Chapter 11

Riemannian Metrics, Riemannian Manifolds

11.1 Frames

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to the tangent bundle of a manifold.

The idea is to equip the tangent space T_pM at p to the manifold M with an inner product $\langle -, - \rangle_p$, in such a way that these inner products vary smoothly as p varies on M.

It is then possible to define the length of a curve segment on a M and to define the distance between two points on M.

The notion of local (and global) frame plays an important technical role.

Definition 11.1. Let M be an n-dimensional smooth manifold. For any open subset, $U \subseteq M$, an n-tuple of vector fields, $(X_1, \ldots, X_n)$, over U is called a *frame over U* iff $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space, T_pM, for every $p \in U$. If $U = M$, then the X_i are global sections and $(X_1, \ldots, X_n)$ is called a *frame* (of M).

The notion of a frame is due to Élie Cartan who (after Darboux) made extensive use of them under the name of *moving frame* (and the *moving frame method*).

Cartan's terminology is intuitively clear: As a point, p, moves in U, the frame, $(X_1(p), \ldots, X_n(p))$, moves from fibre to fibre. Physicists refer to a frame as a choice of *local gauge*.

If $\dim(M) = n$, then for every chart, (U, φ), since $d\varphi^{-1}_{\varphi(p)} \colon \mathbb{R}^n \to T_pM$ is a bijection for every $p \in U$, the n-tuple of vector fields, $(X_1, \ldots, X_n)$, with $X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, is a frame of TM over U, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.

The following proposition tells us when the tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$):

Proposition 11.1. *The tangent bundle, TM, of a smooth n-dimensional manifold, M, is trivial iff it possesses a frame of global sections (vector fields defined on M).*

As an illustration of Proposition 11.1 we can prove that the tangent bundle, TS^1, of the circle, is trivial.

Indeed, we can find a section that is everywhere nonzero, *i.e.* a non-vanishing vector field, namely

$$X(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta).$$

The reader should try proving that TS^3 is also trivial (use the quaternions).

However, TS^2 is nontrivial, although this not so easy to prove.

More generally, it can be shown that TS^n is nontrivial for all even $n \geq 2$. It can even be shown that S^1, S^3 and S^7 are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Remark: A manifold, M, such that its tangent bundle, TM, is trivial is called *parallelizable*.

We now define Riemannian metrics and Riemannian manifolds.

11.2 Riemannian Metrics

Definition 11.2. Given a smooth n-dimensional manifold, M, a *Riemannian metric on M (or TM)* is a family, $(\langle -, - \rangle_p)_{p \in M}$, of inner products on each tangent space, T_pM, such that $\langle -, - \rangle_p$ depends smoothly on p, which means that for every chart, $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, for every frame, $(X_1, \ldots, X_n)$, on U_α, the maps

$$p \mapsto \langle X_i(p), X_j(p) \rangle_p, \qquad p \in U_\alpha,\ 1 \leq i, j \leq n$$

are smooth. A smooth manifold, M, with a Riemannian metric is called a *Riemannian manifold*.

If $\dim(M) = n$, then for every chart, (U, φ), we have the frame, $(X_1, \ldots, X_n)$, over U, with $X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Since every vector field over U is a linear combination, $\sum_{i=1}^n f_i X_i$, for some smooth functions, $f_i \colon U \to \mathbb{R}$, the condition of Definition 11.2 is equivalent to the fact that the maps,

$$p \mapsto \langle d\varphi^{-1}_{\varphi(p)}(e_i), d\varphi^{-1}_{\varphi(p)}(e_j) \rangle_p, \qquad p \in U,\ 1 \leq i, j \leq n,$$

are smooth.

If we let $x = \varphi(p)$, the above condition says that the maps,

$$x \mapsto \langle d\varphi_x^{-1}(e_i), d\varphi_x^{-1}(e_j)\rangle_{\varphi^{-1}(x)}, \quad x \in \varphi(U), 1 \leq i, j \leq n,$$

are smooth.

If M is a Riemannian manifold, the metric on TM is often denoted $g = (g_p)_{p\in M}$. In a chart, using local coordinates, we often use the notation $g = \sum_{ij} g_{ij} dx_i \otimes dx_j$ or simply $g = \sum_{ij} g_{ij} dx_i dx_j$, where

$$g_{ij}(p) = \left\langle \left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right\rangle_p.$$

For every $p \in U$, the matrix, $(g_{ij}(p))$, is symmetric, positive definite.

The standard Euclidean metric on $\mathbb{R}^n$, namely,

$$g = dx_1^2 + \cdots + dx_n^2,$$

makes $\mathbb{R}^n$ into a Riemannian manifold.

Then, every submanifold, M, of $\mathbb{R}^n$ inherits a metric by restricting the Euclidean metric to M.

For example, the sphere, S^{n-1}, inherits a metric that makes S^{n-1} into a Riemannian manifold. It is a good exercise to find the local expression of this metric for S^2 in spherical coordinates.

A nontrivial example of a Riemannian manifold is the *Poincaré upper half-space*, namely, the set $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

Consider the Lie group $\mathbf{SO}(n)$.

We know from Section 5.2 that its tangent space at the identity $T_I\mathbf{SO}(n)$ is the vector space $\mathfrak{so}(n)$ of $n \times n$ skew symmetric matrices, and that the tangent space $T_Q\mathbf{SO}(n)$ to $\mathbf{SO}(n)$ at Q is isomorphic to

$$Q\mathfrak{so}(n) = \{QB \mid B \in \mathfrak{so}(n)\}.$$

If we give $\mathfrak{so}(n)$ the inner product

$$\langle B_1, B_2 \rangle = \mathrm{tr}(B_1^\top B_2) = -\mathrm{tr}(B_1 B_2),$$

the inner product on $T_Q\mathbf{SO}(n)$ is given by

$$\langle QB_1, QB_2 \rangle = \mathrm{tr}((QB_1)^\top QB_2) = \mathrm{tr}(B_1^\top B_2).$$

We will see in Chapter 13 that the length $L(\gamma)$ of the curve segment γ from I to e^B given by $t \mapsto e^{tB}$ (with $B \in \mathfrak{so}(n)$) is given by

$$L(\gamma) = \Big(\mathrm{tr}(-B^2)\Big)^{\frac{1}{2}}.$$

More generally, given any Lie group G, any inner product $\langle -, - \rangle$ on its Lie algebra $\mathfrak{g}$ induces by left translation an inner product $\langle -, - \rangle_g$ on T_gG for every $g \in G$, and this yields a Riemannian metric on G (which happens to be left-invariant; see Chapter **??**).

Going back to the second example of Section 5.5, where we computed the differential df_R of the function $f\colon \mathbf{SO}(3) \to \mathbb{R}$ given by

$$f(R) = (u^\top R v)^2,$$

we found that

$$df_R(X) = 2u^\top X v u^\top R v, \quad X \in R\mathfrak{so}(3).$$

Since each tangent space $T_R\mathbf{SO}(3)$ is a Euclidean space under the inner product defined above, by duality (see Proposition **??** applied to the pairing $\langle -, - \rangle$), there is a unique vector $Y \in T_R\mathbf{SO}(3)$ defining the linear form df_R; that is,

$$\langle Y, X \rangle = df_R(X), \quad \text{for all } X \in T_R\mathbf{SO}(3).$$

By definition, the vector Y is the *gradient of f at R*, denoted $(\mathrm{grad}(f))_R$.

We leave it as an exercise to prove that the gradient of f at R is given by

$$(\mathrm{grad}(f))_R = u^\top RvR(R^\top uv^\top - vu^\top R).$$

More generally, if $(M, \langle -, - \rangle)$ is a smooth manifold with a Riemannian metric and if $f \colon M \to \mathbb{R}$ is a smooth function on M, the unique smooth vector field $\mathrm{grad}(f)$ defined such that

$$\langle (\mathrm{grad}(f))_p, u\rangle_p = df_p(u), \quad \text{for all } p \in M \text{ and all } u \in T_pM$$

is called the *gradient of f*.

It is usually complicated to find the gradient of a function.

A way to obtain a metric on a manifold, N, is to pull-back the metric, g, on another manifold, M, along a local diffeomorphism, $\varphi\colon N \to M$.

Recall that φ is a local diffeomorphism iff

$$d\varphi_p\colon T_pN \to T_{\varphi(p)}M$$

is a bijective linear map for every $p \in N$.

Given any metric g on M, if φ is a local diffeomorphism, we define the *pull-back metric*, φ^*g, on N induced by g as follows: For all $p \in N$, for all $u, v \in T_pN$,

$$(\varphi^*g)_p(u, v) = g_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We need to check that $(\varphi^*g)_p$ is an inner product, which is very easy since $d\varphi_p$ is a linear isomorphism.

Our map, φ, between the two Riemannian manifolds (N, φ^*g) and (M, g) is a local isometry, as defined below.

Definition 11.3. Given two Riemannian manifolds, (M_1, g_1) and (M_2, g_2), a *local isometry* is a smooth map, $\varphi\colon M_1 \to M_2$, such that $d\varphi_p\colon T_pM_1 \to T_{\varphi(p)}M_2$ is an isometry between the Euclidean spaces $(T_pM_1, (g_1)_p)$ and $(T_{\varphi(p)}M_2, (g_2)_{\varphi(p)})$, for every $p \in M_1$, that is,

$$(g_1)_p(u, v) = (g_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)),$$

for all $u, v \in T_pM_1$ or, equivalently, $\varphi^* g_2 = g_1$. Moreover, φ is an *isometry* iff it is a local isometry and a diffeomorphism.

The isometries of a Riemannian manifold, (M, g), form a group, $\mathrm{Isom}(M, g)$, called the *isometry group of (M, g)*.

An important theorem of Myers and Steenrod asserts that the isometry group, $\mathrm{Isom}(M, g)$, is a Lie group.

Given a map, $\varphi\colon M_1 \to M_2$, and a metric g_1 on M_1, in general, φ does not induce any metric on M_2.

However, if φ has some extra properties, it does induce a metric on M_2. This is the case when M_2 arises from M_1 as a quotient induced by some group of isometries of M_1. For more on this, see Gallot, Hulin and Lafontaine [17], Chapter 2, Section 2.A.

Now, because a manifold is *paracompact* (see Section 7.1), a Riemannian metric always exists on M. This is a consequence of the existence of partitions of unity (see Theorem 7.4).

Theorem 11.2. *Every smooth manifold admits a Riemannian metric.*