathbb{R}^n \rtimes_{d\widetilde{\tau}_1} \mathfrak{so}(n)$ is the Lie algebra of $\mathbf{SE}(n)$, where $d\widetilde{\tau}_1$ is inclusion map of $\mathfrak{so}(n)$ into $\mathfrak{gl}(n, \mathbb{R})$ (and $\widetilde{\tau}$ is the inclusion of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$).

As a special case of Proposition 16.27, when our semidirect product is just a direct product $H \times K$ (τ is the trivial homomorphism mapping every $k \in K$ to id), we see that the Lie algebra of $H \times K$ is the Lie algebra direct sum $\mathfrak{h} \oplus \mathfrak{k}$ (where the bracket between elements of $\mathfrak{h}$ and elements of $\mathfrak{k}$ is 0).

16.6 Universal Covering Groups

Every connected Lie group G is a manifold, and as such, from results in Section 9.2, it has a universal cover $\pi \colon \widetilde{G} \to G$, where $\widetilde{G}$ is simply connected.

It is possible to make $\widetilde{G}$ into a group so that $\widetilde{G}$ is a Lie group and π is a Lie group homomorphism.

We content ourselves with a sketch of the construction whose details can be found in Warner [47], Chapter 3.

Consider the map $\alpha \colon \widetilde{G} \times \widetilde{G} \to G$, given by

$$\alpha(\widetilde{a}, \widetilde{b}) = \pi(\widetilde{a})\pi(\widetilde{b})^{-1},$$

for all $\widetilde{a}, \widetilde{b} \in \widetilde{G}$, and pick some $\widetilde{e} \in \pi^{-1}(e)$.

Since $\widetilde{G}\times\widetilde{G}$ is simply connected, it follows by Proposition 9.12 that there is a unique map $\widetilde{\alpha}\colon \widetilde{G}\times\widetilde{G}\to\widetilde{G}$ such that

$$\alpha = \pi\circ\widetilde{\alpha} \quad \text{and} \quad \widetilde{e} = \widetilde{\alpha}(\widetilde{e},\widetilde{e}),$$

as illustrated below:

$$\begin{tikzcd} & \widetilde{G} \ni \widetilde{e} \arrow[d, "\pi"] \\ \widetilde{G}\times\widetilde{G} \arrow[ur, dashed, "\widetilde{\alpha}"] \arrow[r, "\alpha"'] & G \ni 1. \end{tikzcd}$$

For all $\widetilde{a},\widetilde{b}\in\widetilde{G}$, define

$$\widetilde{b}^{-1} = \widetilde{\alpha}(\widetilde{e},\widetilde{b}), \qquad \widetilde{a}\widetilde{b} = \widetilde{\alpha}(\widetilde{a},\widetilde{b}^{-1}). \tag{$*$}$$

Using Proposition 9.12, it can be shown that the above operations make $\widetilde{G}$ into a group, and as $\widetilde{\alpha}$ is smooth, into a Lie group. Moreover, π becomes a Lie group homomorphism.

Theorem 16.28. *Every connected Lie group has a simply connected covering map $\pi\colon \widetilde{G} \to G$, where $\widetilde{G}$ is a Lie group and π is a Lie group homomorphism.*

The group $\widetilde{G}$ is called the *universal covering group* of G.

Consider $D = \ker \pi$. Since the fibres of π are countable The group D is a countable closed normal subgroup of $\widetilde{G}$; that is, a discrete normal subgroup of $\widetilde{G}$.

It follows that $G \cong \widetilde{G}/D$, where $\widetilde{G}$ is a simply connected Lie group and D is a discrete normal subgroup of $\widetilde{G}$.

We conclude this section by stating the following useful proposition whose proof can be found in Warner [47] (Chapter 3, Proposition 3.26):

Proposition 16.29. *Let $\phi\colon G \to H$ be a homomorphism of connected Lie groups. Then ϕ is a covering map iff $d\phi_e\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras.*

For example, we know that $\mathfrak{su}(2) = \mathfrak{so}(3)$, so the homomorphism from $\mathbf{SU}(2)$ to $\mathbf{SO}(3)$ provided by the representation of 3D rotations by the quaternions is a covering map.

16.7 The Lie Algebra of Killing Fields ⊛

In Section 15.4 we defined Killing vector fields. Recall that a Killing vector field X on a manifold M satisfies the condition

$$L_X g(Y, Z) = X(\langle Y, Z\rangle) - \langle [X, Y], Z\rangle - \langle Y, [X, Z]\rangle = 0,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

By Proposition 15.9, X is a Killing vector field iff the diffeomorphisms Φ_t induced by the flow Φ of X are isometries (on their domain).

The isometries of a Riemannian manifold (M, g) form a group $\mathrm{Isom}(M, g)$, called the *isometry group of (M, g)*.

An important theorem of Myers and Steenrod asserts that the isometry group $\mathrm{Isom}(M, g)$ is a Lie group.

It turns out that the Lie algebra $\mathfrak{i}(M)$ of the group $\mathrm{Isom}(M, g)$ is closely related to a certain Lie subalgebra of the Lie algebra of Killing fields.

We begin by observing that the Killing fields form a Lie algebra.

Proposition 16.30. *The Killing fields on a smooth manifold M form a Lie subalgebra $\mathcal{K}i(M)$ of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M.*

However, unlike $\mathfrak{X}(M)$, the Lie algebra $\mathcal{K}i(M)$ is finite-dimensional.

In fact, the Lie subalgebra $c\mathcal{K}i(M)$ of complete Killing vector fields is anti-isomorphic to the Lie algebra $\mathfrak{i}(M)$ of the Lie group $\mathrm{Isom}(M)$ of isometries of M (see Section 11.2 for the definition of $\mathrm{Isom}(M)$).

The following result is proved in O'Neill [38] (Chapter 9, Lemma 28) and Sakai [43] (Chapter III, Lemma 6.4 and Proposition 6.5).

Proposition 16.31. *Let (M, g) be a connected Riemannian manifold of dimension n (equip-ped with the Levi–Civita connection on M induced by g). The Lie algebra $\mathcal{K}i(M)$ of Killing vector fields on M has dimension at most $n(n+1)/2$.*

We also have the following result proved in O'Neill [38] (Chapter 9, Proposition 30) and Sakai [43] (Chapter III, Corollary 6.3).

Proposition 16.32. *Let (M, g) be a Riemannian manifold of dimension n (equipped with the Levi–Civita connection on M induced by g). If M is complete, then every Killing vector fields on M is complete.*

The relationship between the Lie algebra $\mathfrak{i}(M)$ and Killing vector fields is obtained as follows.

For every element X in the Lie algebra $\mathfrak{i}(M)$ of $\mathrm{Isom}(M)$ (viewed as a left-invariant vector field), define the vector field X^+ on M by

$$X^+(p) = \frac{d}{dt}(\varphi_t(p))\bigg|_{t=0}, \quad p \in M,$$

where $t \mapsto \varphi_t = \exp(tX)$ is the one-parameter group associated with X.

Because φ_t is an isometry of M, the vector field X^+ is a Killing vector field, and it is also easy to show that (φ_t) is the one-parameter group of X^+.

Since φ_t is defined for all t, the vector field X^+ is complete. The following result is shown in O'Neill [38] (Chapter 9, Proposition 33).

Theorem 16.33. *Let (M, g) be a Riemannian manifold (equipped with the Levi–Civita connection on M induced by g). The following properties hold:*

(1) The set $c\mathcal{K}i(M)$ of complete Killing vector fields on M is a Lie subalgebra of the Lie algebra $\mathcal{K}i(M)$ of Killing vector fields.

(2) The map $X \mapsto X^+$ is a Lie anti-isomorphism between $\mathfrak{i}(M)$ and $c\mathcal{K}i(M)$, which means that

$$[X^+, Y^+] = -[X, Y]^+, \quad X, Y \in \mathfrak{i}(M).$$

For more on Killing vector fields, see Sakai [43] (Chapter III, Section 6).

In particular, complete Riemannian manifolds for which $\mathfrak{i}(M)$ has the maximum dimension $n(n+1)/2$ are characterized.