Chapter 7

Partitions of Unity, Orientability, Covering Maps ⊛

7.1 Partitions of Unity

To study manifolds, it is often necessary to construct various objects such as functions, vector fields, Riemannian metrics, volume forms, etc., by gluing together items constructed on the domains of charts.

Partitions of unity are a crucial technical tool in this gluing process.

The first step is to define "bump functions" (also called plateau functions). For any $r > 0$, we denote by $B(r)$ the open ball

$$B(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\},$$

and by $\overline{B(r)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq r\}$, its closure.

Given a topological space, X, for any function, $f \colon X \to \mathbb{R}$, the *support of f*, denoted $\operatorname{supp} f$, is the closed set

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Proposition 7.1. *There is a smooth function, $b\colon \mathbb{R}^n \to \mathbb{R}$, so that*

$$b(x) = \begin{cases} 1 & \text{if } x \in \overline{B(1)} \\ 0 & \text{if } x \in \mathbb{R}^n - B(2). \end{cases}$$

Proposition 7.1 yields the following useful technical result:

Proposition 7.2. *Let M be a smooth manifold. For any open subset, $U \subseteq M$, any $p \in U$ and any smooth function, $f\colon U \to \mathbb{R}$, there exist an open subset, V, with $p \in V$ and a smooth function, $\widetilde{f}\colon M \to \mathbb{R}$, defined on the whole of M, so that $\overline{V}$ is compact,*

$$\overline{V} \subseteq U, \qquad \operatorname{supp} \widetilde{f} \subseteq U$$

and

$$\widetilde{f}(q) = f(q), \qquad \text{for all } q \in \overline{V}.$$

If X is a (Hausdorff) topological space, a family, $\{U_\alpha\}_{\alpha\in I}$, of subsets U_α of X is a *cover* (or *covering*) of X iff $X = \bigcup_{\alpha\in I} U_\alpha$.

A cover, $\{U_\alpha\}_{\alpha\in I}$, such that each U_α is open is an *open cover*.

If $\{U_\alpha\}_{\alpha\in I}$ is a cover of X, for any subset, $J \subseteq I$, the subfamily $\{U_\alpha\}_{\alpha\in J}$ is a *subcover* of $\{U_\alpha\}_{\alpha\in I}$ if $X = \bigcup_{\alpha\in J} U_\alpha$, i.e., $\{U_\alpha\}_{\alpha\in J}$ is still a cover of X.

Given a cover $\{V_\beta\}_{\beta\in J}$, we say that a family $\{U_\alpha\}_{\alpha\in I}$ is a *refinement* of $\{V_\beta\}_{\beta\in J}$ if it is a cover and if there is a function, $h\colon I \to J$, so that $U_\alpha \subseteq V_{h(\alpha)}$, for all $\alpha \in I$.

A family $\{U_\alpha\}_{\alpha\in I}$ of subsets of X is *locally finite* iff for every point, $p \in X$, there is some open subset, U, with $p \in U$, so that $U \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in I$.

A space, X, is *paracompact* iff every open cover has an open locally finite refinement.

Remark: Recall that a space, X, is *compact* iff it is Hausdorff and if every open cover has a *finite* subcover. Thus, the notion of paracompactness (due to Jean Dieudonné) is a generalization of the notion of compactness.

Recall that a topological space, X, is *second-countable* if it has a countable basis, i.e., if there is a countable family of open subsets, $\{U_i\}_{i\geq 1}$, so that every open subset of X is the union of some of the U_i's.

A topological space, X, if *locally compact* iff it is Hausdorff and for every $a \in X$, there is some compact subset, K, and some open subset, U, with $a \in U$ and $U \subseteq K$.

As we will see shortly, every locally compact and second-countable topological space is paracompact.

It is important to observe that every manifold (even not second-countable) is locally compact.

Definition 7.1. Let M be a (smooth) manifold. A *partition of unity on M* is a family, $\{f_i\}_{i\in I}$, of smooth functions on M (the index set I may be uncountable) such that

(a) The family of supports, $\{\mathrm{supp}\, f_i\}_{i\in I}$, is locally finite.

(b) For all $i \in I$ and all $p \in M$, we have $0 \leq f_i(p) \leq 1$, and
$$\sum_{i\in I} f_i(p) = 1, \quad \text{for every } p \in M.$$

Note that condition (b) implies that $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a cover of M. If $\{U_\alpha\}_{\alpha\in J}$ is a cover of M, we say that the partition of unity $\{f_i\}_{i\in I}$ is *subordinate* to the cover $\{U_\alpha\}_{\alpha\in J}$ if $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a refinement of $\{U_\alpha\}_{\alpha\in J}$.

When $I = J$ and $\mathrm{supp}\, f_i \subseteq U_i$, we say that $\{f_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in I}$ *with the same index set as the partition of unity*.

In Definition 7.1, by (a), for every $p \in M$, there is some open set, U, with $p \in U$ and U meets only finitely many of the supports, $\mathrm{supp}\, f_i$.

So, $f_i(p) \neq 0$ for only finitely many $i \in I$ and the infinite sum $\sum_{i \in I} f_i(p)$ is well defined.

Proposition 7.3. *Let X be a topological space which is second-countable and locally compact (thus, also Hausdorff). Then, X is paracompact. Moreover, every open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

Remarks:

1. Proposition 7.3 implies that a second-countable, locally compact (Hausdorff) topological space is the union of countably many compact subsets. Thus, X is *countable at infinity*, a notion that we already encountered in Proposition 3.11 and Theorem 3.14.

2. A manifold that is countable at infinity has a countable open cover by domains of charts. It follows that M is second-countable. Thus, for manifolds, second-countable is equivalent to countable at infinity.

Recall that we are assuming that our manifolds are Hausdorff and second-countable.

Theorem 7.4. *Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in I}$ be an open cover for M. Then, there is a countable partition of unity, $\{f_i\}_{i\geq 1}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in I}$ and the support,* $\mathrm{supp}\, f_i$, *of each f_i is compact.*

If one does not require compact supports, then there is a partition of unity, $\{f_\alpha\}_{\alpha\in I}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in I}$ with at most countably many of the f_α not identically zero. (In the second case, $\mathrm{supp}\, f_\alpha \subseteq U_\alpha$.*)*

We close this section by stating a famous theorem of Whitney whose proof uses partitions of unity.

Theorem 7.5. *(Whitney, 1935) Any smooth manifold (Hausdorff and second-countable), M, of dimension n is diffeomorphic to a closed submanifold of $\mathbb{R}^{2n+1}$.*

For a proof, see Hirsch [19], Chapter 2, Section 2, Theorem 2.14.

7.2 Orientation of Manifolds

Although the notion of orientation of a manifold is quite intuitive, it is technically rather subtle.

We restrict our discussion to smooth manifolds (although the notion of orientation can also be defined for topological manifolds, but more work is involved).

Intuitively, a manifold, M, is orientable if it is possible to give a consistent orientation to its tangent space, T_pM, at every point, $p \in M$.

So, if we go around a closed curve starting at $p \in M$, when we come back to p, the orientation of T_pM should be the same as when we started.

For exampe, if we travel on a Möbius strip (a manifold with boundary) dragging a coin with us, we will come back to our point of departure with the coin flipped. Try it!

To be rigorous, we have to say what it means to orient T_pM (a vector space) and what consistency of orientation means.

We begin by quickly reviewing the notion of orientation of a vector space.

Let E be a vector space of dimension n. If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of E, a basic and crucial fact of linear algebra says that there is a unique linear map, g, mapping each u_i to the corresponding v_i (i.e., $g(u_i) = v_i$, $i = 1, \ldots, n$).

Then, look at the determinant, $\det(g)$, of this map. We know that $\det(g) = \det(P)$, where P is the matrix whose j-th columns consist of the coordinates of v_j over the basis $u_1, \ldots, u_n$.

Either $\det(g)$ is negative or it is positive.

Thus, we define an equivalence relation on bases by saying that two bases have the *same orientation* iff the determinant of the linear map sending the first basis to the second has positive determinant.

An *orientation* of E is the choice of one of the two equivalence classes, which amounts to picking some basis as an orientation frame.

The above definition is perfectly fine but it turns out that it is more convenient, in the long term, to use a definition of orientation in terms of alternating multilinear maps (in particular, to define the notion of integration on a manifold).

Recall that a function, $h\colon E^k \to \mathbb{R}$, is *alternating multilinear* (or *alternating k-linear*) iff it is linear in each of its arguments (holding the others fixed) and if

$$h(\ldots, x, \ldots, x, \ldots) = 0,$$

that is, h vanishes whenever two of its arguments are identical.

Using multilinearity, we immediately deduce that h vanishes for all k-tuples of arguments, $u_1, \ldots, u_k$, that are linearly dependent and that h is *skew-symmetric*, i.e.,

$$h(\ldots, y, \ldots, x, \ldots) = -h(\ldots, x, \ldots, y, \ldots).$$

In particular, for $k = n$, it is easy to see that if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases, then

$$h(v_1, \ldots, v_n) = \det(g)h(u_1, \ldots, u_n),$$

where g is the unique linear map sending each u_i to v_i.

This shows that any alternating n-linear function is a multiple of the determinant function and that the space of alternating n-linear maps is a one-dimensional vector space that we will denote $\bigwedge^n E^*$.

We also call an alternating n-linear map on E an *n-form on E*. But then, observe that two bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ have the same orientation iff

$\omega(u_1, \ldots, u_n)$ and $\omega(v_1, \ldots, v_n)$ have the same sign for all $\omega \in \bigwedge^n E^* - \{0\}$

(where 0 denotes the zero n-form).

As $\bigwedge^n E^*$ is one-dimensional, *picking an orientation of E is equivalent to picking a generator (a one-element basis), ω, of $\bigwedge^n E^*$, and to say that $u_1, \ldots, u_n$ has positive orientation iff $\omega(u_1, \ldots, u_n) > 0$.*

Given an orientation (say, given by $\omega \in \bigwedge^n E^*$) of E, a linear map, $f\colon E \to E$, is *orientation preserving* iff $\omega(f(u_1), \ldots, f(u_n)) > 0$ whenever $\omega(u_1, \ldots, u_n) > 0$ (or equivalently, iff $\det(f) > 0$).

Now, to define the orientation of an n-dimensional manifold, M, we use charts.

Given any $p \in M$, for any chart, (U, φ), at p, the tangent map, $d\varphi^{-1}_{\varphi(p)}\colon \mathbb{R}^n \to T_pM$ makes sense.

If $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$, as it gives an orientation to $\mathbb{R}^n$, we can orient T_pM by giving it the orientation induced by the basis $d\varphi^{-1}_{\varphi(p)}(e_1), \ldots, d\varphi^{-1}_{\varphi(p)}(e_n)$.

Then, the consistency of orientations of the T_pM's is given by the overlapping of charts.

We require that the Jacobian determinants of all $\varphi_j \circ \varphi_i^{-1}$ have the same sign, whenever (U_i, φ_i) and (U_j, φ_j) are any two overlapping charts.

Thus, we are led to the definition below. All definitions and results stated in the rest of this section apply to manifolds with or without boundary.

Definition 7.2. Given a smooth manifold, M, of dimension n, an *orientation atlas* of M is any atlas so that the transition maps, $\varphi_i^j = \varphi_j \circ \varphi_i^{-1}$, (from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$) all have a positive Jacobian determinant for every point in $\varphi_i(U_i \cap U_j)$.

A manifold is *orientable* iff its has some orientation atlas.

Definition 7.2 can be hard to check in practice and there is an equivalent criterion is terms of n-forms which is often more convenient.

The idea is that a manifold of dimension n is orientable iff there is a map, $p \mapsto \omega_p$, assigning to every point, $p \in M$, a nonzero n-form, $\omega_p \in \bigwedge^n T_p^* M$, so that this map is smooth.

In order to explain rigorously what it means for such a map to be smooth, we can define the *exterior n-bundle*, $\bigwedge^n T^*M$ (also denoted $\bigwedge_n^* M$) in much the same way that we defined the bundles TM and T^*M.

There is an obvious smooth projection map, $\pi\colon \bigwedge^n T^*M \to M$.

Then, leaving the details of the fact that $\bigwedge^n T^*M$ can be made into a smooth manifold (of dimension n) as an exercise, a smooth map, $p \mapsto \omega_p$, is simply a smooth section of the bundle $\bigwedge^n T^*M$, i.e., a smooth map, $\omega\colon M \to \bigwedge^n T^*M$, so that $\pi \circ \omega = \mathrm{id}$.

Definition 7.3. If M is an n-dimensional manifold, a smooth section, $\omega \in \Gamma(M, \bigwedge^n T^*M)$, is called a (smooth) *$n$-form*. The set of n-forms, $\Gamma(M, \bigwedge^n T^*M)$, is also denoted $\mathcal{A}^n(M)$.

An n-form, ω, is a *nowhere-vanishing n-form on M* or *volume form on M* iff ω_p is a nonzero form for every $p \in M$.

This is equivalent to saying that $\omega_p(u_1, \ldots, u_n) \neq 0$, for all $p \in M$ and all bases, $u_1, \ldots, u_n$, of T_pM.

The determinant function, $(u_1, \ldots, u_n) \mapsto \det(u_1, \ldots, u_n)$, where the u_i are expressed over the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, is a volume form on $\mathbb{R}^n$. We will denote this volume form by ω_0.

Observe the justification for the term volume form: the quantity $\det(u_1, \ldots, u_n)$ is indeed the (signed) volume of the parallelepiped

$$\{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid 0 \leq \lambda_i \leq 1,\ 1 \leq i \leq n\}.$$

A volume form on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is obtained as follows:

$$\omega_p(u_1, \ldots u_n) = \det(p, u_1, \ldots u_n),$$

where $p \in S^n$ and $u_1, \ldots u_n \in T_p S^n$. As the u_i are orthogonal to p, this is indeed a volume form.

Observe that if f is a smooth function on M and ω is any n-form, then $f\omega$ is also an n-form.

Definition 7.4. Let $h\colon M \to N$ be a smooth map of manifolds of the same dimension, n, and let $\omega \in \mathcal{A}^n(N)$ be an n-form on N. The *pull-back, $h^*\omega$, of ω to M* is the n-form on M given by

$$h^*\omega_p(u_1, \ldots, u_n) = \omega_{h(p)}(dh_p(u_1), \ldots, dh_p(u_n)),$$

for all $p \in M$ and all $u_1, \ldots, u_n \in T_pM$.

One checks immediately that $h^*\omega$ is indeed an n-form on M. More interesting is the following Proposition:

Proposition 7.6. *(a) If $h\colon M \to N$ is a local diffeomorphism of manifolds, where $\dim M = \dim N = n$, and $\omega \in \mathcal{A}^n(N)$ is a volume form on N, then $h^*\omega$ is a volume form on M. (b) Assume M has a volume form, ω. Then, for every n-form, $\eta \in \mathcal{A}^n(M)$, there is a unique smooth function, $f \in C^\infty(M)$, so that $\eta = f\omega$. If η is a volume form, then $f(p) \neq 0$ for all $p \in M$.*

Remark: If h_1 and h_2 are smooth maps of manifolds, it is easy to prove that

$$(h_2 \circ h_1)^* = h_1^* \circ h_2^*$$

and that for any smooth map $h\colon M \to N$,

$$h^*(f\omega) = (f \circ h)h^*\omega,$$

where f is any smooth function on N and ω is any n-form on N.

The connection between Definition 7.2 and volume forms is given by the following important theorem whose proof contains a wonderful use of partitions of unity.

Theorem 7.7. *A smooth manifold (Hausdorff and second-countable) is orientable iff it possesses a volume form.*

Since we showed that there is a volume form on the sphere, S^n, by Theorem 7.7, the sphere S^n is orientable.

It can be shown that the projective spaces, $\mathbb{RP}^n$, are non-orientable iff n is even and thus, orientable iff n is odd. In particular, $\mathbb{RP}^2$ is not orientable.

Also, even though M may not be orientable, its tangent bundle, $T(M)$, is always orientable! (Prove it).

It is also easy to show that if $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth submersion, then $M = f^{-1}(0)$ is a smooth orientable manifold.

Another nice fact is that every Lie group is orientable.

By Proposition 7.6 (b), given any two volume forms, ω_1 and ω_2 on a manifold, M, there is a function, $f\colon M \to \mathbb{R}$, never 0 on M such that $\omega_2 = f\omega_1$.

This fact suggests the following definition:

Definition 7.5. Given an orientable manifold, M, two volume forms, ω_1 and ω_2, on M are *equivalent* iff $\omega_2 = f\omega_1$ for some smooth function, $f\colon M \to \mathbb{R}$, such that $f(p) > 0$ for all $p \in M$.

An *orientation of M* is the choice of some equivalence class of volume forms on M and an *oriented manifold* is a manifold together with a choice of orientation.

If M is a manifold oriented by the volume form, ω, for every $p \in M$, a basis, $(b_1, \ldots, b_n)$ of T_pM is *posively oriented* iff $\omega_p(b_1, \ldots, b_n) > 0$, else it is *negatively oriented* (where $n = \dim(M)$).

A connected orientable manifold has two orientations.

We will also need the notion of orientation-preserving diffeomorphism.

Definition 7.6. Let $h\colon M \to N$ be a diffeomorphism of oriented manifolds, M and N, of dimension n and say the orientation on M is given by the volume form ω_1 while the orientation on N is given by the volume form ω_2. We say that h is *orientation preserving* iff $h^*\omega_2$ determines the same orientation of M as ω_1.

Using Definition 7.6 we can define the notion of a positive atlas.

Definition 7.7. If M is a manifold oriented by the volume form, ω, an atlas for M is *positive* iff for every chart, (U, φ), the diffeomorphism, $\varphi\colon U \to \varphi(U)$, is orientation preserving, where U has the orientation induced by M and $\varphi(U) \subseteq \mathbb{R}^n$ has the orientation induced by the standard orientation on $\mathbb{R}^n$ (with $\dim(M) = n$).

The proof of Theorem 7.7 shows

Proposition 7.8. *If a manifold, M, has an orientation atlas, then there is a uniquely determined orientation on M such that this atlas is positive.*

7.3 Covering Maps and Universal Covering Manifolds

Covering maps are an important technical tool in algebraic topology and more generally in geometry.

We begin with covering maps.

Definition 7.8. A map, $\pi \colon M \to N$, between two smooth manifolds is a *covering map* (or *cover*) iff

(1) The map π is smooth and surjective.

(2) For any $q \in N$, there is some open subset, $V \subseteq N$, so that $q \in V$ and
$$\pi^{-1}(V) = \bigcup_{i \in I} U_i,$$
where the U_i are pairwise disjoint open subsets, $U_i \subseteq M$, and $\pi \colon U_i \to V$ is a diffeomorphism for every $i \in I$. We say that V is *evenly covered*.

The manifold, M, is called a *covering manifold* of N.

A *homomorphism* of coverings, $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$, is a smooth map, $\phi\colon M_1 \to M_2$, so that

$$\pi_1 = \pi_2 \circ \phi,$$

that is, the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & M_2 \arrow[dl, "\pi_2"] \\ & N & \end{tikzcd}.$$

We say that the coverings $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are *equivalent* iff there is a homomorphism, $\phi\colon M_1 \to M_2$, between the two coverings and ϕ is a diffeomorphism.

As usual, the inverse image, $\pi^{-1}(q)$, of any element $q \in N$ is called the *fibre over q*, the space N is called the *base* and M is called the *covering space*.

As π is a covering map, each fibre is a discrete space.

Note that a homomorphism maps each fibre $\pi_1^{-1}(q)$ in M_1 to the fibre $\pi_2^{-1}(\phi(q))$ in M_2, for every $q \in M_1$.

Proposition 7.9. *Let $\pi\colon M \to N$ be a covering map. If N is connected, then all fibres, $\pi^{-1}(q)$, have the same cardinality for all $q \in N$. Furthermore, if $\pi^{-1}(q)$ is not finite then it is countably infinite.*

When the common cardinality of fibres is finite it is called the *multiplicity* of the covering (or the number of *sheets*).

For any integer, $n > 0$, the map, $z \mapsto z^n$, from the unit circle $S^1 = \mathbf{U}(1)$ to itself is a covering with n sheets. The map,

$$t\colon \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

is a covering, $\mathbb{R} \to S^1$, with infinitely many sheets.

It is also useful to note that a covering map, $\pi\colon M \to N$, is a local diffeomorphism (which means that $d\pi_p\colon T_pM \to T_{\pi(p)}N$ is a bijective linear map for every $p \in M$).

The crucial property of covering manifolds is that curves in N can be lifted to M, in a unique way. For any map, $\phi\colon P \to N$, a *lift of ϕ through π* is a map, $\widetilde{\phi}\colon P \to M$, so that

$$\phi = \pi \circ \widetilde{\phi},$$

as in the following commutative diagram:

$$\begin{tikzcd}
 & M \arrow[d, "\pi"] \\
P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N
\end{tikzcd}$$

We state without proof the following results:

Proposition 7.10. *If* $\pi\colon M \to N$ *is a covering map, then for every smooth curve,* $\alpha\colon I \to N$*, in* N *(with* $0 \in I$*) and for any point,* $q \in M$*, such that* $\pi(q) = \alpha(0)$*, there is a unique smooth curve,* $\widetilde{\alpha}\colon I \to M$*, lifting* α *through* π *such that* $\widetilde{\alpha}(0) = q$*.*

Proposition 7.11. *Let* $\pi\colon M \to N$ *be a covering map and let* $\phi\colon P \to N$ *be a smooth map. For any* $p_0 \in P$*, any* $q_0 \in M$ *and any* $r_0 \in N$ *with* $\pi(q_0) = \phi(p_0) = r_0$*, the following properties hold:*

(1) If P *is connected then there is at most one lift,* $\widetilde{\phi}\colon P \to M$*, of* ϕ *through* π *such that* $\widetilde{\phi}(p_0) = q_0$*.*

(2) If P *is simply connected, then such a lift exists.*

$$\begin{tikzcd} & M \ni q_0 \arrow[d, "\pi"] \\ p_0 \in P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \ni r_0 \end{tikzcd}$$

Theorem 7.12. *Every connected manifold, M, possesses a simply connected covering map, $\pi\colon \widetilde{M} \to M$, that is, with $\widetilde{M}$ simply connected. Any two simply connected coverings of N are equivalent.*

In view of Theorem 7.12, it is legitimate to speak of *the* simply connected cover, $\widetilde{M}$, of M, also called *universal covering* (or *cover*) of M.

Given any point, $p \in M$, let $\pi_1(M, p)$ denote the fundamental group of M with basepoint p.

If $\phi\colon M \to N$ is a smooth map, for any $p \in M$, if we write $q = \phi(p)$, then we have an induced group homomorphism

$$\phi_*\colon \pi_1(M, p) \to \pi_1(N, q).$$

Proposition 7.13. *If $\pi\colon M \to N$ is a covering map, for every $p \in M$, if $q = \pi(p)$, then the induced homomorphism, $\pi_*\colon \pi_1(M, p) \to \pi_1(N, q)$, is injective.*

Proposition 7.14. *Let $\pi\colon M \to N$ be a covering map and let $\phi\colon P \to N$ be a smooth map. For any $p_0 \in P$, any $q_0 \in M$ and any $r_0 \in N$ with $\pi(q_0) = \phi(p_0) = r_0$, if P is connected, then a lift, $\widetilde{\phi}\colon P \to M$, of ϕ such that $\widetilde{\phi}(p_0) = q_0$ exists iff*

$$\phi_*(\pi_1(P, p_0)) \subseteq \pi_*(\pi_1(M, q_0)),$$

as illustrated in the diagram below

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \end{tikzcd} \qquad \text{iff} \qquad \begin{tikzcd} & \pi_1(M, q_0) \arrow[d, "\pi_*"] \\ \pi_1(P, p_0) \arrow[ur, dashed] \arrow[r, "\phi_*"'] & \pi_1(N, r_0) \end{tikzcd}$$

Basic Assumption: For any covering, $\pi\colon M \to N$, if N is connected then we also assume that M is connected.

Using Proposition 7.13, we get

Proposition 7.15. *If $\pi\colon M \to N$ is a covering map and N is simply connected, then π is a diffeomorphism (recall that M is connected); thus, M is diffeomorphic to the universal cover, $\widetilde{N}$, of N.*

The following proposition shows that the universal covering of a space covers every other covering of that space. This justifies the terminology "*universal covering*."

Proposition 7.16. *Say $\pi_1\colon M_1 \to N$ and $\pi_2\colon M_2 \to N$ are two coverings of N, with N connected. Every homomorphism, $\phi\colon M_1 \to M_2$, between these two coverings is a covering map. As a consequence, if $\pi\colon \widetilde{N} \to N$ is a universal covering of N, then for every covering, $\pi'\colon M \to N$, of N, there is a covering, $\phi\colon \widetilde{N} \to M$, of M.*

The notion of deck-transformation group of a covering is also useful because it yields a way to compute the fundamental group of the base space.

Definition 7.9. If $\pi\colon M \to N$ is a covering map, a *deck-transformation* is any diffeomorphism,
$\phi\colon M \to M$, such that $\pi = \pi \circ \phi$, that is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi"] \\ & N & \end{tikzcd}.$$

Note that deck-transformations are just automorphisms of the covering map.

The commutative diagram of Definition 7.9 means that a deck transformation permutes every fibre. It is immediately verified that the set of deck transformations of a covering map is a group denoted Γ_π (or simply, Γ), called the *deck-transformation group* of the covering.

Observe that any deck transformation, ϕ, is a lift of π through π. Consequently, if M is connected, by Proposition 7.11 (1), every deck-transformation is determined by its value at a single point.

So, the deck-transformations are determined by their action on each point of any fixed fibre, $\pi^{-1}(q)$, with $q \in N$.

Since the fibre $\pi^{-1}(q)$ is countable, Γ is also countable, that is, a discrete Lie group.

Moreover, if M is compact, as each fibre, $\pi^{-1}(q)$, is compact and discrete, it must be finite and so, the deck-transformation group is also finite.

The following proposition gives a useful method for determining the fundamental group of a manifold.

Proposition 7.17. *If $\pi \colon \widetilde{M} \to M$ is the universal covering of a connected manifold, M, then the deck-transformation group, $\widetilde{\Gamma}$, is isomorphic to the fundamental group, $\pi_1(M)$, of M.*

Remark: When $\pi \colon \widetilde{M} \to M$ is the universal covering of M, it can be shown that the group $\widetilde{\Gamma}$ acts simply and transitively on every fibre, $\pi^{-1}(q)$.

This means that for any two elements, $x, y \in \pi^{-1}(q)$, there is a unique deck-transformation, $\phi \in \widetilde{\Gamma}$ such that $\phi(x) = y$.

So, there is a bijection between $\pi_1(M) \cong \widetilde{\Gamma}$ and the fibre $\pi^{-1}(q)$.

Proposition 7.12 together with previous observations implies that if the universal cover of a connected (compact) manifold is compact, then M has a finite fundamental group.

We will use this fact later, in particular, in the proof of Myers' Theorem.

Chapter 8

Construction of Manifolds From Gluing Data $\circledast$