Chapter 5

Riemannian Manifolds and Connections

5.1 Riemannian Metrics

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold.

The notion of local (and global) frame plays an important technical role.

Definition 5.1. Let M be an n-dimensional smooth manifold. For any open subset, $U \subseteq M$, an n-tuple of vector fields, $(X_1, \ldots, X_n)$, over U is called a *frame over U* iff $(X_1(p), \ldots, X_n(p))$ is a basis of the tangent space, T_pM, for every $p \in U$. If $U = M$, then the X_i are global sections and $(X_1, \ldots, X_n)$ is called a *frame* (of M).

The notion of a frame is due to Élie Cartan who (after Darboux) made extensive use of them under the name of *moving frame* (and the *moving frame method*).

Cartan's terminology is intuitively clear: As a point, p, moves in U, the frame, $(X_1(p), \ldots, X_n(p))$, moves from fibre to fibre. Physicists refer to a frame as a choice of *local gauge*.

If $\dim(M) = n$, then for every chart, (U, φ), since $d\varphi^{-1}_{\varphi(p)} \colon \mathbb{R}^n \to T_pM$ is a bijection for every $p \in U$, the n-tuple of vector fields, $(X_1, \ldots, X_n)$, with $X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, is a frame of TM over U, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.

The following proposition tells us when the tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$):

Proposition 5.1. *The tangent bundle, TM, of a smooth n-dimensional manifold, M, is trivial iff it possesses a frame of global sections (vector fields defined on M).*

As an illustration of Proposition 5.1 we can prove that the tangent bundle, TS^1, of the circle, is trivial.

Indeed, we can find a section that is everywhere nonzero, *i.e.* a non-vanishing vector field, namely

$$X(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta).$$

The reader should try proving that TS^3 is also trivial (use the quaternions).

However, TS^2 is nontrivial, although this not so easy to prove.

More generally, it can be shown that TS^n is nontrivial for all even $n \geq 2$. It can even be shown that S^1, S^3 and S^7 are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Remark: A manifold, M, such that its tangent bundle, TM, is trivial is called *parallelizable*.

We now define Riemannian metrics and Riemannian manifolds.

Definition 5.2. Given a smooth n-dimensional manifold, M, a *Riemannian metric on M (or TM)* is a family, $(\langle -, - \rangle_p)_{p \in M}$, of inner products on each tangent space, T_pM, such that $\langle -, - \rangle_p$ depends smoothly on p, which means that for every chart, $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, for every frame, $(X_1, \ldots, X_n)$, on U_α, the maps

$$p \mapsto \langle X_i(p), X_j(p) \rangle_p, \qquad p \in U_\alpha,\ 1 \leq i, j \leq n$$

are smooth. A smooth manifold, M, with a Riemannian metric is called a *Riemannian manifold*.

If $\dim(M) = n$, then for every chart, (U, φ), we have the frame, $(X_1, \ldots, X_n)$, over U, with $X_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Since every vector field over U is a linear combination, $\sum_{i=1}^n f_i X_i$, for some smooth functions, $f_i \colon U \to \mathbb{R}$, the condition of Definition 5.2 is equivalent to the fact that the maps,

$$p \mapsto \langle d\varphi^{-1}_{\varphi(p)}(e_i), d\varphi^{-1}_{\varphi(p)}(e_j)\rangle_p, \qquad p \in U,\ 1 \leq i, j \leq n,$$

are smooth. If we let $x = \varphi(p)$, the above condition says that the maps,

$$x \mapsto \langle d\varphi^{-1}_x(e_i), d\varphi^{-1}_x(e_j)\rangle_{\varphi^{-1}(x)}, \quad x \in \varphi(U), 1 \leq i, j \leq n,$$

are smooth.

If M is a Riemannian manifold, the metric on TM is often denoted $g = (g_p)_{p\in M}$. In a chart, using local coordinates, we often use the notation $g = \sum_{ij} g_{ij} dx_i \otimes dx_j$ or simply $g = \sum_{ij} g_{ij} dx_i dx_j$, where

$$g_{ij}(p) = \left\langle \left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right\rangle_p.$$

For every $p \in U$, the matrix, $(g_{ij}(p))$, is symmetric, positive definite.

The standard Euclidean metric on $\mathbb{R}^n$, namely,

$$g = dx_1^2 + \cdots + dx_n^2,$$

makes $\mathbb{R}^n$ into a Riemannian manifold.

Then, every submanifold, M, of $\mathbb{R}^n$ inherits a metric by restricting the Euclidean metric to M.

For example, the sphere, S^{n-1}, inherits a metric that makes S^{n-1} into a Riemannian manifold. It is a good exercise to find the local expression of this metric for S^2 in polar coordinates.

A nontrivial example of a Riemannian manifold is the *Poincaré upper half-space*, namely, the set
$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

A way to obtain a metric on a manifold, N, is to pull-back the metric, g, on another manifold, M, along a local diffeomorphism, $\varphi\colon N \to M$.

Recall that φ is a local diffeomorphism iff

$$d\varphi_p\colon T_pN \to T_{\varphi(p)}M$$

is a bijective linear map for every $p \in N$.

Given any metric g on M, if φ is a local diffeomorphism, we define the *pull-back metric*, $\varphi^* g$, on N induced by g as follows: For all $p \in N$, for all $u, v \in T_pN$,

$$(\varphi^* g)_p(u, v) = g_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We need to check that $(\varphi^* g)_p$ is an inner product, which is very easy since $d\varphi_p$ is a linear isomorphism.

Our map, φ, between the two Riemannian manifolds $(N, \varphi^* g)$ and (M, g) is a local isometry, as defined below.

Definition 5.3. Given two Riemannian manifolds, (M_1, g_1) and (M_2, g_2), a *local isometry* is a smooth map, $\varphi\colon M_1 \to M_2$, such that $d\varphi_p\colon T_pM_1 \to T_{\varphi(p)}M_2$ is an isometry between the Euclidean spaces $(T_pM_1, (g_1)_p)$ and $(T_{\varphi(p)}M_2, (g_2)_{\varphi(p)})$, for every $p \in M_1$, that is,

$$(g_1)_p(u, v) = (g_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)),$$

for all $u, v \in T_pM_1$ or, equivalently, $\varphi^* g_2 = g_1$. Moreover, φ is an *isometry* iff it is a local isometry and a diffeomorphism.

The isometries of a Riemannian manifold, (M, g), form a group, $\mathrm{Isom}(M, g)$, called the *isometry group of (M, g)*.

An important theorem of Myers and Steenrod asserts that the isometry group, $\mathrm{Isom}(M, g)$, is a Lie group.

Given a map, $\varphi\colon M_1 \to M_2$, and a metric g_1 on M_1, in general, φ does not induce any metric on M_2.

However, if φ has some extra properties, it does induce a metric on M_2. This is the case when M_2 arises from M_1 as a quotient induced by some group of isometries of M_1. For more on this, see Gallot, Hulin and Lafontaine [17], Chapter 2, Section 2.A.

Now, because a manifold is *paracompact* (see Section 3.6), a Riemannian metric always exists on M. This is a consequence of the existence of partitions of unity (see Theorem 3.32).

Theorem 5.2. *Every smooth manifold admits a Riemannian metric.*

5.2 Connections on Manifolds

Given a manifold, M, in general, for any two points, $p, q \in M$, there is no "natural" isomorphism between the tangent spaces T_pM and T_qM.

Given a curve, $c\colon [0,1] \to M$, on M as $c(t)$ moves on M, how does the tangent space, $T_{c(t)}M$ change as $c(t)$ moves?

If $M = \mathbb{R}^n$, then the spaces, $T_{c(t)}\mathbb{R}^n$, are canonically isomorphic to $\mathbb{R}^n$ and any vector, $v \in T_{c(0)}\mathbb{R}^n \cong \mathbb{R}^n$, is simply moved along c by *parallel transport*, that is, at $c(t)$, the tangent vector, v, also belongs to $T_{c(t)}\mathbb{R}^n$.

However, if M is curved, for example, a sphere, then it is not obvious how to "parallel transport" a tangent vector at $c(0)$ along a curve c.

A way to achieve this is to define the notion of *parallel vector field* along a curve and this, in turn, can be defined in terms of the notion of *covariant derivative* of a vector field.

Assume for simplicity that M is a surface in $\mathbb{R}^3$. Given any two vector fields, X and Y defined on some open subset, $U \subseteq \mathbb{R}^3$, for every $p \in U$, the *directional derivative, $D_X Y(p)$, of Y with respect to X* is defined by

$$D_X Y(p) = \lim_{t \to 0} \frac{Y(p + tX(p)) - Y(p)}{t}.$$

If $f \colon U \to \mathbb{R}$ is a differentiable function on U, for every $p \in U$, the *directional derivative, $X[f](p)$ (or $X(f)(p)$), of f with respect to X* is defined by

$$X[f](p) = \lim_{t \to 0} \frac{f(p + tX(p)) - f(p)}{t}.$$

We know that $X[f](p) = df_p(X(p))$.

It is easily shown that $D_X Y(p)$ is $\mathbb{R}$-bilinear in X and Y, is $C^\infty(U)$-linear in X and satisfies the Leibniz derivation rule with respect to Y, that is:

Proposition 5.3. *The directional derivative of vector fields satisfies the following properties:*

$$\begin{aligned}
D_{X_1+X_2}Y(p) &= D_{X_1}Y(p) + D_{X_2}Y(p) \\
D_{fX}Y(p) &= fD_XY(p) \\
D_X(Y_1+Y_2)(p) &= D_XY_1(p) + D_XY_2(p) \\
D_X(fY)(p) &= X[f](p)Y(p) + f(p)D_XY(p),
\end{aligned}$$

for all $X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(U)$ and all $f \in C^\infty(U)$.

Now, if $p \in U$ where $U \subseteq M$ is an open subset of M, for any vector field, Y, defined on U ($Y(p) \in T_pM$, for all $p \in U$), for every $X \in T_pM$, the directional derivative, $D_XY(p)$, makes sense and it has an orthogonal decomposition,

$$D_XY(p) = \nabla_XY(p) + (D_n)_XY(p),$$

where its *horizontal (or tangential) component* is $\nabla_XY(p) \in T_pM$ and its normal component is $(D_n)_XY(p)$.

The component, $\nabla_X Y(p)$, is the *covariant derivative* of Y with respect to $X \in T_pM$ and it allows us to define the covariant derivative of a vector field, $Y \in \mathfrak{X}(U)$, with respect to a vector field, $X \in \mathfrak{X}(M)$, on M.

We easily check that $\nabla_X Y$ satisfies the four equations of Proposition 5.3.

In particular, Y, may be a vector field associated with a curve, $c\colon [0,1] \to M$.

A *vector field along a curve, c,* is a vector field, Y, such that $Y(c(t)) \in T_{c(t)}M$, for all $t \in [0,1]$. We also write $Y(t)$ for $Y(c(t))$.

Then, we say that *Y is parallel along c* iff $\nabla_{c'(t)}Y = 0$ along c.

The notion of *parallel transport* on a surface can be defined using parallel vector fields along curves. Let p, q be any two points on the surface M and assume there is a curve, $c\colon [0,1] \to M$, joining $p = c(0)$ to $q = c(1)$.

Then, using the uniqueness and existence theorem for ordinary differential equations, it can be shown that for any initial tangent vector, $Y_0 \in T_pM$, there is a unique parallel vector field, Y, along c, with $Y(0) = Y_0$.

If we set $Y_1 = Y(1)$, we obtain a linear map, $Y_0 \mapsto Y_1$, from T_pM to T_qM which is also an isometry.

As a summary, given a surface, M, if we can define a notion of covariant derivative, $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, satisfying the properties of Proposition 5.3, then we can define the notion of parallel vector field along a curve and the notion of parallel transport, which yields a natural way of relating two tangent spaces, T_pM and T_qM, using curves joining p and q.

This can be generalized to manifolds using the notion of connection. We will see that the notion of connection induces the notion of curvature. Moreover, if M has a Riemannian metric, we will see that this metric induces a unique connection with two extra properties (the *Levi-Civita* connection).

Definition 5.4. Let M be a smooth manifold. A *connection* on M is a $\mathbb{R}$-bilinear map,

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

where we write $\nabla_X Y$ for $\nabla(X, Y)$, such that the following two conditions hold:

$$\begin{aligned} \nabla_{fX} Y &= f \nabla_X Y \\ \nabla_X(fY) &= X[f]Y + f \nabla_X Y, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and all $f \in C^\infty(M)$. The vector field, $\nabla_X Y$, is called the *covariant derivative of Y with respect to X*.

A connection on M is also known as an *affine connection* on M.

A basic property of ∇ is that it is a *local operator*.

Proposition 5.4. *Let M be a smooth manifold and let ∇ be a connection on M. For every open subset, $U \subseteq M$, for every vector field, $Y \in \mathfrak{X}(M)$, if $Y \equiv 0$ on U, then $\nabla_X Y \equiv 0$ on U for all $X \in \mathfrak{X}(M)$, that is, ∇ is a local operator.*

Proposition 5.4 implies that a connection, ∇, on M, restricts to a connection, $\nabla \restriction U$, on every open subset, $U \subseteq M$.

It can also be shown that $(\nabla_X Y)(p)$ only depends on $X(p)$, that is, for any two vector fields, $X, Y \in \mathfrak{X}(M)$, if $X(p) = Y(p)$ for some $p \in M$, then

$$(\nabla_X Z)(p) = (\nabla_Y Z)(p) \qquad \text{for every } Z \in \mathfrak{X}(M).$$

Consequently, for any $p \in M$, the covariant derivative, $(\nabla_u Y)(p)$, is well defined for any tangent vector, $u \in T_p M$, and any vector field, Y, defined on some open subset, $U \subseteq M$, with $p \in U$.

Observe that on U, the n-tuple of vector fields, $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, is a local frame.

We can write

$$\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right) = \sum_{k=1}^{n} \Gamma^k_{ij}\, \frac{\partial}{\partial x_k},$$

for some unique smooth functions, Γ^k_{ij}, defined on U, called the *Christoffel symbols*.

We say that a connection, ∇, is *flat* on U iff

$$\nabla_X\left(\frac{\partial}{\partial x_i}\right) = 0, \quad \text{for all} \quad X \in \mathfrak{X}(U),\ 1 \leq i \leq n.$$

Proposition 5.5. *Every smooth manifold, M, possesses a connection.*

Proof. We can find a family of charts, $(U_\alpha, \varphi_\alpha)$, such that $\{U_\alpha\}_\alpha$ is a locally finite open cover of M. If (f_α) is a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ and if ∇^α is the flat connection on U_α, then it is immediately verified that

$$\nabla = \sum_\alpha f_\alpha \nabla^\alpha$$

is a connection on M. $\square$

Remark: A connection on TM can be viewed as a linear map,

$$\nabla \colon \mathfrak{X}(M) \longrightarrow \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), (\mathfrak{X}(M)),$$

such that, for any fixed $Y \in \mathcal{X}(M)$, the map,
$\nabla Y \colon X \mapsto \nabla_X Y$, is $C^\infty(M)$-linear, which implies that ∇Y is a $(1, 1)$ tensor.

5.3 Parallel Transport

The notion of connection yields the notion of parallel transport. First, we need to define the covariant derivative of a vector field along a curve.

Definition 5.5. Let M be a smooth manifold and let $\gamma\colon [a, b] \to M$ be a smooth curve in M. A *smooth vector field along the curve γ* is a smooth map,
$X\colon [a, b] \to TM$, such that $\pi(X(t)) = \gamma(t)$, for all
$t \in [a, b]$ ($X(t) \in T_{\gamma(t)}M$).

Recall that the curve, $\gamma\colon [a, b] \to M$, is smooth iff γ is the restriction to $[a, b]$ of a smooth curve on some open interval containing $[a, b]$.

Since a vector X field along a curve γ does not necessarily extend to an open subset of M (for example, if the image of γ is dense in M), the covariant derivative $(\nabla_{\gamma'(t_0)} X)_{\gamma(t_0)}$ may not be defined, so we need a proposition showing that the covariant derivative of a vector field along a curve makes sense.

Proposition 5.6. *Let M be a smooth manifold, let ∇ be a connection on M and $\gamma\colon [a,b] \to M$ be a smooth curve in M. There is a $\mathbb{R}$-linear map, D/dt, defined on the vector space of smooth vector fields, X, along γ, which satisfies the following conditions:*

(1) For any smooth function, $f\colon [a,b] \to \mathbb{R}$,

$$\frac{D(fX)}{dt} = \frac{df}{dt}X + f\frac{DX}{dt}$$

(2) If X is induced by a vector field, $Z \in \mathfrak{X}(M)$, that is, $X(t_0) = Z(\gamma(t_0))$ for all $t_0 \in [a,b]$, then $\frac{DX}{dt}(t_0) = (\nabla_{\gamma'(t_0)} Z)_{\gamma(t_0)}$.

Proof. Since $\gamma([a,b])$ is compact, it can be covered by a finite number of open subsets, U_α, such that $(U_\alpha, \varphi_\alpha)$ is a chart. Thus, we may assume that $\gamma\colon [a,b] \to U$ for some chart, (U, φ). As $\varphi\circ\gamma\colon [a,b] \to \mathbb{R}^n$, we can write

$$\varphi\circ\gamma(t) = (u_1(t), \ldots, u_n(t)),$$

where each $u_i = pr_i \circ \varphi \circ \gamma$ is smooth. Now, it is easy to see that

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{du_i}{dt} \left(\frac{\partial}{\partial x_i}\right)_{\gamma(t_0)}.$$

If $(s_1, \ldots, s_n)$ is a frame over U, we can write

$$X(t) = \sum_{i=1}^{n} X_i(t) s_i(\gamma(t)),$$

for some smooth functions, X_i.

Then, conditions (1) and (2) imply that

$$\frac{DX}{dt} = \sum_{j=1}^{n} \left(\frac{dX_j}{dt}\, s_j(\gamma(t)) + X_j(t) \nabla_{\gamma'(t)}(s_j(\gamma(t))) \right)$$

and since

$$\gamma'(t) = \sum_{i=1}^{n} \frac{du_i}{dt} \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)},$$

there exist some smooth functions, Γ_{ij}^k, so that

$$\begin{aligned} \nabla_{\gamma'(t)}(s_j(\gamma(t))) &= \sum_{i=1}^{n} \frac{du_i}{dt}\, \nabla_{\frac{\partial}{\partial x_i}}(s_j(\gamma(t))) \\ &= \sum_{i,k} \frac{du_i}{dt}\, \Gamma_{ij}^k s_k(\gamma(t)). \end{aligned}$$

It follows that

$$\frac{DX}{dt} = \sum_{k=1}^{n} \left(\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt}\, X_j \right) s_k(\gamma(t)).$$

Conversely, the above expression defines a linear operator, D/dt, and it is easy to check that it satisfies (1) and (2). $\square$

The operator, D/dt is often called *covariant derivative along γ* and it is also denoted by $\nabla_{\gamma'(t)}$ or simply $\nabla_{\gamma'}$.

Definition 5.6. Let M be a smooth manifold and let ∇ be a connection on M. For every curve, $\gamma\colon [a, b] \to M$, in M, a vector field, X, along γ is *parallel (along γ)* iff

$$\frac{DX}{dt}(s) = 0 \quad \text{for all } s \in [a, b].$$

If M was embedded in $\mathbb{R}^d$, for some d, then to say that X is parallel along γ would mean that the directional derivative, $(D_{\gamma'}X)(\gamma(t))$, is normal to $T_{\gamma(t)}M$.

The following proposition can be shown using the existence and uniqueness of solutions of ODE's (in our case, linear ODE's) and its proof is omitted:

Proposition 5.7. *Let M be a smooth manifold and let ∇ be a connection on M. For every C^1 curve, $\gamma\colon [a,b] \to M$, in M, for every $t \in [a,b]$ and every $v \in T_{\gamma(t)}M$, there is a unique parallel vector field, X, along γ such that $X(t) = v$.*

For the proof of Proposition 5.7 it is sufficient to consider the portions of the curve γ contained in some chart. In such a chart, (U, φ), as in the proof of Proposition 5.6, using a local frame, $(s_1, \ldots, s_n)$, over U, we have

$$\frac{DX}{dt} = \sum_{k=1}^{n} \left(\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^{k} \, \frac{du_i}{dt} \, X_j \right) s_k(\gamma(t)),$$

with $u_i = pr_i \circ \varphi \circ \gamma$. Consequently, X is parallel along our portion of γ iff the system of linear ODE's in the unknowns, X_k,

$$\frac{dX_k}{dt} + \sum_{ij} \Gamma_{ij}^{k} \, \frac{du_i}{dt} \, X_j = 0, \qquad k = 1, \ldots, n,$$

is satisfied.

Remark: Proposition 5.7 can be extended to piecewise C^1 curves.

Definition 5.7. Let M be a smooth manifold and let ∇ be a connection on M. For every curve, $\gamma\colon [a,b] \to M$, in M, for every $t \in [a,b]$, the *parallel transport from $\gamma(a)$ to $\gamma(t)$ along γ* is the linear map from $T_{\gamma(a)}M$ to $T_{\gamma(t)}M$, which associates to any $v \in T_{\gamma(a)}M$ the vector, $X_v(t) \in T_{\gamma(t)}M$, where X_v is the unique parallel vector field along γ with $X_v(a) = v$.

The following proposition is an immediate consequence of properties of linear ODE's:

Proposition 5.8. *Let M be a smooth manifold and let ∇ be a connection on M. For every C^1 curve, $\gamma\colon [a,b] \to M$, in M, the parallel transport along γ defines for every $t \in [a,b]$ a linear isomorphism, $P_\gamma\colon T_{\gamma(a)}M \to T_{\gamma(t)}M$, between the tangent spaces, $T_{\gamma(a)}M$ and $T_{\gamma(t)}M$.*

In particular, if γ is a closed curve, that is, if $\gamma(a) = \gamma(b) = p$, we obtain a linear isomorphism, P_γ, of the tangent space, T_pM, called the *holonomy of γ*. The *holonomy group of ∇ based at p*, denoted $\mathrm{Hol}_p(\nabla)$, is the subgroup of $\mathrm{GL}(n, \mathbb{R})$ (where n is the dimension of the manifold M) given by

$$\begin{aligned}\mathrm{Hol}_p(\nabla) \;=\; \{P_\gamma \in \mathrm{GL}(n, \mathbb{R}) \mid \\ \gamma \text{ is a closed curve based at } p\}.\end{aligned}$$

If M is connected, then $\mathrm{Hol}_p(\nabla)$ depends on the basepoint $p \in M$ up to conjugation and so $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}_q(\nabla)$ are isomorphic for all $p, q \in M$. In this case, it makes sense to talk about the *holonomy group of ∇*. By abuse of language, we call $\mathrm{Hol}_p(\nabla)$ the *holonomy group of M*.

5.4 Connections Compatible with a Metric; Levi-Civita Connections

If a Riemannian manifold, M, has a metric, then it is natural to define when a connection, ∇, on M is compatible with the metric.

Given any two vector fields, $Y, Z \in \mathfrak{X}(M)$, the smooth function $\langle Y, Z\rangle$ is defined by

$$\langle Y, Z\rangle(p) = \langle Y_p, Z_p\rangle_p,$$

for all $p \in M$.

Definition 5.8. Given any metric, $\langle -, -\rangle$, on a smooth manifold, M, a connection, ∇, on M is *compatible with the metric*, for short, a *metric connection* iff

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for all vector fields, $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 5.9. *Let M be a Riemannian manifold with a metric, $\langle -, - \rangle$. Then, M, possesses metric connections.*

Proof. For every chart, $(U_\alpha, \varphi_\alpha)$, we use the Gram-Schmidt procedure to obtain an orthonormal frame over U_α and we let ∇^α be the trivial connection over U_α. By construction, ∇^α is compatible with the metric. We finish the argumemt by using a partition of unity, leaving the details to the reader. $\square$

We know from Proposition 5.9 that metric connections on TM exist. However, there are many metric connections on TM and none of them seems more relevant than the others.

It is remarkable that if we require a certain kind of symmetry on a metric connection, then it is uniquely determined.

Such a connection is known as the *Levi-Civita connection*. The Levi-Civita connection can be characterized in several equivalent ways, a rather simple way involving the notion of torsion of a connection.

There are two *error terms* associated with a connection. The first one is the *curvature*,

$$R(X,Y) = \nabla_{[X,Y]} + \nabla_Y \nabla_X - \nabla_X \nabla_Y.$$

The second natural error term is the *torsion*, $T(X,Y)$, of the connection, ∇, given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

which measures the failure of the connection to behave like the Lie bracket.

Proposition 5.10. *(Levi-Civita, Version 1) Let M be any Riemannian manifold. There is a unique, metric, torsion-free connection, ∇, on M, that is, a connection satisfying the conditions*

$$\begin{aligned} X(\langle Y, Z\rangle) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ \nabla_X Y - \nabla_Y X &= [X, Y], \end{aligned}$$

for all vector fields, $X, Y, Z \in \mathfrak{X}(M)$. This connection is called the Levi-Civita connection (or canonical connection) on M. Furthermore, this connection is determined by the Koszul formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) \\ &\quad - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle - \langle Z, [Y, X]\rangle. \end{aligned}$$

Proof. First, we prove uniqueness. Since our metric is a non-degenerate bilinear form, it suffices to prove the Koszul formula. As our connection is compatible with the metric, we have

$$\begin{aligned} X(\langle Y, Z\rangle) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y(\langle X, Z\rangle) &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\ -Z(\langle X, Y\rangle) &= -\langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \end{aligned}$$

and by adding up the above equations, we get

$$\begin{aligned} X(\langle Y, Z\rangle) + Y(\langle X, Z\rangle) &- Z(\langle X, Y\rangle) = \\ &\langle Y, \nabla_X Z - \nabla_Z X\rangle \\ &+ \langle X, \nabla_Y Z - \nabla_Z Y\rangle \\ &+ \langle Z, \nabla_X Y + \nabla_Y X\rangle. \end{aligned}$$

Then, using the fact that the torsion is zero, we get

$$\begin{aligned} X(\langle Y, Z\rangle) &+ Y(\langle X, Z\rangle) - Z(\langle X, Y\rangle) = \\ &\langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \\ &+ \langle Z, [Y, X]\rangle + 2\langle Z, \nabla_X Y\rangle \end{aligned}$$

which yields the Koszul formula.

We will not prove existence here. The reader should consult the standard texts for a proof. $\square$

Remark: In a chart, (U, φ), if we set

$$\partial_k g_{ij} = \frac{\partial}{\partial x_k}(g_{ij})$$

then it can be shown that the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

where (g^{kl}) is the inverse of the matrix (g_{kl}).

It can be shown that a connection is torsion-free iff

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \qquad \text{for all } i, j, k.$$

We conclude this section with various useful facts about torsion-free or metric connections.

First, there is a nice characterization for the Levi-Civita connection induced by a Riemannian manifold over a submanifold.

Proposition 5.11. *Let M be any Riemannian manifold and let N be any submanifold of M equipped with the induced metric. If ∇^M and ∇^N are the Levi-Civita connections on M and N, respectively, induced by the metric on M, then for any two vector fields, X and Y in $\mathfrak{X}(M)$ with $X(p), Y(p) \in T_pN$, for all $p \in N$, we have*

$$\nabla^N_X Y = (\nabla^M_X Y)^{\parallel},$$

where $(\nabla^M_X Y)^{\parallel}(p)$ is the orthogonal projection of $\nabla^M_X Y(p)$ onto T_pN, for every $p \in N$.

In particular, if γ is a curve on a surface, $M \subseteq \mathbb{R}^3$, then a vector field, $X(t)$, along γ is parallel iff $X'(t)$ is normal to the tangent plane, $T_{\gamma(t)}M$.

If ∇ is a metric connection, then we can say more about the parallel transport along a curve. Recall from Section 5.3, Definition 5.6, that a vector field, X, along a curve, γ, is parallel iff

$$\frac{DX}{dt} = 0.$$

Proposition 5.12. *Given any Riemannian manifold, M, and any metric connection, ∇, on M, for every curve, $\gamma\colon [a,b] \to M$, on M, if X and Y are two vector fields along γ, then*

$$\frac{d}{dt}\langle X(t), Y(t)\rangle = \left\langle \frac{DX}{dt}, Y \right\rangle + \left\langle X, \frac{DY}{dt} \right\rangle.$$

Using Proposition 5.12 we get

Proposition 5.13. *Given any Riemannian manifold, M, and any metric connection, ∇, on M, for every curve, $\gamma\colon [a,b] \to M$, on M, if X and Y are two vector fields along γ that are parallel, then*

$$\langle X, Y\rangle = C,$$

for some constant, C. In particular, $\|X(t)\|$ is constant. Furthermore, the linear isomorphism, $P_\gamma\colon T_{\gamma(a)} \to T_{\gamma(b)}$, is an isometry.

In particular, Proposition 5.13 shows that the holonomy group, $\mathrm{Hol}_p(\nabla)$, based at p, is a subgroup of $\mathbf{O}(n)$.