Chapter 9

Lie Groups, Lie Algebras and the Exponential Map

9.1 Lie Groups and Lie Algebras

In Chapter 1, we defined the notion of a Lie group as a certain type of manifold embedded in $\mathbb{R}^N$, for some $N \geq 1$. Now we can define Lie groups in more generality.

Definition 9.1. A *Lie group* is a nonempty subset, G, satisfying the following conditions:

(a) G is a group (with identity element denoted e or 1).

(b) G is a smooth manifold.

(c) G is a topological group. In particular, the group operation, $\cdot : G \times G \to G$, and the inverse map, ${}^{-1}\colon G \to G$, are smooth.

We have already met a number of Lie groups: $\mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$, $\mathbf{SL}(n, \mathbb{R})$, $\mathbf{SL}(n, \mathbb{C})$, $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{E}(n, \mathbb{R})$. Also, every linear Lie group (i.e., a closed subgroup of $\mathbf{GL}(n, \mathbb{R})$) is a Lie group.

We saw in the case of linear Lie groups that the tangent space to G at the identity, $\mathfrak{g} = T_1 G$, plays a very important role. This is again true in this more general setting.

Definition 9.2. A *(real) Lie algebra*, $\mathcal{A}$, is a real vector space together with a bilinear map, $[\cdot, \cdot]\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the *Lie bracket* on $\mathcal{A}$ such that the following two identities hold for all $a, b, c \in \mathcal{A}$:

$$[a,\, a] = 0,$$

and the so-called *Jacobi identity*

$$[a,\, [b,\, c]] + [c,\, [a,\, b]] + [b,\, [c,\, a]] = 0.$$

It is immediately verified that $[b, a] = -[a, b]$.

Let us also recall the definition of homomorphisms of Lie groups and Lie algebras.

Definition 9.3. Given two Lie groups G_1 and G_2, a *homomorphism (or map) of Lie groups* is a function, $f\colon G_1 \to G_2$, that is a homomorphism of groups and a smooth map (between the manifolds G_1 and G_2). Given two Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *homomorphism (or map) of Lie algebras* is a function, $f\colon \mathcal{A}_1 \to \mathcal{A}_2$, that is a linear map between the vector spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ and that preserves Lie brackets, i.e.,

$$f([A, B]) = [f(A), f(B)]$$

for all $A, B \in \mathcal{A}_1$.

An *isomorphism of Lie groups* is a bijective function f such that both f and f^{-1} are maps of Lie groups, and an *isomorphism of Lie algebras* is a bijective function f such that both f and f^{-1} are maps of Lie algebras.

The Lie bracket operation on $\mathfrak{g}$ can be defined in terms of the so-called adjoint representation.

Given a Lie group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Because multiplication and the inverse maps are smooth, the maps L_a and R_a are diffeomorphisms, and their derivatives play an important role.

The inner automorphisms $R_{a^{-1}} \circ L_a$ (also written $R_{a^{-1}} L_a$ or $\mathbf{Ad}_a$) also play an important role. Note that

$$\mathbf{Ad}_a(b) = R_{a^{-1}} L_a(b) = aba^{-1}.$$

The derivative

$$d(\mathbf{Ad}_a)_1\colon T_1G \to T_1G$$

of $\mathbf{Ad}_a\colon G \to G$ at 1 is an isomorphism of Lie algebras, denoted by $\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$.

The map $a \mapsto \mathrm{Ad}_a$ is a map of Lie groups

$$\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g}),$$

called the *adjoint representation of G* (where $\mathbf{GL}(\mathfrak{g})$ denotes the Lie group of all bijective linear maps on $\mathfrak{g}$).

In the case of a linear group, we showed that

$$\mathrm{Ad}(a)(X) = \mathrm{Ad}_a(X) = aXa^{-1}$$

for all $a \in G$ and all $X \in \mathfrak{g}$.

The derivative

$$dAd_1 \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

of Ad at 1 is map of Lie algebras, denoted by

$$\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

called the *adjoint representation of* $\mathfrak{g}$ (where $\mathfrak{gl}(\mathfrak{g})$ denotes the Lie algebra, $\mathrm{End}(\mathfrak{g}, \mathfrak{g})$, of all linear maps on $\mathfrak{g}$).

In the case of a linear group, we showed that

$$\mathrm{ad}(A)(B) = [A,\ B]$$

for all $A, B \in \mathfrak{g}$.

One can also check (in general) that the Jacobi identity on $\mathfrak{g}$ is equivalent to the fact that ad preserves Lie brackets, i.e., ad is a map of Lie algebras:

$$\mathrm{ad}([u, v]) = [\mathrm{ad}(u), \mathrm{ad}(v)],$$

for all $u, v \in \mathfrak{g}$ (where on the right, the Lie bracket is the commutator of linear maps on $\mathfrak{g}$).

This is the key to the definition of the Lie bracket in the case of a general Lie group (not just a linear Lie group).

Definition 9.4. Given a Lie group, G, the tangent space, $\mathfrak{g} = T_1 G$, at the identity with the Lie bracket defined by

$$[u, v] = \mathrm{ad}(u)(v), \quad \text{for all } u, v \in \mathfrak{g},$$

is the *Lie algebra of the Lie group G*.

Actually, we have to justify why $\mathfrak{g}$ really is a Lie algebra. For this, we have

Proposition 9.1. *Given a Lie group, G, the Lie bracket, $[u, v] = \mathrm{ad}(u)(v)$, of Definition 9.4 satisfies the axioms of a Lie algebra (given in Definition 9.2). Therefore, $\mathfrak{g}$ with this bracket is a Lie algebra.*

Remark: After proving that $\mathfrak{g}$ is isomorphic to the vector space of left-invariant vector fields on G, we get another proof of Proposition 9.1.

9.2 Left and Right Invariant Vector Fields, the Exponential Map

A fairly convenient way to define the exponential map is to use left-invariant vector fields.

Definition 9.5. If G is a Lie group, a vector field, X, on G is *left-invariant* (resp. *right-invariant*) iff

$$d(L_a)_b(X(b)) = X(L_a(b)) = X(ab), \quad \text{for all } a, b \in G.$$

(resp.

$$d(R_a)_b(X(b)) = X(R_a(b)) = X(ba), \quad \text{for all } a, b \in G.)$$

Equivalently, a vector field, X, is left-invariant iff the following diagram commutes (and similarly for a right-invariant vector field):

$$\begin{tikzcd}
T_bG \arrow[r, "d(L_a)_b"] & T_{ab}G \\
G \arrow[u, "X"] \arrow[r, "L_a"'] & G \arrow[u, "X"']
\end{tikzcd}$$

If X is a left-invariant vector field, setting $b = 1$, we see that

$$X(a) = d(L_a)_1(X(1)),$$

which shows that X is determined by its value, $X(1) \in \mathfrak{g}$, at the identity (and similarly for right-invariant vector fields).

Conversely, given any $v \in \mathfrak{g}$, we can define the vector field, v^L, by

$$v^L(a) = d(L_a)_1(v), \quad \text{for all } a \in G.$$

We claim that v^L is left-invariant. This follows by an easy application of the chain rule:

$$\begin{aligned} v^L(ab) &= d(L_{ab})_1(v) \\ &= d(L_a \circ L_b)_1(v) \\ &= d(L_a)_b(d(L_b)_1(v)) \\ &= d(L_a)_b(v^L(b)). \end{aligned}$$

Furthermore, $v^L(1) = v$.

Therefore, we showed that the map, $X \mapsto X(1)$, establishes an isomorphism between the space of left-invariant vector fields on G and $\mathfrak{g}$.

We denote the vector space of left-invariant vector fields on G by $\mathfrak{g}^L$.

Because the derivative of any Lie group homomorphism is a Lie algebra homomorphism, $(dL_a)_b$ is a Lie algebra homomorphism, so $\mathfrak{g}^L$ is a Lie algebra.

In fact, the map $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^L(a)$ is an isomorphism between $G \times \mathfrak{g}$ and the tangent bundle, TG.

Remark: Given any $v \in \mathfrak{g}$, we can also define the vector field, v^R, by

$$v^R(a) = d(R_a)_1(v), \quad \text{for all } a \in G.$$

It is easily shown that v^R is right-invariant and we also have an isomorphism $G \times \mathfrak{g} \longrightarrow TG$ given by $(a, v) \mapsto v^R(a)$.

We denote the vector space of right-invariant vector fields on G by $\mathfrak{g}^R$.

The vector space $\mathfrak{g}^R$ is also a Lie algebra.

Another reason left-invariant (resp. right-invariant) vector fields on a Lie group are important is that they are complete, i.e., they define a flow whose domain is $\mathbb{R} \times G$. To prove this, we begin with the following easy proposition:

Proposition 9.2. *Given a Lie group, G, if X is a left-invariant (resp. right-invariant) vector field and Φ is its flow, then*

$$\Phi(t, g) = g\Phi(t, 1) \quad (\textit{resp.} \quad \Phi(t, g) = \Phi(t, 1)g),$$

for all $(t, g) \in \mathcal{D}(X)$.

Proposition 9.3. *Given a Lie group, G, for every $v \in \mathfrak{g}$, there is a unique smooth homomorphism, $h_v \colon (\mathbb{R}, +) \to G$, such that $\dot{h}_v(0) = v$. Furthermore, $h_v(t)$ is the maximal integral curve of both v^L and v^R with initial condition 1 and the flows of v^L and v^R are defined for all $t \in \mathbb{R}$.*

Since $h_v \colon (\mathbb{R}, +) \to G$ is a homomorphism, the integral curve, h_v, is often referred to as a *one-parameter group*.

Proposition 9.3 yields the definition of the exponential map.

Definition 9.6. Given a Lie group, G, the *exponential map*, $\exp\colon \mathfrak{g} \to G$, is given by

$$\exp(v) = h_v(1) = \Phi_1^v(1), \quad \text{for all } v \in \mathfrak{g},$$

where Φ_t^v denotes the flow of v^L.

It is not difficult to prove that exp is smooth.

Observe that for any fixed $t \in \mathbb{R}$, the map

$$s \mapsto h_v(st)$$

is a smooth homomorphism, h, such that $\dot{h}(0) = tv$.

By uniqueness, we have

$$h_v(st) = h_{tv}(s).$$

Setting $s = 1$, we find that

$$h_v(t) = \exp(tv), \quad \text{for all } v \in \mathfrak{g} \text{ and all } t \in \mathbb{R}.$$

Then, differentiating with respect to t at $t = 0$, we get

$$v = d\exp_0(v),$$

i.e., $d\exp_0 = \mathrm{id}_{\mathfrak{g}}$.

By the inverse function theorem, exp is a local diffeomorphism at 0. This means that there is some open subset, $U \subseteq \mathfrak{g}$, containing 0, such that the restriction of exp to U is a diffeomorphism onto $\exp(U) \subseteq G$, with $1 \in \exp(U)$.

In fact, by left-translation, the map $v \mapsto g\exp(v)$ is a local diffeomorphism between some open subset, $U \subseteq \mathfrak{g}$, containing 0 and the open subset, $\exp(U)$, containing g.

The exponential map is also natural in the following sense:

Proposition 9.4. *Given any two Lie groups, G and H, for every Lie group homomorphism, $f\colon G \to H$, the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "df_1"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

As useful corollary of Proposition 9.4 is:

Proposition 9.5. *Let G be a connected Lie group and H be any Lie group. For any two homomorphisms, $\phi_1\colon G \to H$ and $\phi_2\colon G \to H$, if $d(\phi_1)_1 = d(\phi_2)_1$, then $\phi_1 = \phi_2$.*

The above proposition shows that if G is connected, then a homomorphism of Lie groups, $\phi\colon G \to H$, is uniquely determined by the Lie algebra homomorphism,
$d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$.

We obtain another useful corollary of Proposition 9.4 when we apply it to the adjoint representation of G,

$$\mathrm{Ad}\colon G \to \mathbf{GL}(\mathfrak{g})$$

and to the conjugation map,

$$\mathbf{Ad}_a\colon G \to G,$$

where $\mathbf{Ad}_a(b) = aba^{-1}$.

In the first case, $d\mathrm{Ad}_1 = \mathrm{ad}$, with $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, and in the second case, $d(\mathbf{Ad}_a)_1 = \mathrm{Ad}_a$.

Proposition 9.6. *Given any Lie group, G, the following properties hold:*

(1)
$$\mathrm{Ad}(\exp(u)) = e^{\mathrm{ad}(u)}, \qquad \text{for all } u \in \mathfrak{g},$$

where $\exp\colon \mathfrak{g} \to G$ *is the exponential of the Lie group, G, and $f \mapsto e^f$ is the exponential map given by*

$$e^f = \sum_{k=0}^{\infty} \frac{f^k}{k!},$$

for any linear map (matrix), $f \in \mathfrak{gl}(\mathfrak{g})$. Equivalently, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}"] & \mathbf{GL}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{ad}"'] & \mathfrak{gl}(\mathfrak{g}). \arrow[u, "f \mapsto e^f"'] \end{tikzcd}$$

(2)

$$\exp(t\mathrm{Ad}_g(u)) = g\exp(tu)g^{-1},$$

for all $u \in \mathfrak{g}$*, all* $g \in G$ *and all* $t \in \mathbb{R}$*. Equivalently, the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\mathbf{Ad}_g} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[\mathrm{Ad}_g]{} & \mathfrak{g}. \end{array}$$

Since the Lie algebra $\mathfrak{g} = T_1G$ is isomorphic to the vector space of left-invariant vector fields on G and since the Lie bracket of vector fields makes sense (see Definition 6.3), it is natural to ask if there is any relationship between, $[u, v]$, where $[u, v] = \mathrm{ad}(u)(v)$, and the Lie bracket, $[u^L, v^L]$, of the left-invariant vector fields associated with $u, v \in \mathfrak{g}$.

The answer is: Yes, they coincide (*via* the correspondence $u \mapsto u^L$).

Proposition 9.7. *Given a Lie group, G, we have*

$$[u^L, v^L](1) = \mathrm{ad}(u)(v), \quad \textit{for all } u, v \in \mathfrak{g}.$$

Proposition 9.7 shows that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^L$ are isomorphic (where $\mathfrak{g}^L$ is the Lie algebra of left-invariant vector fields on G).

In view of this isomorphism, if X and Y are any two left-invariant vector fields on G, we define $\mathrm{ad}(X)(Y)$ by

$$\mathrm{ad}(X)(Y) = [X, Y],$$

where the Lie bracket on the right-hand side is the Lie bracket on vector fields.

If G is a Lie group, let G_0 be the connected component of the identity. We know G_0 is a topological normal subgroup of G and it is a submanifold in an obvious way, so it is a Lie group.

Proposition 9.8. *If G is a Lie group and G_0 is the connected component of 1, then G_0 is generated by* $\exp(\mathfrak{g})$. *Moreover, G_0 is countable at infinity.*

9.3 Homomorphisms of Lie Groups and Lie Algebras, Lie Subgroups

If G and H are two Lie groups and $\phi\colon G \to H$ is a homomorphism of Lie groups, then $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is a linear map between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H.

In fact, it is a Lie algebra homomorphism.

Proposition 9.9. *If G and H are two Lie groups and $\phi\colon G \to H$ is a homomorphism of Lie groups, then*

$$d\phi_1 \circ \mathrm{Ad}_g = \mathrm{Ad}_{\phi(g)} \circ d\phi_1, \quad \textit{for all } g \in G,$$

that is, the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d\phi_1"] \arrow[d, "\mathrm{Ad}_g"'] & \mathfrak{h} \arrow[d, "\mathrm{Ad}_{\phi(g)}"] \\ \mathfrak{g} \arrow[r, "d\phi_1"'] & \mathfrak{h} \end{tikzcd}$$

and $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Remark: If we identify the Lie algebra, $\mathfrak{g}$, of G with the space of left-invariant vector fields on G, the map $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is viewed as the map such that, for every left-invariant vector field, X, on G, the vector field $d\phi_1(X)$ is the unique left-invariant vector field on H such that

$$d\phi_1(X)(1) = d\phi_1(X(1)),$$

i.e., $d\phi_1(X) = d\phi_1(X(1))^L$. Then, we can give another proof of the fact that $d\phi_1$ is a Lie algebra homomorphism.

Proposition 9.10. *If G and H are two Lie groups and $\phi\colon G \to H$ is a homomorphism of Lie groups, if we identify $\mathfrak{g}$ (resp. $\mathfrak{h}$) with the space of left-invariant vector fields on G (resp. left-invariant vector fields on H), then,*

(a) X and $d\phi_1(X)$ are ϕ-related, for every left-invariant vector field, X, on G;

(b) $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

We now consider Lie subgroups. As a preliminary result, note that if $\phi\colon G \to H$ is an injective Lie group homomorphism, then $d\phi_g\colon T_gG \to T_{\phi(g)}H$ is injective for all $g \in G$.

As $\mathfrak{g} = T_1G$ and T_gG are isomorphic for all $g \in G$ (and similarly for $\mathfrak{h} = T_1H$ and T_hH for all $h \in H$), it is sufficient to check that $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$ is injective.

However, by Proposition 9.4, the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "d\phi_1"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

commutes, and since the exponential map is a local diffeomorphism at 0, as ϕ is injective, then $d\phi_1$ is injective, too.

Therefore, if $\phi\colon G \to H$ is injective, it is automatically an immersion.

Definition 9.7. Let G be a Lie group. A set, H, is an *immersed (Lie) subgroup* of G iff

(a) H is a Lie group;

(b) There is an injective Lie group homomorphism, $\phi\colon H \to G$ (and thus, ϕ is an immersion, as noted above).

We say that H is a *Lie subgroup* (or *closed Lie subgroup*) of G iff H is a Lie group that is a subgroup of G and also a submanifold of G.

Observe that an immersed Lie subgroup, H, is an immersed submanifold, since ϕ is an injective immersion.

However, $\phi(H)$ may *not* have the subspace topology inherited from G and $\phi(H)$ may not be closed.

An example of this situation is provided by the 2-torus, $T^2 \cong \mathbf{SO}(2) \times \mathbf{SO}(2)$, which can be identified with the group of 2×2 complex diagonal matrices of the form

$$\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}$$

where $\theta_1, \theta_2 \in \mathbb{R}$.

For any $c \in \mathbb{R}$, let S_c be the subgroup of T^2 consisting of all matrices of the form

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{ict} \end{pmatrix}, \quad t \in \mathbb{R}.$$

It is easily checked that S_c is an immersed Lie subgroup of T^2 iff c is irrational.

However, when c is irrational, one can show that S_c is dense in T^2 but not closed.

As we will see below, a Lie subgroup is always closed.

We borrowed the terminology "immersed subgroup" from Fulton and Harris [21] (Chapter 7), but we warn the reader that most books call such subgroups "Lie subgroups" and refer to the second kind of subgroups (that are submanifolds) as "closed subgroups."

Theorem 9.11. *Let G be a Lie group and let (H, ϕ) be an immersed Lie subgroup of G. Then, ϕ is an embedding iff $\phi(H)$ is closed in G. As as consequence, any Lie subgroup of G is closed.*

Proof. The proof can be found in Warner [53] (Chapter 1, Theorem 3.21) and uses a little more machinery than we have introduced. $\square$

However, we can prove easily that a Lie subgroup, H, of G is closed.

If G is a Lie group, say that a subset, $H \subseteq G$, is an *abstract subgroup* iff it is just a subgroup of the underlying group of G (i.e., we forget the topology and the manifold structure).

Theorem 9.12. *Let G be a Lie group. An abstract subgroup, H, of G is a submanifold (i.e., a Lie subgroup) of G iff H is closed (i.e, H with the induced topology is closed in G).*

9.4 The Correspondence Lie Groups–Lie Algebras

Historically, Lie was the first to understand that a lot of the structure of a Lie group is captured by its Lie algebra, a simpler object (since it is a vector space).

In this short section, we state without proof some of the "Lie theorems," although not in their original form.

Definition 9.8. If $\mathfrak{g}$ is a Lie algebra, a *subalgebra*, $\mathfrak{h}$, of $\mathfrak{g}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$, for all $u, v \in \mathfrak{h}$. If $\mathfrak{h}$ is a (linear) subspace of $\mathfrak{g}$ such that $[u, v] \in \mathfrak{h}$ for all $u \in \mathfrak{h}$ and all $v \in \mathfrak{g}$, we say that $\mathfrak{h}$ is an *ideal* in $\mathfrak{g}$.

For a proof of the theorem below, see Warner [53] (Chapter 3) or Duistermaat and Kolk [19] (Chapter 1, Section 10).

Theorem 9.13. *Let G be a Lie group with Lie algebra, $\mathfrak{g}$, and let (H, ϕ) be an immersed Lie subgroup of G with Lie algebra $\mathfrak{h}$, then $d\phi_1\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*

Conversely, for each subalgebra, $\widetilde{\mathfrak{h}}$, of $\mathfrak{g}$, there is a unique connected immersed subgroup, (H, ϕ), of G so that $d\phi_1\mathfrak{h} = \widetilde{\mathfrak{h}}$. In fact, as a group, $\phi(H)$ is the subgroup of G generated by $\exp(\widetilde{\mathfrak{h}})$.

Furthermore, normal subgroups correspond to ideals.

Theorem 9.13 shows that there is a one-to-one correspondence between connected immersed subgroups of a Lie group and subalgebras of its Lie algebra.

Theorem 9.14. *Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For every homomorphism, $\psi\colon \mathfrak{g} \to \mathfrak{h}$, there is a unique Lie group homomorphism, $\phi\colon G \to H$, so that $d\phi_1 = \psi$.*

Again a proof of the theorem above is given in Warner [53] (Chapter 3) or Duistermaat and Kolk [19] (Chapter 1, Section 10).

Corollary 9.15. *If G and H are connected and simply connected Lie groups, then G and H are isomorphic iff $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

It can also be shown that for every finite-dimensional Lie algebra, $\mathfrak{g}$, there is a connected and simply connected Lie group, G, such that $\mathfrak{g}$ is the Lie algebra of G.

This is a consequence of deep theorem (whose proof is quite hard) known as *Ado's theorem*. For more on this, see Knapp [29], Fulton and Harris [21], or Bourbaki [8].

In summary, following Fulton and Harris, we have the following two principles of the Lie group/Lie algebra correspondence:

First Principle: If G and H are Lie groups, with G connected, then a homomorphism of Lie groups, $\phi\colon G \to H$, is uniquely determined by the Lie algebra homomorphism, $d\phi_1\colon \mathfrak{g} \to \mathfrak{h}$.

Second Principle: Let G and H be Lie groups with G connected and simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras.

A linear map, $\psi\colon \mathfrak{g} \to \mathfrak{h}$, is a Lie algebra map iff there is a unique Lie group homomorphism, $\phi\colon G \to H$, so that $d\phi_1 = \psi$.

9.5 Pseudo-Algebraic Groups

The topological structure of certain linear groups determined by equations among the real and the imaginary parts of their entries can be determined by refining the polar form of matrices.

Such groups are called pseudo-algebraic groups. For example, the groups $\mathbf{SO}(p,q)$ ands $\mathbf{SU}(p,q)$ are pseudo-algebraic.

Consider the group $\mathbf{GL}(n,\mathbb{C})$ of invertible $n\times n$ matrices with complex coefficients. If $A=(a_{kl})$ is such a matrix, denote by x_{kl} the real part (resp. y_{kl}, the imaginary part) of a_{kl} (so, $a_{kl}=x_{kl}+iy_{kl}$).

Definition 9.9. A subgroup G of $\mathbf{GL}(n,\mathbb{C})$ is *pseudo-algebraic* iff there is a finite set of polynomials in $2n^2$ variables with real coefficients
$\{P_j(X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2})\}_{j=1}^t$, so that

$$\begin{aligned}&A=(x_{kl}+iy_{kl})\in G\quad \text{iff}\\ &\quad P_j(x_{11},\ldots,x_{nn},y_{11},\ldots,y_{nn})=0,\quad \text{for } j=1,\ldots,t.\end{aligned}$$

Since a pseudo-algebraic subgroup is the zero locus of a set of polynomials, it is a closed subgroup, and thus a Lie group.

Recall that if A is a complex $n \times n$-matrix, its *adjoint* A^* is defined by $A^* = (\overline{A})^\top$.

Also, $\mathbf{U}(n)$ denotes the group of unitary matrices, i.e., those matrices $A \in \mathbf{GL}(n, \mathbb{C})$ so that $AA^* = A^*A = I$, and $\mathbf{H}(n)$ denotes the vector space of Hermitian matrices i.e., those matrices A so that $A^* = A$.

Proposition 9.16. *Let $P(x_1, \ldots, x_n)$ be a polynomial with real coefficients. For any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, assume that $P(e^{ka_1}, \ldots, e^{ka_n}) = 0$ for all $k \in \mathbb{N}$. Then,*

$$P(e^{ta_1}, \ldots, e^{ta_n}) = 0 \quad \textit{for all } t \in \mathbb{R}.$$

Then, we have the following theorem which is essentially a refined version of the polar decomposition of matrices:

Theorem 9.17. *Let G be a pseudo-algebraic subgroup of $\mathbf{GL}(n, \mathbb{C})$ stable under adjunction (i.e., we have $A^* \in G$ whenever $A \in G$). Then, there is some integer $d \in \mathbb{N}$ so that G is homeomorphic to $(G \cap \mathbf{U}(n)) \times \mathbb{R}^d$. Moreover, if $\mathfrak{g}$ is the Lie algebra of G, the map*

$$(\mathbf{U}(n) \cap G) \times (\mathbf{H}(n) \cap \mathfrak{g}) \longrightarrow G \qquad \text{given by} \quad (U, H) \mapsto Ue^H,$$

is a homeomorphism onto G.

Observe that if G is also compact then $d = 0$, and $G \subseteq \mathbf{U}(n)$.

Remark: A subgroup G of $\mathbf{GL}(n, \mathbb{R})$ is called *algebraic* if there is a finite set of polynomials in n^2 variables with real coefficients $\{P_j(X_1, \ldots, X_{n^2}\}_{j=1}^{t}$, so that

$$A = (x_{kl}) \in G \text{ iff } P_j(x_{11}, \ldots, x_{nn}) = 0, \text{ for } j = 1, \ldots, t.$$

Then, it can be shown that every compact subgroup of $\mathbf{GL}(n, \mathbb{R})$ is algebraic.

The proof is quite involved and uses the existence of the Haar measure on a compact Lie group; see Mneimné and Testard [41] (Theorem 3.7).

9.6 Semidirect Products of Lie Algebras and Lie Groups

If $\mathfrak{a}$ and $\mathfrak{b}$ are two Lie algebras, recall that the direct sum $\mathfrak{a} \oplus \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is $\mathfrak{a} \times \mathfrak{b}$ with the product vector space structure where

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

for all $a_1, a_2 \in \mathfrak{a}$ and all $b_1, b_2 \in \mathfrak{b}$, and

$$\lambda(a, b) = (\lambda a, \lambda b)$$

for all $\lambda \in \mathbb{R}$, all $a \in \mathfrak{a}$, and all $b \in \mathfrak{b}$.

The map $a \mapsto (a, 0)$ is an isomorphism of $\mathfrak{a}$ with the subspace $\{(a, 0) \mid a \in \mathfrak{a}\}$ of $\mathfrak{a} \oplus \mathfrak{b}$ and the map $b \mapsto (0, b)$ is an isomorphism of $\mathfrak{b}$ with the subspace $\{(0, b) \mid b \in \mathfrak{b}\}$ of $\mathfrak{a} \oplus \mathfrak{b}$.

These isomorphisms allow us to identify $\mathfrak{a}$ with the subspace $\{(a, 0) \mid a \in \mathfrak{a}\}$ and $\mathfrak{b}$ with the subspace $\{(0, b) \mid b \in \mathfrak{b}\}$.

We can make the direct sum $\mathfrak{a} \oplus \mathfrak{b}$ into a Lie algebra by defining the Lie bracket $[-, -]$ such that $[a_1, a_2]$ agrees with the Lie bracket on $\mathfrak{a}$ for all $a_1, a_2, \in \mathfrak{a}$, $[b_1, b_2]$ agrees with the Lie bracket on $\mathfrak{b}$ for all $b_1, b_2, \in \mathfrak{b}$, and $[a, b] = [b, a] = 0$ for all $a \in \mathfrak{a}$ and all $b \in \mathfrak{b}$.

This Lie algebra is called the *Lie algebra direct sum* of $\mathfrak{a}$ and $\mathfrak{b}$. Observe that with this Lie algebra structure, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals.

The above construction is sometimes called an "external direct sum" because it does not assume that the constituent Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of some given Lie algebra $\mathfrak{g}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of a given Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum as a vector space and if both $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then for all $a \in \mathfrak{a}$ and all $b \in \mathfrak{b}$, we have $[a, b] \in \mathfrak{a} \cap \mathfrak{b} = (0)$, so $\mathfrak{a} \oplus \mathfrak{b}$ is the Lie algebra direct sum of $\mathfrak{a}$ and $\mathfrak{b}$.

Sometimes, it is called an "internal direct sum."

We now would like to generalize this construction to the situation where the Lie bracket $[a, b]$ of some $a \in \mathfrak{a}$ and some $b \in \mathfrak{b}$ is given in terms of a map from $\mathfrak{b}$ to $\mathrm{Hom}(\mathfrak{a}, \mathfrak{a})$. For this to work, we need to consider derivations.

Definition 9.10. Given a Lie algebra $\mathfrak{g}$, a *derivation* is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ satisfying the following condition:

$$D([X, Y]) = [D(X), Y] + [X, D(Y)], \quad \text{for all } X, Y \in \mathfrak{g}.$$

The vector space of all derivations on $\mathfrak{g}$ is denoted by $\mathrm{Der}(\mathfrak{g})$.

The first thing to observe is that the Jacobi identity can be expressed as

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]],$$

which holds iff

$$(\mathrm{ad}\, Z)[X, Y] = [(\mathrm{ad}\, Z)X, Y] + [X, (\mathrm{ad}\, Z)Y],$$

and the above equation means that $\mathrm{ad}(Z)$ is a derivation.

In fact, it is easy to check that the Jacobi identity holds iff $\mathrm{ad}\, Z$ is a derivation for every $Z \in \mathfrak{g}$.

It tuns out that the vector space of derivations $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra under the commutator bracket.

Proposition 9.18. *For any Lie algebra* $\mathfrak{g}$, *the vector space* $\mathrm{Der}(\mathfrak{g})$ *is a Lie algebra under the commutator bracket. Furthermore, the map* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ *is a Lie algebra homomorphism.*

If $D \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, it is easy to show that

$$[D, \mathrm{ad}\, X] = \mathrm{ad}\,(DX).$$

If $\mathfrak{g}$ is a Lie algebra and if $\mathfrak{a}$ and $\mathfrak{b}$ are subspaces of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b},$$

$\mathfrak{a}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, then for every $B \in \mathfrak{b}$, because $\mathfrak{a}$ is an ideal, the restriction of $\mathrm{ad}\, B$ to $\mathfrak{a}$ leaves $\mathfrak{a}$ invariant, so by Proposition 9.18, the map $B \mapsto \mathrm{ad}\, B \restriction \mathfrak{a}$ is a Lie algebra homomorphism $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$.

Observe that $[B, A] = \tau(B)(A)$, for all $A \in \mathfrak{a}$ and all $B \in \mathfrak{b}$, so the Lie bracket on $\mathfrak{g}$ is completely determined by the Lie brackets on $\mathfrak{a}$ and $\mathfrak{b}$ and the homomorphism τ.

We say that $\mathfrak{g}$ is the *semidirect product* of $\mathfrak{b}$ and $\mathfrak{a}$ and we write

$$\mathfrak{g} = \mathfrak{a} \oplus_\tau \mathfrak{b}.$$

The above is an internal construction. The corresponding external construction is given by the following proposition.

Proposition 9.19. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras, and suppose τ is a Lie algebra homomorphism $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$. Then there is a unique Lie algebra structure on the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ whose Lie bracket agrees with the Lie bracket on $\mathfrak{a}$ and the Lie bracket on $\mathfrak{b}$, and such that*

$$[(0, B), (A, 0)]_{\mathfrak{g}} = \tau(B)(A) \qquad \text{for all } A \in \mathfrak{a} \text{ and all } B \in \mathfrak{b}.$$

The Lie bracket on $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is given by

$$[(A, B), (A', B')]_{\mathfrak{g}} = ([A, A']_{\mathfrak{a}} + \tau(B)(A') - \tau(B')(A),\ [B, B']_{\mathfrak{b}}),$$

for all $A, A' \in \mathfrak{a}$ and all $B, B' \in \mathfrak{b}$. In particular,

$$[(0, B), (A', 0)]_{\mathfrak{g}} = \tau(B)(A') \in \mathfrak{a}.$$

With this Lie algebra structure, $\mathfrak{a}$ is an ideal and $\mathfrak{b}$ is a subalgebra.

The Lie algebra obtained in Proposition 9.19 is denoted by

$$\mathfrak{a} \oplus_\tau \mathfrak{b} \quad \text{or} \quad \mathfrak{a} \rtimes_\tau \mathfrak{b}$$

and is called the *semidirect product of* $\mathfrak{b}$ *by* $\mathfrak{a}$ *with respect to* $\tau\colon \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$.

When τ is the zero map, we get back the Lie algebra direct sum.

Remark: A sequence of Lie algebra maps

$$\mathfrak{a} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\psi} \mathfrak{b}$$

with φ injective, ψ surjective, and with $\mathrm{Im}\, \varphi = \mathrm{Ker}\, \psi = \mathfrak{n}$, is called an *extension of* $\mathfrak{b}$ *by* $\mathfrak{a}$ *with kernel* $\mathfrak{n}$.

If there is a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{p}$, then we say that this extension is *inessential*.

Given a semidirect product $\mathfrak{g} = \mathfrak{a} \rtimes_\tau \mathfrak{b}$ of $\mathfrak{b}$ by $\mathfrak{a}$, if $\varphi\colon \mathfrak{a} \to \mathfrak{g}$ is the map given $\varphi(a) = (a, 0)$ and ψ is the map $\psi\colon \mathfrak{g} \to \mathfrak{b}$ given by $\psi(a, b) = b$, then $\mathfrak{g}$ is an inessential extension of $\mathfrak{b}$ by $\mathfrak{a}$.

Conversely, it is easy to see that every inessential extension of of $\mathfrak{b}$ by $\mathfrak{a}$ is a semidirect product of of $\mathfrak{b}$ by $\mathfrak{a}$.

Let $\mathfrak{g}$ be any Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}_n(\mathbb{R})$, let $\mathfrak{a} = \mathbb{R}^n$ with the zero bracket making $\mathbb{R}^n$ into an abelian Lie algebra.

Then, $\mathrm{Der}(\mathfrak{a}) = \mathfrak{gl}(n, \mathbb{R})$, and we let $\tau\colon \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ be the inclusion map.

The resulting semidirect product $\mathbb{R}^n \rtimes \mathfrak{g}$ is the affine Lie algebra associated with $\mathfrak{g}$. Its Lie bracket is defined by

$$[(u, A), (v, B)] = (Av - Bu, [A, B]).$$

In particular, if $\mathfrak{g} = \mathfrak{so}(n)$, the Lie algebra of $\mathbf{SO}(n)$, then $\mathbb{R}^n \rtimes \mathfrak{so}(n) = \mathfrak{se}(n)$, the Lie algebra of $\mathbf{SE}(n)$.

Before turning our attention to semidirect products of Lie groups, let us consider the group $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra isomorphisms of a Lie algebra $\mathfrak{g}$.

The group $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of the groups $\mathbf{GL}(\mathfrak{g})$ of linear automorphisms of $\mathfrak{g}$, and it is easy to see that it is closed, so it is a Lie group.

Proposition 9.20. *For any (real) Lie algebra $\mathfrak{g}$, the Lie algebra $\mathrm{L}(\mathrm{Aut}(\mathfrak{g}))$ of the group $\mathrm{Aut}(\mathfrak{g})$ is $\mathrm{Der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$.*

We know that Ad is a Lie group homomorphism

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}),$$

and Proposition 9.20 implies that ad is a Lie algebra homomorphism

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}).$$

We now define semidirect products of Lie groups and show how their algebras are semidirect products of Lie algebras.

Proposition 9.21. *Let H and K be two groups and let $\tau\colon K \to \mathrm{Aut}(H)$ be a homomorphism of K into the automorphism group of H. Let $G = H \times K$ with multiplication defined as follows:*

$$(h_1, k_1)(h_2, k_2) = (h_1\tau(k_1)(h_2), k_1k_2),$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K$. Then, the following properties hold:

(1) This multiplication makes G into a group with identity $(1, 1)$ and with inverse given by

$$(h, k)^{-1} = (\tau(k^{-1})(h^{-1}), k^{-1}).$$

(2) The maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms from H to the subgroup $\{(h, 1) \mid h \in H\}$ of G and from K to the subgroup $\{(1, k) \mid k \in K\}$ of G.

(3) Using the isomorphisms from (2), the group H is a normal subgroup of G.

(4) Using the isomorphisms from (2), $H \cap K = (1)$.

(5) For all $h \in H$ an all $k \in K$, we have

$$(1, k)(h, 1)(1, k)^{-1} = (\tau(k)(h), 1).$$

In view of Proposition 9.21, we make the following definition.

Definition 9.11. Let H and K be two groups and let $\tau\colon K \to \mathrm{Aut}(H)$ be a homomorphism of K into the automorphism group of H. The group defined in Proposition 9.21 is called the *semidirect product of K by H with respect to τ*, and it is denoted $H \rtimes_\tau K$ (or even $H \rtimes K$).

Note that $\tau\colon K \to \mathrm{Aut}(H)$ can be viewed as a left action $\cdot\colon K \times H \to H$ of K on H "acting by automorphisms," which means that for every $k \in K$, the map $h \mapsto \tau(k, h)$ is an automorphism of H.

Note that when τ is the trivial homomorphism (that is, $\tau(k) = \mathrm{id}$ for all $k \in K$), the semidirect product is just the direct product $H \times K$ of the groups H and K, and K is also a normal subgroup of G.

Let $H = \mathbb{R}^n$ under addition, let $K = \mathbf{SO}(n)$, and let τ be the inclusion map of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$.

In other words, τ is the action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$ given by $R \cdot u = Ru$.

Then, the semidirect product $\mathbb{R}^n \rtimes \mathbf{SO}(n)$ is isomorphic to the group $\mathbf{SE}(n)$ of direct affine rigid motions of $\mathbb{R}^n$ (translations and rotations), since the multiplication is given by

$$(u, R)(v, S) = (Rv + u, RS).$$

We obtain other affine groups by letting K be $\mathbf{SL}(n)$, $\mathbf{GL}(n)$, *etc.*

As in the case of Lie algebras, a sequence of groups homomorphisms

$$H \xrightarrow{\varphi} G \xrightarrow{\psi} K$$

with φ injective, ψ surjective, and with $\operatorname{Im}\varphi = \operatorname{Ker}\psi = N$, is called an *extension of K by H with kernel N*.

If $H \rtimes_\tau K$ is a semidirect product, we have the homomorphisms $\varphi\colon H \to G$ and $\psi\colon G \to K$ given by

$$\varphi(h) = (h, 1), \qquad \psi(h, k) = k,$$

and it is clear that we have an extension of K by H with kernel $N = \{(h, 1) \mid h \in H\}$. Note that we have a homomorphism $\gamma\colon K \to G$ (a section of ψ) given by

$$\gamma(k) = (1, k),$$

and that

$$\psi \circ \gamma = \mathrm{id}.$$

Conversely, it can be shown that if an extension of K by H has a section $\gamma\colon K \to G$, then G is isomorphic to a semidirect product of K by H with respect to a certain homomorphism τ; find it!

I claim that if H and K are two Lie groups and if the map from $H \times K$ to H given by $(h, k) \mapsto \tau(k)(h)$ is smooth, then the semidirect product $H \rtimes_\tau K$ is a Lie group (see Varadarajan [52] (Section 3.15), Bourbaki [8], (Chapter 3, Section 1.4)). This is because

$$\begin{aligned}
(h_1, k_1)(h_2, k_2)^{-1} &= (h_1, k_1)(\tau(k_2^{-1})(h_2^{-1}), k_2^{-1}) \\
&= (h_1\tau(k_1)(\tau(k_2^{-1})(h_2^{-1})), k_1k_2^{-1}) \\
&= (h_1\tau(k_1k_2^{-1})(h_2^{-1}), k_1k_2^{-1}),
\end{aligned}$$

which shows that multiplication and inversion in $H \rtimes_\tau K$ are smooth.

For every $k \in K$, the derivative of $d(\tau(k))_1$ of $\tau(k)$ at 1 is a Lie algebra isomorphism of $\mathfrak{h}$, and just like Ad, it can be shown that the map $\widetilde{\tau}\colon K \to \mathrm{Aut}(\mathfrak{h})$ given by

$$\widetilde{\tau}(k) = d(\tau(k))_1 \quad k \in K$$

is a smooth homomorphism from K into $\mathrm{Aut}(\mathfrak{h})$.

It follows by Proposition 9.20 that its derivative $d\widetilde{\tau}_1\colon \mathfrak{k} \to \mathrm{Der}(\mathfrak{h})$ at 1 is a homomorphism of $\mathfrak{k}$ into $\mathrm{Der}(\mathfrak{h})$.

Proposition 9.22. *Using the notations just introduced, the Lie algebra of the semidirect product $H \rtimes_\tau K$ of K by H with respect to τ is the semidirect product $\mathfrak{h} \rtimes_{d\widetilde{\tau}_1} \mathfrak{k}$ of $\mathfrak{k}$ by $\mathfrak{h}$ with respect to $d\widetilde{\tau}_1$.*

Proposition 9.22 applied to the semidirect product $\mathbb{R}^n \rtimes_\tau \mathbf{SO}(n) \cong \mathbf{SE}(n)$ where τ is the inclusion map of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$ confirms that $\mathbb{R}^n \rtimes_{d\widetilde{\tau}_1} \mathfrak{so}(n)$ is the Lie algebra of $\mathbf{SE}(n)$, where $d\widetilde{\tau}_1$ is inclusion map of $\mathfrak{so}(n)$ into $\mathfrak{gl}(n, \mathbb{R})$ (and $\widetilde{\tau}$ is the inclusion of $\mathbf{SO}(n)$ into $\mathrm{Aut}(\mathbb{R}^n)$).

As a special case of Proposition 9.22, when our semidirect product is just a direct product $H \times K$ (τ is the trivial homomorphism mapping every $k \in K$ to id), we see that the Lie algebra of $H \times K$ is the Lie algebra direct sum $\mathfrak{h} \oplus \mathfrak{k}$ (where the bracket between elements of $\mathfrak{h}$ and elements of $\mathfrak{k}$ is 0).

9.7 Universal Covering Groups

Every connected Lie group G is a manifold, and as such, from results in Section 7.3, it has a universal cover $\pi\colon \widetilde{G} \to G$, where $\widetilde{G}$ is simply connected.

It is possible to make $\widetilde{G}$ into a group so that $\widetilde{G}$ is a Lie group and π is a Lie group homomorphism.

We content ourselves with a sketch of the construction whose details can be found in Warner [53], Chapter 3.

Consider the map $\alpha\colon \widetilde{G} \times \widetilde{G} \to G$, given by

$$\alpha(\widetilde{a}, \widetilde{b}) = \pi(\widetilde{a})\pi(\widetilde{b})^{-1},$$

for all $\widetilde{a}, \widetilde{b} \in \widetilde{G}$, and pick some $\widetilde{e} \in \pi^{-1}(e)$.

Since $\widetilde{G} \times \widetilde{G}$ is simply connected, it follows by Proposition 7.11 that there is a unique map $\widetilde{\alpha}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ such that

$$\alpha = \pi \circ \widetilde{\alpha} \quad \text{and} \quad \widetilde{e} = \widetilde{\alpha}(\widetilde{e}, \widetilde{e}).$$

For all $\widetilde{a}, \widetilde{b} \in \widetilde{G}$, define

$$\widetilde{b}^{-1} = \widetilde{\alpha}(\widetilde{e}, \widetilde{b}), \qquad \widetilde{a}\widetilde{b} = \widetilde{\alpha}(\widetilde{a}, \widetilde{b}^{-1}). \tag{$*$}$$

Using Proposition 7.11, it can be shown that the above operations make $\widetilde{G}$ into a group, and as $\widetilde{\alpha}$ is smooth, into a Lie group. Moreover, π becomes a Lie group homomorphism.

Theorem 9.23. *Every connected Lie group has a simply connected covering map $\pi\colon \widetilde{G} \to G$, where $\widetilde{G}$ is a Lie group and π is a Lie group homomorphism.*

The group $\widetilde{G}$ is called the *universal covering group* of G.

Consider $D = \ker \pi$. Since the fibres of π are countable The group D is a countable closed normal subgroup of $\widetilde{G}$; that is, a discrete normal subgroup of $\widetilde{G}$.

It follows that $G \cong \widetilde{G}/D$, where $\widetilde{G}$ is a simply connected Lie group and D is a discrete normal subgroup of $\widetilde{G}$.

We conclude this section by stating the following useful proposition whose proof can be found in Warner [53] (Chapter 3, Proposition 3.26):

Proposition 9.24. *Let $\phi\colon G \to H$ be a homomorphism of connected Lie groups. Then ϕ is a covering map iff $d\phi_e\colon \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras.*

For example, we know that $\mathfrak{su}(2) = \mathfrak{so}(3)$, so the homomorphism from $\mathbf{SU}(2)$ to $\mathbf{SO}(3)$ provided by the representation of 3D rotations by the quaternions is a covering map.

9.8 More on the Lorentz Group $\mathbf{SO}_0(n,1)$ ⊛

In this section, we take a closer look at the Lorentz group $\mathbf{SO}_0(n,1)$ and, in particular, at the relationship between $\mathbf{SO}_0(n,1)$ and its Lie algebra, $\mathfrak{so}(n,1)$.

The Lie algebra of $\mathbf{SO}_0(n,1)$ is easily determined by computing the tangent vectors to curves, $t \mapsto A(t)$, on $\mathbf{SO}_0(n,1)$ through the identity, I. Since $A(t)$ satisfies

$$A^\top JA = J,$$

differentiating and using the fact that $A(0) = I$, we get

$$A'^\top J + JA' = 0.$$

Therefore,

$$\mathfrak{so}(n,1) = \{A \in \mathrm{Mat}_{n+1,n+1}(\mathbb{R}) \mid A^\top J + JA = 0\}.$$

This means that JA is skew-symmetric and so,

$$\mathfrak{so}(n,1) = \left\{ \begin{pmatrix} B & u \\ u^\top & 0 \end{pmatrix} \mid u \in \mathbb{R}^n, \quad B^\top = -B \right\}.$$

Observe that every matrix $A \in \mathfrak{so}(n,1)$ can be written uniquely as

$$\begin{pmatrix} B & u \\ u^\top & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0^\top & 0 \end{pmatrix} + \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix},$$

where the first matrix is skew-symmetric, the second one is symmetric, and both belong to $\mathfrak{so}(n,1)$.

Thus, it is natural to define

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0^\top & 0 \end{pmatrix} \mid B^\top = -B \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix} \mid u \in \mathbb{R}^n \right\}.$$

It is immediately verified that both $\mathfrak{k}$ and $\mathfrak{p}$ are subspaces of $\mathfrak{so}(n,1)$ (as vector spaces) and that $\mathfrak{k}$ is a Lie subalgebra isomorphic to $\mathfrak{so}(n)$, but $\mathfrak{p}$ is *not* a Lie subalgebra of $\mathfrak{so}(n,1)$ because it is not closed under the Lie bracket.

Still, we have

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}.$$

Clearly, we have the direct sum decomposition

$$\mathfrak{so}(n,1) = \mathfrak{k} \oplus \mathfrak{p},$$

known as *Cartan decomposition*.

There is also an automorphism of $\mathfrak{so}(n,1)$ known as the *Cartan involution*, namely,

$$\theta(A) = -A^{\top},$$

and we see that

$$\mathfrak{k} = \{A \in \mathfrak{so}(n,1) \mid \theta(A) = A\}$$

and

$$\mathfrak{p} = \{A \in \mathfrak{so}(n,1) \mid \theta(A) = -A\}.$$

Unfortunately, there does not appear to be any simple way of obtaining a formula for $\exp(A)$, where $A \in \mathfrak{so}(n,1)$ (except for small n—there is such a formula for $n=3$ due to Chris Geyer).

However, it is possible to obtain an explicit formula for the matrices in $\mathfrak{p}$. This is because for such matrices, A, if we let $\omega = \|u\| = \sqrt{u^\top u}$, we have

$$A^3 = \omega^2 A.$$

Proposition 9.25. *For every matrix, $A \in \mathfrak{p}$, of the form*

$$A = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix},$$

we have

$$\begin{aligned} e^A &= \begin{pmatrix} I + \frac{(\cosh\omega - 1)}{\omega^2} uu^\top & \frac{\sinh\omega}{\omega} u \\ \frac{\sinh\omega}{\omega} u^\top & \cosh\omega \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{I + \frac{\sinh^2\omega}{\omega^2} uu^\top} & \frac{\sinh\omega}{\omega} u \\ \frac{\sinh\omega}{\omega} u^\top & \cosh\omega \end{pmatrix}. \end{aligned}$$

Now, it clear from the above formula that each e^B, with $B \in \mathfrak{p}$ is a Lorentz boost. Conversely, every Lorentz boost is the exponential of some $B \in \mathfrak{p}$, as shown below.

Proposition 9.26. *Every Lorentz boost,*

$$A = \begin{pmatrix} \sqrt{I + vv^\top} & v \\ v^\top & c \end{pmatrix},$$

with $c = \sqrt{\|v\|^2 + 1}$, *is of the form* $A = e^B$, *for* $B \in \mathfrak{p}$, *i.e., for some* $B \in \mathfrak{so}(n, 1)$ *of the form*

$$B = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}.$$

Remarks:

(1) It is easy to show that the eigenvalues of matrices

$$B = \begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}$$

are 0, with multiplicity $n - 1$, $\|u\|$ and $-\|u\|$. Eigenvectors are also easily determined.

(2) The matrices $B \in \mathfrak{so}(n,1)$ of the form

$$B = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha \\ 0 & \cdots & \alpha & 0 \end{pmatrix}$$

are easily seen to form an abelian Lie subalgebra, $\mathfrak{a}$, of $\mathfrak{so}(n,1)$ (which means that for all $B, C \in \mathfrak{a}$, $[B,C] = 0$, i.e., $BC = CB$).

One will easily check that for any $B \in \mathfrak{a}$, as above, we get

$$e^B = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & \cosh\alpha & \sinh\alpha \\ 0 & \cdots & 0 & \sinh\alpha & \cosh\alpha \end{pmatrix}$$

The matrices of the form e^B, with $B \in \mathfrak{a}$, form an abelian subgroup, A, of $\mathbf{SO}_0(n,1)$ isomorphic to $\mathbf{SO}_0(1,1)$.

As we already know, the matrices $B \in \mathfrak{so}(n,1)$ of the form

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where B is skew-symmetric, form a Lie subalgebra, $\mathfrak{k}$, of $\mathfrak{so}(n,1)$.

Clearly, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(n)$ and using the exponential, we get a subgroup, K, of $\mathbf{SO}_0(n,1)$ isomorphic to $\mathbf{SO}(n)$.

It is also clear that $\mathfrak{k} \cap \mathfrak{a} = (0)$, but $\mathfrak{k} \oplus \mathfrak{a}$ is *not* equal to $\mathfrak{so}(n,1)$. What is the missing piece?

Consider the matrices $N \in \mathfrak{so}(n,1)$ of the form

$$N = \begin{pmatrix} 0 & -u & u \\ u^\top & 0 & 0 \\ u^\top & 0 & 0 \end{pmatrix},$$

where $u \in \mathbb{R}^{n-1}$.

The reader should check that these matrices form an abelian Lie subalgebra, $\mathfrak{n}$, of $\mathfrak{so}(n,1)$ and that

$$\mathfrak{so}(n,1) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

This is the *Iwasawa decomposition* of the Lie algebra $\mathfrak{so}(n,1)$.

Furthermore, the reader should check that every $N \in \mathfrak{n}$ is nilpotent; in fact, $N^3 = 0$. (It turns out that $\mathfrak{n}$ is a nilpotent Lie algebra, see Knapp [29]).

The connected Lie subgroup of $\mathbf{SO}_0(n,1)$ associated with $\mathfrak{n}$ is denoted N and it can be shown that we have the *Iwasawa decomposition* of the Lie group $\mathbf{SO}_0(n,1)$:

$$\mathbf{SO}_0(n,1) = KAN.$$

It is easy to check that $[\mathfrak{a}, \mathfrak{n}] \subseteq \mathfrak{n}$, so $\mathfrak{a} \oplus \mathfrak{n}$ is a Lie subalgebra of $\mathfrak{so}(n,1)$ and $\mathfrak{n}$ is an ideal of $\mathfrak{a} \oplus \mathfrak{n}$.

This implies that N is normal in the group corresponding to $\mathfrak{a} \oplus \mathfrak{n}$, so AN is a subgroup (in fact, solvable) of $\mathbf{SO}_0(n, 1)$.

For more on the Iwasawa decomposition, see Knapp [29].

Observe that the image, $\overline{\mathfrak{n}}$, of $\mathfrak{n}$ under the Cartan involution, θ, is the Lie subalgebra

$$\overline{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & u & u \\ -u^\top & 0 & 0 \\ u^\top & 0 & 0 \end{pmatrix} \mid u \in \mathbb{R}^{n-1} \right\}.$$

It is easy to see that the centralizer of $\mathfrak{a}$ is the Lie subalgebra

$$\mathfrak{m} = \left\{ \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{n+1,n+1}(\mathbb{R}) \mid B \in \mathfrak{so}(n-1) \right\}$$

and the reader should check that

$$\mathfrak{so}(n, 1) = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}.$$

We also have
$$[\mathfrak{m}, \mathfrak{n}] \subseteq \mathfrak{n},$$
so $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{so}(n,1)$.

The group, M, associated with $\mathfrak{m}$ is isomorphic to $\mathbf{SO}(n-1)$ and it can be shown that $B = MAN$ is a subgroup of $\mathbf{SO}_0(n,1)$. In fact,

$$\begin{aligned}\mathbf{SO}_0(n,1)/(MAN) &= KAN/MAN = K/M \\ &= \mathbf{SO}(n)/\mathbf{SO}(n-1) = S^{n-1}.\end{aligned}$$

It is customary to denote the subalgebra $\mathfrak{m} \oplus \mathfrak{a}$ by $\mathfrak{g}_0$, the algebra $\mathfrak{n}$ by $\mathfrak{g}_1$ and $\overline{\mathfrak{n}}$ by $\mathfrak{g}_{-1}$, so that $\mathfrak{so}(n,1) = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$ is also written

$$\mathfrak{so}(n,1) = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

By the way, if $N \in \mathfrak{n}$, then

$$e^N = I + N + \frac{1}{2}N^2,$$

and since $N + \frac{1}{2}N^2$ is also nilpotent, e^N can't be diagonalized when $N \neq 0$.

This provides a simple example of matrices in $\mathbf{SO}_0(n, 1)$ that can't be diagonalized.

Combining Proposition 4.1 and Proposition 9.26, we have the corollary:

Corollary 9.27. *Every matrix $A \in \mathbf{O}(n, 1)$ can be written as*

$$A = \begin{pmatrix} Q & 0 \\ 0 & \epsilon \end{pmatrix} e^{\begin{pmatrix} 0 & u \\ u^\top & 0 \end{pmatrix}}$$

where $Q \in \mathbf{O}(n)$, $\epsilon = \pm 1$ and $u \in \mathbb{R}^n$.

Observe that Corollary 9.27 proves that every matrix, $A \in \mathbf{SO}_0(n,1)$, can be written as

$$A = Pe^S, \quad \text{with } P \in K \cong \mathbf{SO}(n) \text{ and } S \in \mathfrak{p},$$

i.e.,

$$\mathbf{SO}_0(n,1) = K \exp(\mathfrak{p}),$$

a version of the polar decomposition for $\mathbf{SO}_0(n,1)$.

Now, it is known that the exponential map, $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$, is surjective.

So, when $A \in \mathbf{SO}_0(n,1)$, since then $Q \in \mathbf{SO}(n)$ and $\epsilon = +1$, the matrix

$$\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

is the exponential of some skew symmetric matrix

$$C = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{so}(n,1),$$

and we can write $A = e^C e^Z$, with $C \in \mathfrak{k}$ and $Z \in \mathfrak{p}$.

Unfortunately, C and Z generally don't commute, so it is generally not true that $A = e^{C+Z}$.

Thus, we don't get an "easy" proof of the surjectivity of the exponential $\exp\colon \mathfrak{so}(n,1) \to \mathbf{SO}_0(n,1)$.

This is not too surprising because, to the best of our knowledge, proving surjectivity for all n is not a simple matter.

One proof is due to Nishikawa [43] (1983). Nishikawa's paper is rather short, but this is misleading.

Indeed, Nishikawa relies on a classic paper by Djokovic [15], which itself relies heavily on another fundamental paper by Burgoyne and Cushman [11], published in 1977.

Burgoyne and Cushman determine the conjugacy classes for some linear Lie groups and their Lie algebras, where the linear groups arise from an inner product space (real or complex).

This inner product is nondegenerate, symmetric, or hermitian or skew-symmetric of skew-hermitian. Altogether, one has to read over 40 pages to fully understand the proof of surjectivity.

In his introduction, Nishikawa states that he is not aware of any other proof of the surjectivity of the exponential for $\mathbf{SO}_0(n, 1)$.

However, such a proof was also given by Marcel Riesz as early as 1957, in some lectures notes that he gave while visiting the University of Maryland in 1957-1958.

These notes were probably not easily available until 1993, when they were published in book form, with commentaries, by Bolinder and Lounesto [47].

Interestingly, these two proofs use very different methods. The Nishikawa–Djokovic–Burgoyne and Cushman proof makes heavy use of methods in Lie groups and Lie algebra, although not far beyond linear algebra.

Riesz's proof begins with a deep study of the structure of the minimal polynomial of a Lorentz isometry (Chapter III). This is a beautiful argument that takes about 10 pages.

The story is not over, as it takes most of Chapter IV (some 40 pages) to prove the surjectivity of the exponential (actually, Riesz proves other things along the way). In any case, the reader can see that both proofs are quite involved.

It is worth noting that Milnor (1969) also uses techniques very similar to those used by Riesz (in dealing with minimal polynomials of isometries) in his paper on isometries of inner product spaces [38].

What we will do to close this section is to give a relatively simple proof that the exponential map,
$\exp\colon \mathfrak{so}(1, 3) \to \mathbf{SO}_0(1, 3)$, is surjective.

In the case of $\mathbf{SO}_0(1,3)$, we can use the fact that $\mathbf{SL}(2,\mathbb{C})$ is a two-sheeted covering space of $\mathbf{SO}_0(1,3)$, which means that there is a homomorphism, $\phi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$, which is surjective and that $\mathrm{Ker}\ \phi = \{-I, I\}$.

Then, the small miracle is that, although the exponential, $\exp\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$, is *not* surjective, for every $A \in \mathbf{SL}(2,\mathbb{C})$, *either A or $-A$ is in the image of the exponential!*

Proposition 9.28. *Given any matrix*

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

let ω be any of the two complex roots of $a^2 + bc$. If $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Furthermore, every matrix $A \in \mathbf{SL}(2, \mathbb{C})$ is in the image of the exponential map, unless $A = -I + N$, where N is a nonzero nilpotent (i.e., $N^2 = 0$ with $N \neq 0$). Consequently, for any $A \in \mathbf{SL}(2, \mathbb{C})$, either A or $-A$ is of the form e^B, for some $B \in \mathfrak{sl}(2, \mathbb{C})$.

Remark: If we restrict our attention to $\mathbf{SL}(2, \mathbb{R})$, then we have the following proposition that can be used to prove that the exponential map $\exp \colon \mathfrak{so}(1, 2) \to \mathbf{SO}_0(1, 2)$ is surjective:

Proposition 9.29. *Given any matrix*

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

if $a^2 + b > 0$, *then let* $\omega = \sqrt{a^2 + bc} > 0$ *and if* $a^2 + b < 0$, *then let* $\omega = \sqrt{-(a^2 + bc)} > 0$ *(i.e.,* $\omega^2 = -(a^2 + bc)$*). In the first case (*$a^2 + bc > 0$*), we have*

$$e^B = \cosh \omega \, I + \frac{\sinh \omega}{\omega} \, B,$$

and in the second case ($a^2 + bc < 0$*), we have*

$$e^B = \cos \omega \, I + \frac{\sin \omega}{\omega} \, B.$$

If $a^2 + bc = 0$, *then* $e^B = I + B$. *Furthermore, every matrix* $A \in \mathbf{SL}(2, \mathbb{R})$ *whose trace satisfies* $\mathrm{tr}(A) \geq -2$ *is in the image of the exponential map, unless* $A = -I + N$, *where* N *is a nonzero nilpotent (i.e.,* $N^2 = 0$ *with* $N \neq 0$*). Consequently, for any* $A \in \mathbf{SL}(2, \mathbb{R})$, *either* A *or* $-A$ *is of the form* e^B, *for some* $B \in \mathfrak{sl}(2, \mathbb{R})$.

We now return to the relationship between $\mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SO}_0(1,3)$.

In order to define a homomorphism
$\phi\colon \mathbf{SL}(2,\mathbb{C}) \to \mathbf{SO}_0(1,3)$, we begin by defining a linear bijection, h, between $\mathbb{R}^4$ and $\mathbf{H}(2)$, the set of complex 2×2 Hermitian matrices, by

$$(t,x,y,z) \mapsto \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}.$$

Those familiar with quantum physics will recognize a linear combination of the Pauli matrices! The inverse map is easily defined and we leave it as an exercise.

For instance, given a Hermitian matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$t = \frac{a+d}{2},\; x = \frac{a-d}{2}, \quad \text{etc.}$$

Next, for any $A \in \mathbf{SL}(2, \mathbb{C})$, we define a map, $l_A \colon \mathbf{H}(2) \to \mathbf{H}(2)$, *via*

$$S \mapsto ASA^*.$$

(Here, $A^* = \overline{A}^\top$.)

Using the linear bijection $h \colon \mathbb{R}^4 \to \mathbf{H}(2)$ and its inverse, we obtain a map $\mathrm{lor}_A \colon \mathbb{R}^4 \to \mathbb{R}^4$, where

$$\mathrm{lor}_A = h^{-1} \circ l_A \circ h.$$

As ASA^* is hermitian, we see that l_A is well defined. It is obviously linear and since $\det(A) = 1$ (recall, $A \in \mathbf{SL}(2, \mathbb{C})$) and

$$\det \begin{pmatrix} t + x & y - iz \\ y + iz & t - x \end{pmatrix} = t^2 - x^2 - y^2 - z^2,$$

we see that lor_A preserves the Lorentz metric!

Furthermore, it is not hard to prove that $\mathbf{SL}(2, \mathbb{C})$ is connected (use the polar form or analyze the eigenvalues of a matrix in $\mathbf{SL}(2, \mathbb{C})$, for example, as in Duistermatt and Kolk [19] (Chapter 1, Section 1.2)) and that the map

$$\phi\colon A \mapsto \mathrm{lor}_A$$

is a continuous group homomorphism. Thus, the range of ϕ is a connected subgroup of $\mathbf{SO}_0(1, 3)$.

This shows that $\phi\colon \mathbf{SL}(2, \mathbb{C}) \to \mathbf{SO}_0(1, 3)$ is indeed a homomorphism. It remains to prove that it is surjective and that its kernel is $\{I, -I\}$.

Proposition 9.30. *The homomorphism,* $\phi\colon \mathbf{SL}(2, \mathbb{C}) \to \mathbf{SO}_0(1, 3)$, *is surjective and its kernel is* $\{I, -I\}$.

Remark: The group $\mathbf{SL}(2, \mathbb{C})$ is isomorphic to the group $\mathbf{Spin}(1, 3)$, which is a (simply-connected) double-cover of $\mathbf{SO}_0(1, 3)$.

This is a standard result of Clifford algebra theory, see Bröcker and tom Dieck [10] or Fulton and Harris [21]. What we just did is to provide a direct proof of this fact.

We just proved that there is an isomorphism

$$\mathbf{SL}(2, \mathbb{C})/\{I, -I\} \cong \mathbf{SO}_0(1, 3).$$

However, the reader may recall that
$\mathbf{SL}(2, \mathbb{C})/\{I, -I\} = \mathbf{PSL}(2, \mathbb{C}) \cong \mathbf{M\ddot{o}b}^+$.

Therefore, the Lorentz group is isomorphic to the Möbius group.

We now have all the tools to prove that the exponential map, $\exp\colon \mathfrak{so}(1, 3) \to \mathbf{SO}_0(1, 3)$, is surjective.

Theorem 9.31. *The exponential map,*
$\exp\colon \mathfrak{so}(1,3) \to \mathbf{SO}_0(1,3)$, *is surjective.*

Proof. First, recall from Proposition 9.4 that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{SL}(2,\mathbb{C}) \arrow[r, "\phi"] & \mathbf{SO}_0(1,3) \\
\mathfrak{sl}(2,\mathbb{C}) \arrow[u, "\exp"] \arrow[r, "d\phi_1"'] & \mathfrak{so}(1,3) \arrow[u, "\exp"']
\end{tikzcd}.$$

Pick any $A \in \mathbf{SO}_0(1,3)$. By Proposition 9.30, the homomorphism ϕ is surjective and as $\mathrm{Ker}\,\phi = \{I, -I\}$, there exists some $B \in \mathbf{SL}(2,\mathbb{C})$ so that

$$\phi(B) = \phi(-B) = A.$$

Now, by Proposition 9.28, for any $B \in \mathbf{SL}(2,\mathbb{C})$, either B or $-B$ is of the form e^C, for some $C \in \mathfrak{sl}(2,\mathbb{C})$. By the commutativity of the diagram, if we let
$D = d\phi_1(C) \in \mathfrak{so}(1,3)$, we get

$$A = \phi(\pm e^C) = e^{d\phi_1(C)} = e^D,$$

with $D \in \mathfrak{so}(1,3)$, as required. $\square$

Remark: We can restrict the bijection $h\colon \mathbb{R}^4 \to \mathbf{H}(2)$ defined earlier to a bijection between $\mathbb{R}^3$ and the space of real symmetric matrices of the form

$$\begin{pmatrix} t+x & y \\ y & t-x \end{pmatrix}.$$

Then, if we also restrict ourselves to $\mathbf{SL}(2,\mathbb{R})$, for any $A \in \mathbf{SL}(2,\mathbb{R})$ and any symmetric matrix, S, as above, we get a map

$$S \mapsto ASA^{\top}.$$

The reader should check that these transformations correspond to isometries in $\mathbf{SO}_0(1,2)$ and we get a homomorphism, $\phi\colon \mathbf{SL}(2,\mathbb{R}) \to \mathbf{SO}_0(1,2)$.

Then, we have a version of Proposition 9.30 for $\mathbf{SL}(2,\mathbb{R})$ and $\mathbf{SO}_0(1,2)$:

Proposition 9.32. *The homomorphism,* $\phi\colon \mathbf{SL}(2, \mathbb{R}) \to \mathbf{SO}_0(1, 2)$, *is surjective and its kernel is* $\{I, -I\}$.

Using Proposition 9.32 and Proposition 9.29, we get a version of Theorem 9.31 for $\mathbf{SO}_0(1, 2)$:

Theorem 9.33. *The exponential map,* $\exp\colon \mathfrak{so}(1, 2) \to \mathbf{SO}_0(1, 2)$, *is surjective.*

Also observe that $\mathbf{SO}_0(1, 1)$ consists of the matrices of the form

$$A = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$$

and a direct computation shows that

$$e^{\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}.$$

Thus, we see that the map $\exp\colon \mathfrak{so}(1,1) \to \mathbf{SO}_0(1,1)$ is also surjective.

Therefore, we have proved that $\exp\colon \mathfrak{so}(1,n) \to \mathbf{SO}_0(1,n)$ is surjective for $n = 1, 2, 3$.

This actually holds for all $n \geq 1$, but the proof is much more involved, as we already discussed earlier.

Chapter 10

The Derivative of $\exp$ and Dynkin's Formula $\circledast$