Chapter 11

Riemannian Metrics, Riemannian Manifolds

11.1 Frames

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to the tangent bundle of a manifold.

The idea is to equip the tangent space $T_pM$ at $p$ to the manifold $M$ with an inner product $\langle -,- \rangle_p$, in such a way that these inner products vary smoothly as $p$ varies on $M$.

It is then possible to define the length of a curve segment on a $M$ and to define the distance between two points on $M$. 
The notion of local (and global) frame plays an important technical role.

**Definition 11.1.** Let $M$ be an $n$-dimensional smooth manifold. For any open subset, $U \subseteq M$, an $n$-tuple of vector fields, $(X_1, \ldots, X_n)$, over $U$ is called a *frame over $U$* iff $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space, $T_p M$, for every $p \in U$. If $U = M$, then the $X_i$ are global sections and $(X_1, \ldots, X_n)$ is called a *frame* (of $M$).

The notion of a frame is due to Élie Cartan who (after Darboux) made extensive use of them under the name of *moving frame* (and the *moving frame method*).

Cartan’s terminology is intuitively clear: As a point, $p$, moves in $U$, the frame, $(X_1(p), \ldots, X_n(p))$, moves from fibre to fibre. Physicists refer to a frame as a choice of *local gauge*. 
If \( \dim(M) = n \), then for every chart, \((U, \varphi)\), since \(d\varphi^{-1}_{\varphi(p)}: \mathbb{R}^n \to T_pM\) is a bijection for every \( p \in U \), the \( n \)-tuple of vector fields, \((X_1, \ldots, X_n)\), with \(X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)\), is a frame of \( TM \) over \( U \), where \((e_1, \ldots, e_n)\) is the canonical basis of \( \mathbb{R}^n \).

The following proposition tells us when the tangent bundle is trivial (that is, isomorphic to the product, \( M \times \mathbb{R}^n \)):

**Proposition 11.1.** The tangent bundle, \( TM \), of a smooth \( n \)-dimensional manifold, \( M \), is trivial iff it possesses a frame of global sections (vector fields defined on \( M \)).

As an illustration of Proposition 11.1 we can prove that the tangent bundle, \( TS^1 \), of the circle, is trivial.
Indeed, we can find a section that is everywhere nonzero, \textit{i.e.} a non-vanishing vector field, namely

\[ X(\cos \theta, \sin \theta) = (-\sin \theta, \cos \theta). \]

The reader should try proving that $\mathcal{T} S^3$ is also trivial (use the quaternions).

However, $\mathcal{T} S^2$ is nontrivial, although this not so easy to prove.

More generally, it can be shown that $\mathcal{T} S^n$ is nontrivial for all even $n \geq 2$. It can even be shown that $S^1$, $S^3$ and $S^7$ are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

\textbf{Remark:} A manifold, $M$, such that its tangent bundle, $TM$, is trivial is called \textit{parallelizable}.

We now define Riemannian metrics and Riemannian manifolds.
11.2 Riemannian Metrics

**Definition 11.2.** Given a smooth $n$-dimensional manifold, $M$, a *Riemannian metric on $M$ (or $TM$)* is a family, $(\langle -, - \rangle_p)_{p \in M}$, of inner products on each tangent space, $T_pM$, such that $\langle -, - \rangle_p$ depends smoothly on $p$, which means that for every chart, $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$, for every frame, $(X_1, \ldots, X_n)$, on $U_\alpha$, the maps

$$ p \mapsto \langle X_i(p), X_j(p) \rangle_p, \quad p \in U_\alpha, \; 1 \leq i, j \leq n $$

are smooth. A smooth manifold, $M$, with a Riemannian metric is called a *Riemannian manifold*.

If $\dim(M) = n$, then for every chart, $(U, \varphi)$, we have the frame, $(X_1, \ldots, X_n)$, over $U$, with $X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Since every vector field over $U$ is a linear combination, $\sum_{i=1}^n f_i X_i$, for some smooth functions, $f_i: U \to \mathbb{R}$, the condition of Definition 11.2 is equivalent to the fact that the maps,

$$ p \mapsto \langle d\varphi^{-1}_{\varphi(p)}(e_i), d\varphi^{-1}_{\varphi(p)}(e_j) \rangle_p, \quad p \in U, \; 1 \leq i, j \leq n, $$

are smooth.
If we let $x = \varphi(p)$, the above condition says that the maps,

$$x \mapsto \langle d\varphi_x^{-1}(e_i), d\varphi_x^{-1}(e_j) \rangle_{\varphi^{-1}(x)}, \quad x \in \varphi(U), 1 \leq i, j \leq n,$$

are smooth.

If $M$ is a Riemannian manifold, the metric on $TM$ is often denoted $g = (g_p)_{p \in M}$. In a chart, using local coordinates, we often use the notation $g = \sum_{ij} g_{ij} dx_i \otimes dx_j$ or simply $g = \sum_{ij} g_{ij} dx_i dx_j$, where

$$g_{ij}(p) = \left\langle \left( \frac{\partial}{\partial x_i} \right)_p, \left( \frac{\partial}{\partial x_j} \right)_p \right\rangle_p.$$
For every \( p \in U \), the matrix, \((g_{ij}(p))\), is symmetric, positive definite.

The standard Euclidean metric on \( \mathbb{R}^n \), namely,

\[
g = dx_1^2 + \cdots + dx_n^2,
\]

makes \( \mathbb{R}^n \) into a Riemannian manifold.

Then, every submanifold, \( M \), of \( \mathbb{R}^n \) inherits a metric by restricting the Euclidean metric to \( M \).

For example, the sphere, \( S^{n-1} \), inherits a metric that makes \( S^{n-1} \) into a Riemannian manifold. It is a good exercise to find the local expression of this metric for \( S^2 \) in spherical coordinates.

A nontrivial example of a Riemannian manifold is the \textit{Poincaré upper half-space}, namely, the set \( H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \) equipped with the metric

\[
g = \frac{dx^2 + dy^2}{y^2}.
\]
Consider the Lie group $\text{SO}(n)$.

We know from Section 5.2 that its tangent space at the identity $T_I\text{SO}(n)$ is the vector space $\mathfrak{so}(n)$ of $n \times n$ skew symmetric matrices, and that the tangent space $T_Q\text{SO}(n)$ to $\text{SO}(n)$ at $Q$ is isomorphic to

$$Q\mathfrak{so}(n) = \{QB \mid B \in \mathfrak{so}(n)\}.$$ 

If we give $\mathfrak{so}(n)$ the inner product

$$\langle B_1, B_2 \rangle = \text{tr}(B_1^T B_2) = -\text{tr}(B_1 B_2),$$

the inner product on $T_Q\text{SO}(n)$ is given by

$$\langle QB_1, QB_2 \rangle = \text{tr}((QB_1)^TQB_2) = \text{tr}(B_1^T B_2).$$
We will see in Chapter 13 that the length $L(\gamma)$ of the curve segment $\gamma$ from $I$ to $e^B$ given by $t \mapsto e^{tB}$ (with $B \in \mathfrak{so}(n)$) is given by

$$L(\gamma) = \left( \text{tr}(-B^2) \right)^{\frac{1}{2}}.$$

More generally, given any Lie group $G$, any inner product $\langle - , - \rangle$ on its Lie algebra $\mathfrak{g}$ induces by left translation an inner product $\langle - , - \rangle_g$ on $T_gG$ for every $g \in G$, and this yields a Riemannian metric on $G$ (which happens to be left-invariant; see Chapter ??).
Going back to the second example of Section 5.5, where we computed the differential $df_R$ of the function $f : \mathbf{SO}(3) \to \mathbb{R}$ given by

$$f(R) = (u^\top Rv)^2,$$

we found that

$$df_R(X) = 2u^\top Xvu^\top Rv, \quad X \in R_{\mathbf{so}}(3).$$

Since each tangent space $T_R\mathbf{SO}(3)$ is a Euclidean space under the inner product defined above, by duality (see Proposition ?? applied to the pairing $\langle -, - \rangle$), there is a unique vector $Y \in T_R\mathbf{SO}(3)$ defining the linear form $df_R$; that is,

$$\langle Y, X \rangle = df_R(X), \quad \text{for all } X \in T_R\mathbf{SO}(3).$$

By definition, the vector $Y$ is the \emph{gradient of $f$ at $R$}, denoted $(\text{grad}(f))_R$. 

We leave it as an exercise to prove that the gradient of $f$ at $R$ is given by

$$(\text{grad}(f))_R = u^\top R v R (R^\top u v^\top - vu^\top R).$$

More generally, if $(M, \langle \cdot, \cdot \rangle)$ is a smooth manifold with a Riemannian metric and if $f: M \to \mathbb{R}$ is a smooth function on $M$, the unique smooth vector field $\text{grad}(f)$ defined such that

$$\langle (\text{grad}(f))_p, u \rangle_p = df_p(u),$$

for all $p \in M$ and all $u \in T_p M$

is called the \textit{gradient of $f$}.

It is usually complicated to find the gradient of a function.
A way to obtain a metric on a manifold, $N$, is to pull-back the metric, $g$, on another manifold, $M$, along a local diffeomorphism, $\varphi: N \to M$.

Recall that $\varphi$ is a local diffeomorphism iff

$$d\varphi_p: T_pN \to T_{\varphi(p)}M$$

is a bijective linear map for every $p \in N$.

Given any metric $g$ on $M$, if $\varphi$ is a local diffeomorphism, we define the pull-back metric, $\varphi^*g$, on $N$ induced by $g$ as follows: For all $p \in N$, for all $u, v \in T_pN$,

$$(\varphi^*g)_p(u, v) = g_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We need to check that $(\varphi^*g)_p$ is an inner product, which is very easy since $d\varphi_p$ is a linear isomorphism.

Our map, $\varphi$, between the two Riemannian manifolds $(N, \varphi^*g)$ and $(M, g)$ is a local isometry, as defined below.
Definition 11.3. Given two Riemannian manifolds, \((M_1, g_1)\) and \((M_2, g_2)\), a \textit{local isometry} is a smooth map, \(\varphi: M_1 \rightarrow M_2\), such that \(d\varphi_p: T_pM_1 \rightarrow T_{\varphi(p)}M_2\) is an isometry between the Euclidean spaces \((T_pM_1, (g_1)_p)\) and \((T_{\varphi(p)}M_2, (g_2)_{\varphi(p)})\), for every \(p \in M_1\), that is,

\[
(g_1)_p(u, v) = (g_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)),
\]

for all \(u, v \in T_pM_1\) or, equivalently, \(\varphi^*g_2 = g_1\). Moreover, \(\varphi\) is an \textit{isometry} iff it is a local isometry and a diffeomorphism.

The isometries of a Riemannian manifold, \((M, g)\), form a group, \(\text{Isom}(M, g)\), called the \textit{isometry group of \((M, g)\)}.

An important theorem of Myers and Steenrod asserts that the isometry group, \(\text{Isom}(M, g)\), is a Lie group.
Given a map, \( \varphi : M_1 \to M_2 \), and a metric \( g_1 \) on \( M_1 \), in general, \( \varphi \) does not induce any metric on \( M_2 \).

However, if \( \varphi \) has some extra properties, it does induce a metric on \( M_2 \). This is the case when \( M_2 \) arises from \( M_1 \) as a quotient induced by some group of isometries of \( M_1 \). For more on this, see Gallot, Hulin and Lafontaine [17], Chapter 2, Section 2.A.

Now, because a manifold is \textit{paracompact} (see Section 7.1), a Riemannian metric always exists on \( M \). This is a consequence of the existence of partitions of unity (see Theorem 7.4).

\textbf{Theorem 11.2.} \textit{Every smooth manifold admits a Riemannian metric.}