Chapter 3

Manifolds, Tangent Spaces, Cotangent Spaces, Vector Fields, Flow, Integral Curves

3.1 Manifolds

In Chapter 1 we defined the notion of a manifold embedded in some ambient space, $\mathbb{R}^N$.

In order to maximize the range of applications of the theory of manifolds it is necessary to generalize the concept of a manifold to spaces that are not a priori embedded in some $\mathbb{R}^N$.

The basic idea is still that, whatever a manifold is, it is a topological space that can be covered by a collection of open subsets, U_α, where each U_α is isomorphic to some "standard model," *e.g.*, some open subset of Euclidean space, $\mathbb{R}^n$.

Of course, manifolds would be very dull without functions defined on them and between them.

This is a general fact learned from experience: *Geometry arises not just from spaces but from spaces and interesting classes of functions between them*.

In particular, we still would like to "do calculus" on our manifold and have good notions of curves, tangent vectors, differential forms, etc.

The small drawback with the more general approach is that the definition of a tangent vector is more abstract.

We can still define the notion of a curve on a manifold, but such a curve does not live in any given $\mathbb{R}^n$, so it it not possible to define tangent vectors in a simple-minded way using derivatives.

Instead, we have to resort to the notion of chart. This is not such a strange idea.

For example, a geography atlas gives a set of maps of various portions of the earth and this provides a very good description of what the earth is, without actually imagining the earth embedded in 3-space.

Given $\mathbb{R}^n$, recall that the *projection functions*, $pr_i\colon \mathbb{R}^n \to \mathbb{R}$, are defined by

$$pr_i(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n.$$

For technical reasons, in particular to ensure that partitions of unity exist (a crucial tool in manifold theory) from now on, all topological spaces under consideration will be assumed to be Hausdorff and second-countable (which means that the topology has a countable basis).

Definition 3.1. Given a topological space, M, a *chart* (or *local coordinate map*) is a pair, (U, φ), where U is an open subset of M and $\varphi\colon U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{R}^{n_\varphi}$ (for some $n_\varphi \geq 1$).

For any $p \in M$, a chart, (U, φ), is a *chart at p* iff $p \in U$. If (U, φ) is a chart, then the functions $x_i = pr_i \circ \varphi$ are called *local coordinates* and for every $p \in U$, the tuple $(x_1(p), \ldots, x_n(p))$ is the set of *coordinates of p* w.r.t. the chart.

The inverse, (Ω, φ^{-1}), of a chart is called a *local parametrization*.

Given any two charts, (U_i, φ_i) and (U_j, φ_j), if $U_i \cap U_j \neq \emptyset$, we have the *transition maps*, $\varphi_i^j\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ and $\varphi_j^i\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, defined by

$$\varphi_i^j = \varphi_j \circ \varphi_i^{-1} \quad \text{and} \quad \varphi_j^i = \varphi_i \circ \varphi_j^{-1}.$$

Clearly, $\varphi_j^i = (\varphi_i^j)^{-1}$.

Observe that the transition maps φ_i^j (resp. φ_j^i) are maps between *open subsets of* $\mathbb{R}^n$.

This is good news! Indeed, the whole arsenal of calculus is available for functions on $\mathbb{R}^n$, and we will be able to promote many of these results to manifolds by imposing suitable conditions on transition functions.

Definition 3.2. Given a topological space, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) $\varphi_i(U_i) \subseteq \mathbb{R}^n$ for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism. When $k = \infty$, the φ_i^j are smooth diffeomorphisms.

We must insure that we have enough charts in order to carry out our program of generalizing calculus on $\mathbb{R}^n$ to manifolds.

For this, we must be able to add new charts whenever necessary, provided that they are consistent with the previous charts in an existing atlas.

Technically, given a C^k n-atlas, $\mathcal{A}$, on M, for any other chart, (U, φ), we say that (U, φ) is *compatible* with the atlas $\mathcal{A}$ iff every map $\varphi_i \circ \varphi^{-1}$ and $\varphi \circ \varphi_i^{-1}$ is C^k (whenever $U \cap U_i \neq \emptyset$).

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* iff every chart of one is compatible with the other atlas.

This is equivalent to saying that the union of the two atlases is still an atlas.

It is immediately verified that compatibility induces an equivalence relation on C^k n-atlases on M.

In fact, given an atlas, $\mathcal{A}$, for M, the collection, $\widetilde{\mathcal{A}}$, of all charts compatible with $\mathcal{A}$ is a maximal atlas in the equivalence class of charts compatible with $\mathcal{A}$.

Definition 3.3. Given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k-manifold of dimension n* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M. Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on* $\mathbb{R}^n$. When $k = \infty$, we say that M is a *smooth manifold*.

Remark: It might have been better to use the terminology *abstract manifold* rather than manifold, to emphasize the fact that the space M is not a priori a subspace of $\mathbb{R}^N$, for some suitable N.

We can allow $k = 0$ in the above definitions. Condition (3) in Definition 3.2 is void, since a C^0-diffeomorphism is just a homeomorphism, but φ_i^j is always a homeomorphism.

In this case, M is called a *topological manifold of dimension n*.

We do not require a manifold to be connected but we require all the components to have the same dimension, n.

Actually, on every connected component of M, it can be shown that the dimension, n_φ, of the range of every chart is the same. This is quite easy to show if $k \geq 1$ but for $k = 0$, this requires a deep theorem of Brouwer.

What happens if $n = 0$? In this case, every one-point subset of M is open, so every subset of M is open, i.e., M is any (countable if we assume M to be second-countable) set with the discrete topology!

Observe that since $\mathbb{R}^n$ is locally compact and locally connected, so is every manifold.

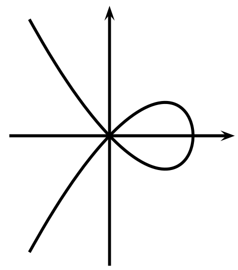

Figure 3.1: A nodal cubic; not a manifold

In order to get a better grasp of the notion of manifold it is useful to consider examples of non-manifolds.

First, consider the curve in $\mathbb{R}^2$ given by the zero locus of the equation

$$y^2 = x^2 - x^3,$$

namely, the set of points

$$M_1 = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^2 - x^3\}.$$

This curve showed in Figure 3.1 and called a *nodal cubic* is also defined as the parametric curve

$$\begin{aligned} x &= 1 - t^2 \\ y &= t(1 - t^2). \end{aligned}$$

We claim that M_1 is not even a topological manifold. The problem is that the nodal cubic has a self-intersection at the origin.

If M_1 was a topological manifold, then there would be a connected open subset, $U \subseteq M_1$, containing the origin, $O = (0, 0)$, namely the intersection of a small enough open disc centered at O with M_1, and a local chart, $\varphi \colon U \to \Omega$, where Ω is some connected open subset of $\mathbb{R}$ (that is, an open interval), since φ is a homeomorphism.

However, $U - \{O\}$ consists of four disconnected components and $\Omega - \varphi(O)$ of two disconnected components, contradicting the fact that φ is a homeomorphism.

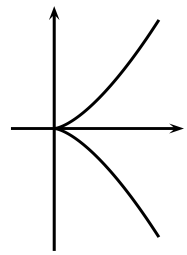

Figure 3.2: A Cuspidal Cubic

Let us now consider the curve in $\mathbb{R}^2$ given by the zero locus of the equation

$$y^2 = x^3,$$

namely, the set of points

$$M_2 = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3\}.$$

This curve showed in Figure 3.2 and called a *cuspidal cubic* is also defined as the parametric curve

$$\begin{aligned} x &= t^2 \\ y &= t^3. \end{aligned}$$

Consider the map, $\varphi\colon M_2 \to \mathbb{R}$, given by
$$\varphi(x, y) = y^{1/3}.$$
Since $x = y^{2/3}$ on M_2, we see that φ^{-1} is given by
$$\varphi^{-1}(t) = (t^2, t^3)$$
and clearly, φ is a homeomorphism, so M_2 is a topological manifold.

However, in the atlas consisting of the single chart, $\{\varphi\colon M_2 \to \mathbb{R}\}$, the space M_2 is also a smooth manifold!

Indeed, as there is a single chart, condition (3) of Definition 3.2 holds vacuously.

This fact is somewhat unexpected because the cuspidal cubic is usually not considered smooth at the origin, since the tangent vector of the parametric curve, $c\colon t \mapsto (t^2, t^3)$, at the origin is the zero vector (the velocity vector at t, is $c'(t) = (2t, 3t^2)$).

However, this apparent paradox has to do with the fact that, as a parametric curve, M_2 is *not immersed* in $\mathbb{R}^2$ since c' is not injective (see Definition 3.23 (a)), whereas as an abstract manifold, with this single chart, M_2 is diffeomorphic to $\mathbb{R}$.

Now, we also have the chart, $\psi\colon M_2 \to \mathbb{R}$, given by

$$\psi(x, y) = y,$$

with ψ^{-1} given by

$$\psi^{-1}(u) = (u^{2/3}, u).$$

Then, observe that

$$\varphi \circ \psi^{-1}(u) = u^{1/3},$$

a map that is *not* differentiable at $u = 0$. Therefore, the atlas $\{\varphi\colon M_2 \to \mathbb{R}, \psi\colon M_2 \to \mathbb{R}\}$ is not C^1 and thus, with respect to that atlas, M_2 is not a C^1-manifold.

The example of the cuspidal cubic shows a peculiarity of the definition of a C^k (or C^∞) manifold:

If a space, M, happens to be a topological manifold because it has an atlas consisting of a single chart, then it is automatically a smooth manifold!

In particular, if $f\colon U \to \mathbb{R}^m$ is any *continuous* function from some open subset, U, of $\mathbb{R}^n$, to $\mathbb{R}^m$, then the graph, $\Gamma(f) \subseteq \mathbb{R}^{n+m}$, of f given by

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} \mid x \in U\}$$

is a smooth manifold with respect to the atlas consisting of the single chart, $\varphi\colon \Gamma(f) \to U$, given by

$$\varphi(x, f(x)) = x,$$

with its inverse, $\varphi^{-1}\colon U \to \Gamma(f)$, given by

$$\varphi^{-1}(x) = (x, f(x)).$$

The notion of a submanifold using the concept of "adapted chart" (see Definition 3.22 in Section 3.4) gives a more satisfactory treatment of C^k (or smooth) submanifolds of $\mathbb{R}^n$.

The example of the cuspidal cubic also shows clearly that whether a topological space is a C^k-manifold or a smooth manifold depends on the choice of atlas.

In some cases, M does not come with a topology in an obvious (or natural) way and a slight variation of Definition 3.2 is more convenient in such a situation:

Definition 3.4. Given a set, M, given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*), $\mathcal{A}$, is a family of charts, $\{(U_i, \varphi_i)\}$, such that

(1) Each U_i is a subset of M and $\varphi_i \colon U_i \to \varphi_i(U_i)$ is a bijection onto an open subset, $\varphi_i(U_i) \subseteq \mathbb{R}^n$, for all i;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the set $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$ and the transition map φ_i^j (and φ_j^i) is a C^k-diffeomorphism.

Then, the notion of a chart being compatible with an atlas and of two atlases being compatible is just as before and we get a new definition of a manifold, analogous to Definition 3.2.

But, this time, we give M the topology in which the open sets are arbitrary unions of domains of charts (the U_i's in a maximal atlas).

It is not difficult to verify that the axioms of a topology are verified and M is indeed a topological space with this topology.

It can also be shown that when M is equipped with the above topology, then the maps $\varphi_i \colon U_i \to \varphi_i(U_i)$ are homeomorphisms, so M is a manifold according to Definition 3.3.

We require M to be Hausdorff and second-countable with this topology.

Thus, we are back to the original notion of a manifold where it is assumed that M is already a topological space.

If the underlying topological space of a manifold is compact, then M has some finite atlas.

Also, if $\mathcal{A}$ is some atlas for M and (U, φ) is a chart in $\mathcal{A}$, for any (nonempty) open subset, $V \subseteq U$, we get a chart, $(V, \varphi \upharpoonright V)$, and it is obvious that this chart is compatible with $\mathcal{A}$.

Thus, $(V, \varphi \upharpoonright V)$ is also a chart for M. This observation shows that if U is any open subset of a C^k-manifold, M, then U is also a C^k-manifold whose charts are the restrictions of charts on M to U.

Interesting manifolds often occur as the result of a quotient construction.

For example, real projective spaces and Grassmannians are obtained this way.

In this situation, the natural topology on the quotient object is the quotient topology but, unfortunately, even if the original space is Hausdorff, the quotient topology may not be.

Therefore, it is useful to have criteria that insure that a quotient topology is Hausdorff (or second-countable). We will present two criteria.

First, let us review the notion of quotient topology.

Definition 3.5. Given any topological space, X, and any set, Y, for any surjective function, $f\colon X \to Y$, we define the *quotient topology on Y determined by f* (also called the *identification topology on Y determined by f*), by requiring a subset, V, of Y to be open if $f^{-1}(V)$ is an open set in X.

Given an equivalence relation R on a topological space X, if $\pi\colon X \to X/R$ is the projection sending every $x \in X$ to its equivalence class $[x]$ in X/R, the space X/R equipped with the quotient topology determined by π is called the *quotient space of X modulo R*.

Thus a set, V, of equivalence classes in X/R is open iff $\pi^{-1}(V)$ is open in X, which is equivalent to the fact that $\bigcup_{[x]\in V}[x]$ is open in X.

It is immediately verified that Definition 3.5 defines topologies and that $f\colon X \to Y$ and $\pi\colon X \to X/R$ are continuous when Y and X/R are given these quotient topologies.

One should be careful that if X and Y are topological spaces and $f\colon X \to Y$ is a continuous surjective map, Y *does not* necessarily have the quotient topology determined by f.

Indeed, it may not be true that a subset V of Y is open when $f^{-1}(V)$ is open. However, this will be true in two important cases.

Definition 3.6. A continuous map, $f\colon X \to Y$, is an *open map* (or simply *open*) if $f(U)$ is open in Y whenever U is open in X, and similarly, $f\colon X \to Y$, is a *closed map* (or simply *closed*) if $f(F)$ is closed in Y whenever F is closed in X.

Then, Y has the quotient topology induced by the continuous surjective map f if either f is open or f is closed.

Unfortunately, the Hausdorff separation property is not necessarily preserved under quotient. Nevertheless, it is preserved in some special important cases.

Proposition 3.1. *Let X and Y be topological spaces, let $f\colon X \to Y$ be a continuous surjective map, and assume that X is compact and that Y has the quotient topology determined by f. Then Y is Hausdorff iff f is a closed map.*

Another simple criterion uses continuous open maps.

Proposition 3.2. *Let $f\colon X \to Y$ be a surjective continuous map between topological spaces. If f is an open map then Y is Hausdorff iff the set*

$$\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

Note that the hypothesis of Proposition 3.2 implies that Y has the quotient topology determined by f.

A special case of Proposition 3.2 is discussed in Tu.

Given a topological space, X, and an equivalence relation, R, on X, we say that R is *open* if the projection map, $\pi\colon X \to X/R$, is an open map, where X/R is equipped with the quotient topology.

Then, if R is an open equivalence relation on X, the topological quotient space X/R is Hausdorff iff R is closed in $X \times X$.

The following proposition yields a sufficient condition for second-countability:

Proposition 3.3. *If X is a topological space and R is an open equivalence relation on X, then for any basis, $\{B_\alpha\}$, for the topology of X, the family $\{\pi(B_\alpha)\}$ is a basis for the topology of X/R, where $\pi\colon X \to X/R$ is the projection map. Consequently, if X is second-countable, then so is X/R.*

Example 1. The sphere S^n.

Using the stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $\sigma_N\colon S^n - \{N\} \to \mathbb{R}^n$ and $\sigma_S\colon S^n - \{S\} \to \mathbb{R}^n$, where $N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ (the north pole) and $S = (0, \cdots, 0, -1) \in \mathbb{R}^{n+1}$ (the south pole) be the maps called respectively *stereographic projection from the north pole* and *stereographic projection from the south pole* given by

$$\sigma_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n)$$

and

$$\sigma_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\sigma_N^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, \left(\sum_{i=1}^n x_i^2\right) - 1\right)$$

and

$$\sigma_S^{-1}(x_1, \ldots, x_n) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right) + 1} \left(2x_1, \ldots, 2x_n, -\left(\sum_{i=1}^n x_i^2\right) + 1\right).$$

Thus, if we let $U_N = S^n - \{N\}$ and $U_S = S^n - \{S\}$, we see that U_N and U_S are two open subsets covering S^n, both homeomorphic to $\mathbb{R}^n$.

Furthermore, it is easily checked that on the overlap, $U_N \cap U_S = S^n - \{N, S\}$, the transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^{n} x_i^2} (x_1, \ldots, x_n),$$

that is, the inversion of center $O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n - \{O\}$, so we conclude that (U_N, σ_N) and (U_S, σ_S) form a smooth atlas for S^n.

Example 2. The projective space $\mathbb{RP}^n$.

To define an atlas on $\mathbb{RP}^n$ it is convenient to view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}.$$

Given any $p = [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$, we call $(x_1, \ldots, x_{n+1})$ the *homogeneous coordinates* of p.

It is customary to write $(x_1 \colon \cdots : x_{n+1})$ instead of $[x_1, \ldots, x_{n+1}]$. (Actually, in most books, the indexing starts with 0, i.e., homogeneous coordinates for $\mathbb{RP}^n$ are written as $(x_0 \colon \cdots : x_n)$.)

Now, $\mathbb{RP}^n$ can also be viewed as the quotient of the sphere, S^n, under the equivalence relation where any two antipodal points, x and $-x$, are identified.

It is not hard to show that the projection $\pi\colon S^n \to \mathbb{RP}^n$ is both open and closed.

Since S^n is compact and second-countable, we can apply our previous results to prove that under the quotient topology, $\mathbb{RP}^n$ is Hausdorff, second-countable, and compact.

We define charts in the following way. For any i, with $1 \leq i \leq n+1$, let

$$U_i = \{(x_1\colon \cdots : x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Observe that U_i is well defined, because if $(y_1\colon \cdots : y_{n+1}) = (x_1\colon \cdots : x_{n+1})$, then there is some $\lambda \neq 0$ so that $y_i = \lambda z_i$, for $i = 1, \ldots, n+1$.

We can define a homeomorphism, φ_i, of U_i onto $\mathbb{R}^n$, as follows:

$$\varphi_i(x_1\colon \cdots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right),$$

where the ith component is omitted. Again, it is clear that this map is well defined since it only involves ratios.

We can also define the maps, ψ_i, from $\mathbb{R}^n$ to $U_i \subseteq \mathbb{RP}^n$, given by

$$\psi_i(x_1, \ldots, x_n) = (x_1 \colon \cdots : x_{i-1} \colon 1 \colon x_i \colon \cdots : x_n),$$

where the 1 goes in the ith slot, for $i = 1, \ldots, n+1$.

One easily checks that φ_i and ψ_i are mutual inverses, so the φ_i are homeomorphisms. On the overlap, $U_i \cap U_j$, (where $i \neq j$), as $x_j \neq 0$, we have

$$(\varphi_j \circ \varphi_i^{-1})(x_1, \ldots, x_n) = \\ \left(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

(We assumed that $i < j$; the case $j < i$ is similar.) This is clearly a smooth function from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

As the U_i cover $\mathbb{RP}^n$, see conclude that the (U_i, φ_i) are $n+1$ charts making a smooth atlas for $\mathbb{RP}^n$.

Intuitively, the space $\mathbb{RP}^n$ is obtained by gluing the open subsets U_i on their overlaps. Even for $n = 3$, this is not easy to visualize!

Example 3. The Grassmannian $G(k, n)$.

Recall that $G(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$, also called k-planes.

Every k-plane, W, is the linear span of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$; furthermore, $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ both span W iff there is an invertible $k \times k$-matrix, $\Lambda = (\lambda_{ij})$, such that

$$v_j = \sum_{i=1}^{k} \lambda_{ij} u_i, \quad 1 \leq j \leq k.$$

Obviously, there is a bijection between the collection of k linearly independent vectors, $u_1, \ldots, u_k$, in $\mathbb{R}^n$ and the collection of $n \times k$ matrices of rank k.

Furthermore, two $n \times k$ matrices A and B of rank k represent the same k-plane iff

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda.$$

(Note the analogy with projective spaces where two vectors u, v represent the same point iff $v = \lambda u$ for some invertible $\lambda \in \mathbb{R}$.)

The set of $n \times k$ matrices of rank k is a subset of $\mathbb{R}^{n\times k}$, in fact, an open subset.

One can show that the equivalence relation on $n \times k$ matrices of rank k given by

$$B = A\Lambda, \quad \text{for some invertible } k \times k \text{ matrix, } \Lambda,$$

is open and that the graph of this equivalence relation is closed.

By Proposition 3.2, the Grassmannian $G(k, n)$ is Hausdorff and second-countable.

We can define the domain of charts (according to Definition 3.2) on $G(k,n)$ as follows: For every subset, $S = \{i_1,\ldots,i_k\}$ of $\{1,\ldots,n\}$, let U_S be the subset of $n \times k$ matrices, A, of rank k whose rows of index in $S = \{i_1,\ldots,i_k\}$ form an invertible $k \times k$ matrix denoted A_S.

Observe that the $k \times k$ matrix consisting of the rows of the matrix AA_S^{-1} whose index belong to S is the identity matrix, I_k.

Therefore, we can define a map, $\varphi_S \colon U_S \to \mathbb{R}^{(n-k)\times k}$, where $\varphi_S(A) =$ the $(n-k)\times k$ matrix obtained by deleting the rows of index in S from AA_S^{-1}.

We need to check that this map is well defined, i.e., that it does not depend on the matrix, A, representing W.

Let us do this in the case where $S = \{1, \ldots, k\}$, which is notationally simpler. The general case can be reduced to this one using a suitable permutation.

If $B = A\Lambda$, with Λ invertible, if we write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

as $B = A\Lambda$, we get $B_1 = A_1\Lambda$ and $B_2 = A_2\Lambda$, from which we deduce that

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} B_1^{-1} = \begin{pmatrix} I_k \\ B_2 B_1^{-1} \end{pmatrix} = \\ \begin{pmatrix} I_k \\ A_2 \Lambda \Lambda^{-1} A_1^{-1} \end{pmatrix} = \begin{pmatrix} I_k \\ A_2 A_1^{-1} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} A_1^{-1}.$$

Therefore, our map is indeed well-defined.

It is clearly injective and we can define its inverse, ψ_S, as follows: Let π_S be the permutation of $\{1, \ldots, n\}$ swaping $\{1, \ldots, k\}$ and S and leaving every other element fixed (i.e., if $S = \{i_1, \ldots, i_k\}$, then $\pi_S(j) = i_j$ and $\pi_S(i_j) = j$, for $j = 1, \ldots, k$).

If P_S is the permutation matrix associated with π_S, for any $(n-k) \times k$ matrix, M, let

$$\psi_S(M) = P_S \begin{pmatrix} I_k \\ M \end{pmatrix}.$$

The effect of ψ_S is to "insert into M" the rows of the identity matrix I_k as the rows of index from S.

At this stage, we have charts that are bijections from subsets, U_S, of $G(k, n)$ to open subsets, namely, $\mathbb{R}^{(n-k)\times k}$.

Then, the reader can check that the transition map $\varphi_T \circ \varphi_S^{-1}$ from $\varphi_S(U_S \cap U_T)$ to $\varphi_T(U_S \cap U_T)$ is given by

$$M \mapsto (C + DM)(A + BM)^{-1},$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = P_T P_S,$$

is the matrix of the permutation $\pi_T \circ \pi_S$ (this permutation "shuffles" S and T).

This map is smooth, as it is given by determinants, and so, the charts (U_S, φ_S) form a smooth atlas for $G(k, n)$.

Finally, it is easy to check that the conditions of Definition 3.2 are satisfied, so the atlas just defined makes $G(k, n)$ into a topological space and a smooth manifold.

The Grassmannian $G(k, n)$ is actually compact. To see this, observe that if W is any k-plane, then using the Gram-Schmidt orthonormalization procedure, every basis $B = (b_1, \ldots, b_k)$ for W yields an orthonormal basis $U = (u_1, \ldots, u_k)$ and there is an invertible matrix, Λ, such that

$$U = B\Lambda,$$

where the the columns of B are the b_js and the columns of U are the u_js.

The matrices U have orthonormal columns and are characterized by the equation

$$U^\top U = I_k.$$

Consequently, the space of such matrices is closed and clearly bounded in $\mathbb{R}^{n \times k}$ and thus, compact.

The Grassmannian $G(k, n)$ is the quotient of this space under our usual equivalence relation and $G(k, n)$ is the image of a compact set under the projection map, which is clearly continuous, so $G(k, n)$ is compact.

Remark: The reader should have no difficulty proving that the collection of k-planes represented by matrices in U_S is precisely the set of k-planes, W, supplementary to the $(n-k)$-plane spanned by the $n-k$ canonical basis vectors $e_{j_{k+1}}, \ldots, e_{j_n}$ (i.e., $\mathrm{span}(W \cup \{e_{j_{k+1}}, \ldots, e_{j_n}\}) = \mathbb{R}^n$, where $S = \{i_1, \ldots, i_k\}$ and $\{j_{k+1}, \ldots, j_n\} = \{1, \ldots, n\} - S$).

Example 4. Product Manifolds.

Let M_1 and M_2 be two C^k-manifolds of dimension n_1 and n_2, respectively.

The topological space, $M_1 \times M_2$, with the product topology (the opens of $M_1 \times M_2$ are arbitrary unions of sets of the form $U \times V$, where U is open in M_1 and V is open in M_2) can be given the structure of a C^k-manifold of dimension $n_1 + n_2$ by defining charts as follows:

For any two charts, (U_i, φ_i) on M_1 and (V_j, ψ_j) on M_2, we declare that $(U_i \times V_j, \varphi_i \times \psi_j)$ is a chart on $M_1 \times M_2$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbb{R}^{n_1+n_2}$ is defined so that

$$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q)), \quad \text{for all } (p, q) \in U_i \times V_j.$$

We define C^k-maps between manifolds as follows:

Definition 3.7. Given any two C^k-manifolds, M and N, of dimension m and n respectively, a *C^k-map* if a continuous functions, $h\colon M \to N$, so that for every $p \in M$, there is some chart, (U, φ), at p and some chart, (V, ψ), at $q = h(p)$, with $h(U) \subseteq V$ and

$$\psi \circ h \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \psi(V)$$

a C^k-function.

It is easily shown that Definition 3.7 does not depend on the choice of charts. In particular, if $N = \mathbb{R}$, we obtain a *C^k-function on M*.

One checks immediately that a function, $f\colon M \to \mathbb{R}$, is a C^k-map iff for every $p \in M$, there is some chart, (U, φ), at p so that

$$f \circ \varphi^{-1}\colon \varphi(U) \longrightarrow \mathbb{R}$$

is a C^k-function.

If U is an open subset of M, the set of C^k-functions on U is denoted by $\mathcal{C}^k(U)$. In particular, $\mathcal{C}^k(M)$ denotes the set of C^k-functions on the manifold, M.

Observe that $\mathcal{C}^k(U)$ is a ring.

On the other hand, if M is an open interval of $\mathbb{R}$, say $M =]a, b[$, then $\gamma\colon]a, b[\to N$ is called a *C^k-curve* in N.

One checks immediately that a function, $\gamma\colon]a, b[\to N$, is a C^k-map iff for every $q \in N$, there is some chart (V, ψ) at q and some open subinterval $]c, d[$ of $]a, b[$, so that $\gamma(]c, d[) \subseteq V$ and

$$\psi \circ \gamma\colon]c, d[\longrightarrow \psi(V)$$

is a C^k-function.

It is clear that the composition of C^k-maps is a C^k-map. A C^k-map, $h\colon M \to N$, between two manifolds is a *C^k-diffeomorphism* iff h has an inverse, $h^{-1}\colon N \to M$ (i.e., $h^{-1} \circ h = \mathrm{id}_M$ and $h \circ h^{-1} = \mathrm{id}_N$), and both h and h^{-1} are C^k-maps (in particular, h and h^{-1} are homeomorphisms).

Next, we define tangent vectors.

3.2 Tangent Vectors, Tangent Spaces, Cotangent Spaces

Let M be a C^k manifold of dimension n, with $k \geq 1$.

The most intuitive method to define tangent vectors is to use curves.

Let $p \in M$ be any point on M and let $\gamma\colon\,]-\epsilon, \epsilon[\, \to M$ be a C^1-curve passing through p, that is, with $\gamma(0) = p$.

Unfortunately, if M is not embedded in any $\mathbb{R}^N$, the derivative $\gamma'(0)$ does not make sense.

However, for any chart, (U, φ), at p, the map $\varphi \circ \gamma$ is a C^1-curve in $\mathbb{R}^n$ and the tangent vector $v = (\varphi \circ \gamma)'(0)$ is well defined.

The trouble is that different curves may yield the same v!

To remedy this problem, we define an equivalence relation on curves through p as follows:

Definition 3.8. Given a C^k manifold, M, of dimension n, for any $p \in M$, two C^1-curves, $\gamma_1\colon\,]-\epsilon_1, \epsilon_1[\to M$ and $\gamma_2\colon\,]-\epsilon_2, \epsilon_2[\to M$, through p (i.e., $\gamma_1(0) = \gamma_2(0) = p$) are *equivalent* iff there is some chart, (U, φ), at p so that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Now, the problem is that this definition seems to depend on the choice of the chart. Fortunately, this is not the case.

Definition 3.9. (Tangent Vectors, Version 1) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any equivalence class of C^1-curves through p on M, modulo the equivalence relation defined in Definition 3.8. The set of all tangent vectors at p is denoted by $T_p(M)$.

It is obvious that $T_p(M)$ is a vector space.

Observe that the map that sends a curve,
$\gamma\colon\,]-\epsilon,\epsilon[\to M$, through p (with $\gamma(0) = p$) to its tangent vector, $(\varphi \circ \gamma)'(0) \in \mathbb{R}^n$ (for any chart (U, φ), at p), induces a map, $\overline{\varphi}\colon T_p(M) \to \mathbb{R}^n$.

It is easy to check that $\overline{\varphi}$ is a linear bijection (by definition of the equivalence relation on curves through p).

This shows that $T_p(M)$ is a vector space of dimension $n =$ dimension of M.

One should observe that unless $M = \mathbb{R}^n$, in which case, for any $p, q \in \mathbb{R}^n$, the tangent space $T_q(M)$ is naturally isomorphic to the tangent space $T_p(M)$ by the translation $q - p$, for an arbitrary manifold, there is no relationship between $T_p(M)$ and $T_q(M)$ when $p \neq q$.

One of the defects of the above definition of a tangent vector is that it has no clear relation to the C^k-differential structure of M.

In particular, the definition does not seem to have anything to do with the functions defined locally at p.

There is another way to define tangent vectors that reveals this connection more clearly. Moreover, such a definition is more intrinsic, i.e., does not refer explicitly to charts.

As a first step, consider the following: Let (U, φ) be a chart at $p \in M$ (where M is a C^k-manifold of dimension n, with $k \geq 1$) and let $x_i = pr_i \circ \varphi$, the ith local coordinate $(1 \leq i \leq n)$.

For any function, f, defined on $U \ni p$, set

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n.$$

(Here, $(\partial g/\partial X_i)|_y$ denotes the partial derivative of a function $g\colon \mathbb{R}^n \to \mathbb{R}$ with respect to the ith coordinate, evaluated at y.)

We would expect that the function that maps f to the above value is a linear map on the set of functions defined locally at p, but there is technical difficulty:

The set of functions defined locally at p is **not** a vector space!

To see this, observe that if f is defined on an open $U \ni p$ and g is defined on a different open $V \ni p$, then we do not know how to define $f + g$.

The problem is that we need to identify functions that agree on a smaller open subset. This leads to the notion of *germs*.

Definition 3.10. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *locally defined function at p* is a pair, (U, f), where U is an open subset of M containing p and f is a function defined on U.

Two locally defined functions, (U, f) and (V, g), at p are *equivalent* iff there is some open subset, $W \subseteq U \cap V$, containing p so that

$$f \upharpoonright W = g \upharpoonright W.$$

The equivalence class of a locally defined function at p, denoted $[f]$ or $\mathbf{f}$, is called a *germ at p*.

One should check that the relation of Definition 3.10 is indeed an equivalence relation.

For example, for every $a \in (-1, 1)$, the locally defined functions $(\mathbb{R} - \{1\}, 1/(1 - x))$ and $((-1, 1), \sum_{n=0}^{\infty} x^n)$ at a are equivalent.

Of course, the value at p of all the functions, f, in any germ, $\mathbf{f}$, is $f(p)$. Thus, we set $\mathbf{f}(p) = f(p)$.

We can define addition of germs, multiplication of a germ by a scalar, and multiplication of germs as follows.

If (U, f) and (V, g) are two locally defined functions at p, we define $(U \cap V, f + g)$, $(U \cap V, fg)$ and $(U, \lambda f)$ as the locally defined functions at p given by $(f + g)(q) = f(q) + g(q)$ and $(fg)(q) = f(q)g(q)$ for all $q \in U \cap V$, and $(\lambda f)(q) = \lambda f(q)$ for all $q \in U$, with $\lambda \in \mathbb{R}$.

If $\mathbf{f} = [f]$ and $\mathbf{g} = [g]$ are two germs at p, and then

$$\begin{aligned}
[f] + [g] &= [f + g] \\
\lambda[f] &= [\lambda f] \\
[f][g] &= [fg].
\end{aligned}$$

Therefore, the germs at p form a ring.

The *ring of germs of C^k-functions at p is denoted* $\mathcal{O}^{(k)}_{M,p}$.

When $k = \infty$, we usually drop the superscript ∞.

Remark: Most readers will most likely be puzzled by the notation $\mathcal{O}^{(k)}_{M,p}$.

In fact, it is standard in algebraic geometry, but it is not as commonly used in differential geometry.

For any open subset, U, of a manifold, M, the ring, $\mathcal{C}^k(U)$, of C^k-functions on U is also denoted $\mathcal{O}^{(k)}_M(U)$ (certainly by people with an algebraic geometry bent!).

Then, it turns out that the map $U \mapsto \mathcal{O}^{(k)}_M(U)$ is a *sheaf*, denoted $\mathcal{O}^{(k)}_M$, and the ring $\mathcal{O}^{(k)}_{M,p}$ is the *stalk* of the sheaf $\mathcal{O}^{(k)}_M$ at p.

Such rings are called *local rings*.

Roughly speaking, all the "local" information about M at p is contained in the local ring $\mathcal{O}^{(k)}_{M,p}$. (This is to be taken with a grain of salt. In the C^k-case where $k < \infty$, we also need the "stationary germs," as we will see shortly.)

Now that we have a rigorous way of dealing with functions locally defined at p, observe that the map

$$v_i\colon f \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f$$

yields the same value for all functions f in a germ $\mathbf{f}$ at p.

Furthermore, the above map is linear on $\mathcal{O}^{(k)}_{M,p}$. More is true.

(1) For any two functions f, g locally defined at p, we have

$$\left(\frac{\partial}{\partial x_i}\right)_p (fg) = f(p)\left(\frac{\partial}{\partial x_i}\right)_p g + g(p)\left(\frac{\partial}{\partial x_i}\right)_p f.$$

(2) If $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, then

$$\left(\frac{\partial}{\partial x_i}\right)_p f = 0.$$

The first property says that v_i is a *point-derivation*.

As to the second property, when $(f \circ \varphi^{-1})'(\varphi(p)) = 0$, we say that *$f$ is stationary at p*.

It is easy to check (using the chain rule) that being stationary at p does not depend on the chart, (U, φ), at p or on the function chosen in a germ, $\mathbf{f}$. Therefore, the notion of a stationary germ makes sense:

Definition 3.11. We say that a germ $\mathbf{f}$ at $p \in M$ is a *stationary germ* iff $(f \circ \varphi^{-1})'(\varphi(p)) = 0$ for some chart, (U, φ), at p and some function, f, in the germ, $\mathbf{f}$.

The C^k-stationary germs form a subring of $\mathcal{O}_{M,p}^{(k)}$ (but not an ideal!) denoted $\mathcal{S}_{M,p}^{(k)}$.

Remarkably, it turns out that the dual of the vector space $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ is isomorphic to the tangent space, $T_p(M)$.

First, we prove that the subspace of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ has $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ as a basis.

Proposition 3.4. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n functions, $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$, defined on $\mathcal{O}^{(k)}_{M,p}$ by*

$$\left(\frac{\partial}{\partial x_i}\right)_p f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}, \qquad 1 \leq i \leq n$$

are linear forms that vanish on $\mathcal{S}^{(k)}_{M,p}$.

Every linear form L on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$ can be expressed in a unique way as

$$L = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

where $\lambda_i \in \mathbb{R}$. Therefore, the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

form a basis of the vector space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

As the subspace of linear forms on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$ is isomorphic to the dual $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$, of the space $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$, we see that the

$$\left(\frac{\partial}{\partial x_i}\right)_p, \qquad i = 1, \ldots, n$$

also form a basis of $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$.

Definition 3.12. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *derivation at p in M* or *point-derivation on $\mathcal{O}_{M,p}^{(k)}$*, is a linear form v, on $\mathcal{O}_{M,p}^{(k)}$, such that

$$v(\mathbf{f}\mathbf{g}) = v(\mathbf{f})\mathbf{g}(p) + \mathbf{f}(p)v(\mathbf{g}),$$

for all germs $\mathbf{f}, \mathbf{g} \in \mathcal{O}_{M,p}^{(k)}$. The above is called the *Leibniz property*.

As expected, point-derivations vanish on constant functions.

Proposition 3.5. *Every point-derivation, v, on $\mathcal{O}^{(k)}_{M,p}$, vanishes on germs of constant functions.*

Recall that we observed earlier that the $\left(\frac{\partial}{\partial x_i}\right)_p$ are point-derivations at p. Therefore, we have

Proposition 3.6. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$ are exactly the point-derivations on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.*

Remarks:

(1) If we let $\mathcal{D}_p^{(k)}(M)$ denote the set of point-derivations on $\mathcal{O}_{M,p}^{(k)}$, then Proposition 3.6 says that any linear form on $\mathcal{O}_{M,p}^{(k)}$ that vanishes on $\mathcal{S}_{M,p}^{(k)}$ belongs to $\mathcal{D}_p^{(k)}(M)$, so we have the inclusion

$$(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^* \subseteq \mathcal{D}_p^{(k)}(M).$$

However, in general, when $k \neq \infty$, a point-derivation on $\mathcal{O}_{M,p}^{(k)}$ does *not* necessarily vanish on $\mathcal{S}_{M,p}^{(k)}$.

We will see in Proposition 3.11 that this is true for $k = \infty$.

(2) In the case of smooth manifolds ($k = \infty$) some authors, including Morita and O'Neil, define point-derivations as linear derivations with domain $\mathcal{C}^\infty(M)$, the set of all smooth funtions on the entire manifold, M.

This definition is simpler in the sense that it does not require the definition of the notion of germ but it is not local, because it is not obvious that if v is a point-derivation at p, then $v(f) = v(g)$ whenever $f, g \in \mathcal{C}^\infty(M)$ agree locally at p.

In fact, if two smooth locally defined functions agree near p it may not be possible to extend both of them to the whole of M.

However, it can proved that this property is local because on smooth manifolds, "bump functions" exist (see Section 3.6, Proposition 3.30).

Unfortunately, this argument breaks down for C^k-manifolds with $k < \infty$ and in this case the ring of germs at p can't be avoided.

Here is now our second definition of a tangent vector.

Definition 3.13. (Tangent Vectors, Version 2) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, a *tangent vector to M at p* is any point-derivation on $\mathcal{O}^{(k)}_{M,p}$ that vanishes on $\mathcal{S}^{(k)}_{M,p}$, the subspace of stationary germs.

Let us consider the simple case where $M = \mathbb{R}$. In this case, for every $x \in \mathbb{R}$, the tangent space, $T_x(\mathbb{R})$, is a one-dimensional vector space isomorphic to $\mathbb{R}$ and $\left(\frac{\partial}{\partial t}\right)_x = \frac{d}{dt}\big|_x$ is a basis vector of $T_x(\mathbb{R})$.

For every C^k-function, f, locally defined at x, we have

$$\left(\frac{\partial}{\partial t}\right)_x f = \left.\frac{df}{dt}\right|_x = f'(x).$$

Thus, $\left(\frac{\partial}{\partial t}\right)_x$ is: compute the derivative of a function at x.

We now prove the equivalence of the two Definitions of a tangent vector.

Proposition 3.7. *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For any $p \in M$, let u be any tangent vector (version 1) given by some equivalence class of C^1-curves, $\gamma \colon]-\epsilon, +\epsilon[\to M$, through p (i.e., $p = \gamma(0)$). Then, the map L_u defined on $\mathcal{O}^{(k)}_{M,p}$ by*

$$L_u(\mathbf{f}) = (f \circ \gamma)'(0)$$

is a point-derivation that vanishes on $\mathcal{S}^{(k)}_{M,p}$.

Furthermore, the map $u \mapsto L_u$ defined above is an isomorphism between $T_p(M)$ and $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$, the space of linear forms on $\mathcal{O}^{(k)}_{M,p}$ that vanish on $\mathcal{S}^{(k)}_{M,p}$.

There is a conceptually clearer way to define a canonical isomorphism between $T_p(M)$ and the dual of $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ in terms of a *nondegenerate pairing* between $T_p(M)$ and $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$

This pairing is described by Serre in [32] (Chapter III, Section 8) for analytic manifolds and can be adapted to our situation.

Define the map, $\omega\colon T_p(M) \times \mathcal{O}^{(k)}_{M,p} \to \mathbb{R}$, so that

$$\omega([\gamma], \mathbf{f}) = (f \circ \gamma)'(0),$$

for all $[\gamma] \in T_p(M)$ and all $\mathbf{f} \in \mathcal{O}^{(k)}_{M,p}$ (with $f \in \mathbf{f}$).

It is easy to check that the above expression does not depend on the representatives chosen in the equivalences classes $[\gamma]$ and $\mathbf{f}$ and that ω is bilinear.

However, as defined, ω is degenerate because $\omega([\gamma], \mathbf{f}) = 0$ if $\mathbf{f}$ is a stationary germ.

Thus, we are led to consider the pairing with domain $T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})$ given by

$$\omega([\gamma], [\mathbf{f}]) = (f \circ \gamma)'(0),$$

where $[\gamma] \in T_p(M)$ and $[\mathbf{f}] \in \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, which we also denote $\omega\colon T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}) \to \mathbb{R}$.

Proposition 3.8. *The map* $\omega\colon T_p(M) \times (\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}) \to \mathbb{R}$ *defined so that*

$$\omega([\gamma], [\mathbf{f}]) = (f \circ \gamma)'(0),$$

for all $[\gamma] \in T_p(M)$ *and all* $[\mathbf{f}] \in \mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$, *is a non-degenerate pairing (with* $f \in \mathbf{f}$*).*

Consequently, there is a canonical isomorphism between $T_p(M)$ *and* $(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^*$ *and a canonical isomorphism between* $T^*_p(M)$ *and* $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$.

In view of Proposition 3.8, we can identify $T_p(M)$ with $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$ and $T_p^*(M)$ with $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$.

Remark: Also recall that if E is a finite dimensional space, the map $i_E\colon E \to E^{**}$ defined so that, for any $v \in E$,

$$v \mapsto \widetilde{v}, \quad \text{where} \quad \widetilde{v}(f) = f(v), \quad \text{for all } f \in E^*$$

is a linear isomorphism.

Observe that we can view $\omega(u, \mathbf{f}) = \omega([\gamma], [\mathbf{f}])$ as the result of computing the directional derivative of the locally defined function $f \in \mathbf{f}$ in the direction u (given by a curve γ).

Proposition 3.8 also suggests the following definition:

Definition 3.14. Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, the *tangent space at p*, denoted $T_p(M)$, is the space of point-derivations on $\mathcal{O}_{M,p}^{(k)}$ that vanish on $\mathcal{S}_{M,p}^{(k)}$.

Thus, $T_p(M)$ can be identified with $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$.

The space $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ is called the *cotangent space at p*; it is isomorphic to the dual $T_p^*(M)$, of $T_p(M)$.

Even though this is just a restatement of Proposition 3.4, we state the following proposition because of its practical usefulness:

Proposition 3.9. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n tangent vectors,*

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p,$$

form a basis of T_pM.

Observe that if $x_i = pr_i \circ \varphi$, as

$$\left(\frac{\partial}{\partial x_i}\right)_p x_j = \delta_{i,j},$$

the images of $x_1, \ldots, x_n$ in $\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p}$ constitute the dual basis of the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

Given any C^k-function f, on U, we denote the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ by df_p.

This is the *differential of f at p*.

Using the isomorphism between $\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ and $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^{**}$ described above, df_p corresponds to the linear map in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}), \quad \text{for all } v \in T_p(M).$$

With this notation, we see that $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T_p^*(M)$, and this basis is dual to the basis $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p$ of $T_p(M)$.

For simplicity of notation, we often omit the subscript p unless confusion arises.

Remark: Strictly speaking, a tangent vector, $v \in T_p(M)$, is defined on the space of germs, $\mathcal{O}_{M,p}^{(k)}$, at p. However, it is often convenient to define v on C^k-functions, $f \in \mathcal{C}^k(U)$, where U is some open subset containing p. This is easy: set

$$v(f) = v(\mathbf{f}).$$

Given any chart, (U, φ), at p, since v can be written in a unique way as

$$v = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p,$$

we get

$$v(f) = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p f.$$

This shows that $v(f)$ is the *directional derivative of f in the direction v*.

When M is a smooth manifold, things get a little simpler.

Indeed, it turns out that in this case, every point-derivation vanishes on stationary germs.

To prove this, we recall the following result from calculus (see Warner [33]):

Proposition 3.10. *If $g\colon \mathbb{R}^n \to \mathbb{R}$ is a C^k-function ($k \geq 2$) on a convex open U, about $p \in \mathbb{R}^n$, then for every $q \in U$, we have*

$$g(q) = g(p) + \sum_{i=1}^{n} \left.\frac{\partial g}{\partial X_i}\right|_p (q_i - p_i)$$
$$+ \sum_{i,j=1}^{n} (q_i - p_i)(q_j - p_j) \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial X_i \partial X_j}\right|_{(1-t)p+tq} dt.$$

In particular, if $g \in C^\infty(U)$, then the integral as a function of q is C^∞.

Proposition 3.11. *Let M be any C^∞-manifold of dimension n. For any $p \in M$, any point-derivation on $\mathcal{O}^{(\infty)}_{M,p}$ vanishes on $\mathcal{S}^{(\infty)}_{M,p}$, the ring of stationary germs. Consequently, $T_p(M) = \mathcal{D}^{(\infty)}_p(M)$.*

Proposition 3.11 shows that in the case of a smooth manifold, in Definition 3.13, we can omit the requirement that point-derivations vanish on stationary germs, since this is automatic.

It is also possible to define $T_p(M)$ just in terms of $\mathcal{O}^{(\infty)}_{M,p}$.

Let $\mathfrak{m}_{M,p} \subseteq \mathcal{O}^{(\infty)}_{M,p}$ be the ideal of germs that vanish at p.

Then, we also have the ideal $\mathfrak{m}^2_{M,p}$, which consists of all finite linear combinations of products of two elements in $\mathfrak{m}_{M,p}$, and it turns out that $T^*_p(M)$ is isomorphic to $\mathfrak{m}_{M,p}/\mathfrak{m}^2_{M,p}$ (see Warner [33], Lemma 1.16).

Actually, if we let $\mathfrak{m}^{(k)}_{M,p} \subseteq \mathcal{O}^{(k)}_{M,p}$ denote the ideal of C^k-germs that vanish at p and $\mathfrak{s}^{(k)}_{M,p} \subseteq \mathcal{S}^{(k)}_{M,p}$ denote the ideal of stationary C^k-germs that vanish at p, adapting Warner's argument, we can prove the following proposition:

Proposition 3.12. *We have the inclusion,* $(\mathfrak{m}^{(k)}_{M,p})^2 \subseteq \mathfrak{s}^{(k)}_{M,p}$ *and the isomorphism*

$$(\mathcal{O}^{(k)}_{M,p}/\mathcal{S}^{(k)}_{M,p})^* \cong (\mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p})^*.$$

As a consequence, $T_p(M) \cong (\mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p})^*$ *and* $T^*_p(M) \cong \mathfrak{m}^{(k)}_{M,p}/\mathfrak{s}^{(k)}_{M,p}$.

When $k = \infty$, Proposition 3.10 shows that every stationary germ that vanishes at p belongs to $\mathfrak{m}^2_{M,p}$.

Therefore, when $k = \infty$, we have

$$\mathfrak{s}_{M,p}^{(\infty)} = \mathfrak{m}_{M,p}^{2}$$

and so, we obtain the result quoted above (from Warner):

$$T_p^*(M) = \mathcal{O}_{M,p}^{(\infty)}/\mathcal{S}_{M,p}^{(\infty)} \cong \mathfrak{m}_{M,p}/\mathfrak{m}_{M,p}^{2}.$$

Remarks:

(1) The isomorphism

$$(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^* \cong (\mathfrak{m}_{M,p}^{(k)}/\mathfrak{s}_{M,p}^{(k)})^*$$

yields another proof that the linear forms in $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$ are point-derivations, using the argument from Warner [33] (Lemma 1.16).

(2) The ideal $\mathfrak{m}_{M,p}^{(k)}$ is in fact the unique *maximal ideal* of $\mathcal{O}_{M,p}^{(k)}$.

This is because if $\mathbf{f} \in \mathcal{O}_{M,p}^{(k)}$ does not vanish at p, then $\mathbf{1}/\mathbf{f}$ belongs to $\mathcal{O}_{M,p}^{(k)}$, and any proper ideal containing $\mathfrak{m}_{M,p}^{(k)}$ and $\mathbf{f}$ would be equal to $\mathcal{O}_{M,p}^{(k)}$, which is absurd.

Thus, $\mathcal{O}_{M,p}^{(k)}$ is a local ring (in the sense of commutative algebra) called the *local ring of germs of C^k-functions at p*. These rings play a crucial role in algebraic geometry.

Yet one more way of defining tangent vectors will make it a little easier to define tangent bundles.

Definition 3.15. (Tangent Vectors, Version 3) Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$, consider the triples, (U, φ, u), where (U, φ) is any chart at p and u is any vector in $\mathbb{R}^n$.

Say that two such triples (U, φ, u) and (V, ψ, v) are *equivalent* iff

$$(\psi \circ \varphi^{-1})'_{\varphi(p)}(u) = v.$$

A *tangent vector* to M at p is an equivalence class of triples, $[(U, \varphi, u)]$, for the above equivalence relation.

The intuition behind Definition 3.15 is quite clear: The vector u is considered as a tangent vector to $\mathbb{R}^n$ at $\varphi(p)$.

If (U, φ) is a chart on M at p, we can define a natural isomorphism, $\theta_{U,\varphi,p}\colon \mathbb{R}^n \to T_p(M)$, between $\mathbb{R}^n$ and $T_p(M)$, as follows: For any $u \in \mathbb{R}^n$,

$$\theta_{U,\varphi,p}\colon u \mapsto [(U, \varphi, u)].$$

One immediately check that the above map is indeed linear and a bijection.

The equivalence of this definition with the definition in terms of curves (Definition 3.9) is easy to prove.

Proposition 3.13. *Let M be any C^k-manifold of dimension n, with $k \geq 1$. For every $p \in M$, for every chart, (U, φ), at p, if $x = [\gamma]$ is any tangent vector (version 1) given by some equivalence class of C^1-curves $\gamma\colon\,]-\epsilon, +\epsilon[\, \to M$ through p (i.e., $p = \gamma(0)$), then the map*

$$x \mapsto [(U, \varphi, (\varphi \circ \gamma)'(0))]$$

is an isomorphism between $T_p(M)$-version 1 and $T_p(M)$-version 3.

For simplicity of notation, we also use the notation T_pM for $T_p(M)$ (resp. T_p^*M for $T_p^*(M)$).

After having explored thorougly the notion of tangent vector, we show how a C^k-map, $h\colon M \to N$, between C^k manifolds, induces a linear map, $dh_p\colon T_p(M) \to T_{h(p)}(N)$, for every $p \in M$.

We find it convenient to use Version 2 of the definition of a tangent vector. So, let $u \in T_p(M)$ be a point-derivation on $\mathcal{O}_{M,p}^{(k)}$ that vanishes on $\mathcal{S}_{M,p}^{(k)}$.

We would like $dh_p(u)$ to be a point-derivation on $\mathcal{O}_{N,h(p)}^{(k)}$ that vanishes on $\mathcal{S}_{N,h(p)}^{(k)}$.

Now, for every germ, $\mathbf{g} \in \mathcal{O}_{N,h(p)}^{(k)}$, if $g \in \mathbf{g}$ is any locally defined function at $h(p)$, it is clear that $g \circ h$ is locally defined at p and is C^k and that if $g_1, g_2 \in \mathbf{g}$ then $g_1 \circ h$ and $g_2 \circ h$ are equivalent.

The germ of all locally defined functions at p of the form $g \circ h$, with $g \in \mathbf{g}$, will be denoted $\mathbf{g} \circ h$.

Then, we set

$$dh_p(u)(\mathbf{g}) = u(\mathbf{g} \circ h).$$

Moreover, if $\mathbf{g}$ is a stationary germ at $h(p)$, then for some chart, (V, ψ) on N at $q = h(p)$, we have $(g \circ \psi^{-1})'(\psi(q)) = 0$ and, for some chart (U, φ) at p on M, we get

$$(g \circ h \circ \varphi^{-1})'(\varphi(p)) = (g \circ \psi^{-1})(\psi(q))((\psi \circ h \circ \varphi^{-1})'(\varphi(p))) \\ = 0,$$

which means that $\mathbf{g} \circ h$ is stationary at p.

Therefore, $dh_p(u) \in T_{h(p)}(M)$. It is also clear that dh_p is a linear map.

Definition 3.16. Given any two C^k-manifolds, M and N, of dimension m and n, respectively, for any C^k-map, $h\colon M \to N$, and for every $p \in M$, the *differential of h at p* or *tangent map* $dh_p\colon T_p(M) \to T_{h(p)}(N)$ (also denoted $T_p h\colon T_p(M) \to T_{h(p)}(N)$), is the linear map defined so that

$$dh_p(u)(\mathbf{g}) = T_p h(u)(\mathbf{g}) = u(\mathbf{g} \circ h),$$

for every $u \in T_p(M)$ and every germ, $\mathbf{g} \in \mathcal{O}^{(k)}_{N,h(p)}$.

The linear map dh_p $(= T_p h)$ is sometimes denoted h'_p or $D_p h$.

The chain rule is easily generalized to manifolds.

Proposition 3.14. *Given any two C^k-maps $f\colon M \to N$ and $g\colon N \to P$ between smooth C^k-manifolds, for any $p \in M$, we have*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

In the special case where $N = \mathbb{R}$, a C^k-map between the manifolds M and $\mathbb{R}$ is just a C^k-function on M.

It is interesting to see what T_pf is explicitly. Since $N = \mathbb{R}$, germs (of functions on $\mathbb{R}$) at $t_0 = f(p)$ are just germs of C^k-functions, $g\colon \mathbb{R} \to \mathbb{R}$, locally defined at t_0.

Then, for any $u \in T_p(M)$ and every germ $\mathbf{g}$ at t_0,

$$T_pf(u)(\mathbf{g}) = u(\mathbf{g} \circ f).$$

If we pick a chart, (U, φ), on M at p, we know that the $\left(\frac{\partial}{\partial x_i}\right)_p$ form a basis of $T_p(M)$, with $1 \leq i \leq n$.

Therefore, it is enough to figure out what $T_pf(u)(\mathbf{g})$ is when $u = \left(\frac{\partial}{\partial x_i}\right)_p$.

In this case,

$$T_pf\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left.\frac{\partial(g\circ f\circ\varphi^{-1})}{\partial X_i}\right|_{\varphi(p)}.$$

Using the chain rule, we find that

$$T_pf\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)(\mathbf{g}) = \left(\frac{\partial}{\partial x_i}\right)_p f\left.\frac{dg}{dt}\right|_{t_0}.$$

Therefore, we have

$$T_pf(u) = u(\mathbf{f})\left.\frac{d}{dt}\right|_{t_0}.$$

This shows that we can identify T_pf with the linear form in $T_p^*(M)$ defined by

$$df_p(u) = u(\mathbf{f}), \qquad u \in T_pM,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$.

This is consistent with our previous definition of df_p as the image of f in $T_p^*(M) = \mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)}$ (as $T_p(M)$ is isomorphic to $(\mathcal{O}_{M,p}^{(k)}/\mathcal{S}_{M,p}^{(k)})^*$).

Proposition 3.15. *Given any C^k-manifold, M, of dimension n, with $k \geq 1$, for any $p \in M$ and any chart (U, φ) at p, the n linear maps,*

$$(dx_1)_p, \ldots, (dx_n)_p,$$

form a basis of T_p^*M, where $(dx_i)_p$, the differential of x_i at p, is identified with the linear form in T_p^*M such that

$$(dx_i)_p(v) = v(\mathbf{x_i}), \quad \textit{for every } v \in T_pM$$

(by identifying $T_\lambda\mathbb{R}$ with $\mathbb{R}$).

In preparation for the definition of the flow of a vector field (which will be needed to define the exponential map in Lie group theory), we need to define the tangent vector to a curve on a manifold.

Given a C^k-curve, $\gamma\colon]a,b[\to M$, on a C^k-manifold, M, for any $t_0 \in]a,b[$, we would like to define the tangent vector to the curve γ at t_0 as a tangent vector to M at $p = \gamma(t_0)$.

We do this as follows: Recall that $\frac{d}{dt}\big|_{t_0}$ is a basis vector of $T_{t_0}(\mathbb{R}) = \mathbb{R}$.

So, define the *tangent vector to the curve γ at t_0*, denoted $\dot{\gamma}(t_0)$ (or $\gamma'(t_0)$, or $\frac{d\gamma}{dt}(t_0)$), by

$$\dot{\gamma}(t_0) = d\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right).$$

Sometime, it is necessary to define curves (in a manifold) whose domain is not an open interval.

A map, $\gamma\colon [a, b] \to M$, is a *$C^k$-curve in M* if it is the restriction of some C^k-curve, $\widetilde{\gamma}\colon]a - \epsilon, b + \epsilon[\to M$, for some (small) $\epsilon > 0$.

Note that for such a curve (if $k \geq 1$) the tangent vector, $\dot{\gamma}(t)$, is defined for all $t \in [a, b]$.

A continuous curve, $\gamma\colon [a, b] \to M$, is *piecewise C^k* iff there a sequence, $a_0 = a, a_1, \ldots, a_m = b$, so that the restriction, γ_i, of γ to each $[a_i, a_{i+1}]$ is a C^k-curve, for $i = 0, \ldots, m - 1$.

This implies that $\gamma_i'(a_{i+1})$ and $\gamma_{i+1}'(a_{i+1})$ are defined for $i = 0, \ldots, m-1$, but there may be a *jump* in the tangent vector to γ at a_{i+1}, that is, we may have
$\gamma_i'(a_{i+1}) \neq \gamma_{i+1}'(a_{i+1})$.

3.3 Tangent and Cotangent Bundles, Vector Fields

Let M be a C^k-manifold (with $k \geq 2$). Roughly speaking, a vector field on M is the assignment, $p \mapsto X(p)$, of a tangent vector $X(p) \in T_p(M)$, to a point $p \in M$.

Generally, we would like such assignments to have some smoothness properties when p varies in M, for example, to be C^l, for some l related to k.

Now, if the collection, $T(M)$, of all tangent spaces, $T_p(M)$, was a C^l-manifold, then it would be very easy to define what we mean by a C^l-vector field: We would simply require the map, $X\colon M \to T(M)$, to be C^l.

If M is a C^k-manifold of dimension n, then we can indeed make $T(M)$ into a C^{k-1}-manifold of dimension $2n$ and we now sketch this construction.

We find it most convenient to use Version 3 of the definition of tangent vectors, i.e., as equivalence classes of triples (U, φ, x), with $x \in \mathbb{R}^n$.

First, we let $T(M)$ be the disjoint union of the tangent spaces $T_p(M)$, for all $p \in M$. Formally,

$$T(M) = \{(p, v) \mid p \in M, v \in T_p(M)\}.$$

There is a *natural projection*,

$$\pi \colon T(M) \to M, \quad \text{with} \quad \pi(p, v) = p.$$

We still have to give $T(M)$ a topology and to define a C^{k-1}-atlas.

For every chart, (U, φ), of M (with U open in M) we define the function, $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$, by

$$\widetilde{\varphi}(p, v) = (\varphi(p), \theta_{U,\varphi,p}^{-1}(v)),$$

where $(p, v) \in \pi^{-1}(U)$ and $\theta_{U,\varphi,p}$ is the isomorphism between $\mathbb{R}^n$ and $T_p(M)$ described just after Definition 3.15.

It is obvious that $\widetilde{\varphi}$ is a bijection between $\pi^{-1}(U)$ and $\varphi(U) \times \mathbb{R}^n$, an open subset of $\mathbb{R}^{2n}$.

We give $T(M)$ the weakest topology that makes all the $\widetilde{\varphi}$ continuous, i.e., we take the collection of subsets of the form $\widetilde{\varphi}^{-1}(W)$, where W is any open subset of $\varphi(U) \times \mathbb{R}^n$, as a basis of the topology of $T(M)$.

One easily checks that $T(M)$ is Hausdorff and second-countable in this topology. If (U, φ) and (V, ψ) are overlapping charts, then the transition function

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

is given by

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(z, x) = (\psi \circ \varphi^{-1}(z), (\psi \circ \varphi^{-1})'_z(x)),$$

with $(z, x) \in \varphi(U \cap V) \times \mathbb{R}^n$.

It is clear that $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a C^{k-1}-map. Therefore, $T(M)$ is indeed a C^{k-1}-manifold of dimension $2n$, called the *tangent bundle*.

Remark: Even if the manifold M is naturally embedded in $\mathbb{R}^N$ (for some $N \geq n = \dim(M)$), it is not at all obvious how to view the tangent bundle, $T(M)$, as embedded in $\mathbb{R}^{N'}$, for some suitable N'. Hence, we see that the definition of an abtract manifold is unavoidable.

A similar construction can be carried out for the cotangent bundle.

In this case, we let $T^*(M)$ be the disjoint union of the cotangent spaces $T_p^*(M)$,

$$T^*(M) = \{(p, \omega) \mid p \in M, \omega \in T_p^*(M)\}.$$

We also have a natural projection, $\pi\colon T^*(M) \to M$.

We can define charts as follows:

For any chart, (U, φ), on M, we define the function $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(p, \omega) = \left(\varphi(p), \omega\left(\left(\frac{\partial}{\partial x_1}\right)_p\right), \ldots, \omega\left(\left(\frac{\partial}{\partial x_n}\right)_p\right)\right),$$

where $(p, \omega) \in \pi^{-1}(U)$ and the $\left(\frac{\partial}{\partial x_i}\right)_p$ are the basis of $T_p(M)$ associated with the chart (U, φ).

Again, one can make $T^*(M)$ into a C^{k-1}-manifold of dimension $2n$, called the *cotangent bundle*.

Another method using Version 3 of the definition of tangent vectors is presented in Section **??**.

For each chart (U, φ) on M, we obtain a chart

$$\widetilde{\varphi}^*\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$$

on $T^*(M)$ given by

$$\widetilde{\varphi}^*(p, \omega) = (\varphi(p), \theta^*_{U,\varphi,\pi(\omega)}(\omega))$$

for all $(p, \omega) \in \pi^{-1}(U)$, where

$$\theta^*_{U,\varphi,p} = \iota \circ \theta^\top_{U,\varphi,p}\colon T^*_p(M) \to \mathbb{R}^n.$$

Here, $\theta^\top_{U,\varphi,p}\colon T^*_p(M) \to (\mathbb{R}^n)^*$ is obtained by dualizing the map, $\theta_{U,\varphi,p}\colon \mathbb{R}^n \to T_p(M)$, and $\iota\colon (\mathbb{R}^n)^* \to \mathbb{R}^n$ is the isomorphism induced by the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ and its dual basis.

For simplicity of notation, we also use the notation TM for $T(M)$ (resp. T^*M for $T^*(M)$).

Observe that for every chart, (U, φ), on M, there is a bijection

$$\tau_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

given by

$$\tau_U(p, v) = (p, \theta^{-1}_{U,\varphi,p}(v)).$$

Clearly, $pr_1 \circ \tau_U = \pi$, on $\pi^{-1}(U)$ as illustrated by the following commutative diagram:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\tau_U"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}$$

Thus, locally, that is, over U, the bundle $T(M)$ looks like the product manifold $U \times \mathbb{R}^n$.

We say that $T(M)$ is *locally trivial* (over U) and we call τ_U a *trivializing map*.

For any $p \in M$, the vector space $\pi^{-1}(p) = \{p\} \times T_p(M) \cong T_p(M)$ is called the *fibre above p*.

Observe that the restriction of τ_U to $\pi^{-1}(p)$ is an isomorphism between $\{p\} \times T_p(M) \cong T_p(M)$ and $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$, for any $p \in M$.

Furthermore, for any two overlapping charts (U, φ) and (V, ψ), there is a function $g_{UV} \colon U \cap V \to \mathrm{GL}(n, \mathbb{R})$ such that

$$(\tau_U \circ \tau_V^{-1})(p, x) = (p, g_{UV}(p)(x))$$

for all $p \in U \cap V$ and all $x \in \mathbb{R}^n$, with $g_{UV}(p)$ given by

$$g_{UV}(p) = (\varphi \circ \psi^{-1})'_{\psi(p)}.$$

Obviously, $g_{UV}(p)$ is a linear isomorphism of $\mathbb{R}^n$ for all $p \in U \cap V$.

The maps $g_{UV}(p)$ are called the *transition functions* of the tangent bundle.

All these ingredients are part of being a *vector bundle*.

For more on bundles, see Lang [23], Gallot, Hulin and Lafontaine [17], Lafontaine [21] or Bott and Tu [5].

When $M = \mathbb{R}^n$, observe that
$T(M) = M \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, i.e., the bundle $T(M)$ is (globally) trivial.

Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds, we can define the function, $dh\colon T(M) \to T(N)$, (also denoted Th, or h_*, or Dh) by setting

$$dh(u) = dh_p(u), \quad \text{iff} \quad u \in T_p(M).$$

We leave the next proposition as an exercise to the reader (A proof can be found in Berger and Gostiaux [4]).

Proposition 3.16. *Given a C^k-map, $h\colon M \to N$, between two C^k-manifolds M and N (with $k \geq 1$), the map $dh\colon T(M) \to T(N)$ is a C^{k-1} map.*

We are now ready to define vector fields.

Definition 3.17. Let M be a C^{k+1} manifold, with $k \geq 1$. For any open subset, U of M, a *vector field on U* is any *section X of $T(M)$ over U*, that is, any function $X\colon U \to T(M)$ such that $\pi \circ X = \mathrm{id}_U$ (i.e., $X(p) \in T_p(M)$, for every $p \in U$). We also say that X is a *lifting of U into $T(M)$*.

We say that X is a *C^k-vector field on U* iff X is a section over U and a C^k-map.

The set of C^k-vector fields over U is denoted $\Gamma^{(k)}(U, T(M))$.

Given a curve, $\gamma\colon [a,b] \to M$, a *vector field X along γ* is any section of $T(M)$ over γ, i.e., a C^k-function, $X\colon [a,b] \to T(M)$, such that $\pi \circ X = \gamma$. We also say that *X lifts γ into $T(M)$*.

Clearly, $\Gamma^{(k)}(U, T(M))$ is a real vector space.

For short, the space $\Gamma^{(k)}(M, T(M))$ is also denoted by $\Gamma^{(k)}(T(M))$ (or $\mathfrak{X}^{(k)}(M)$, or even $\Gamma(T(M))$ or $\mathfrak{X}(M)$).

Remark: We can also define a *C^j-vector field on U* as a section, X, over U which is a C^j-map, where $0 \leq j \leq k$. Then, we have the vector space $\Gamma^{(j)}(U, T(M))$, *etc.*

If $M = \mathbb{R}^n$ and U is an open subset of M, then $T(M) = \mathbb{R}^n \times \mathbb{R}^n$ and a section of $T(M)$ over U is simply a function, X, such that

$$X(p) = (p, u), \quad \text{with} \quad u \in \mathbb{R}^n,$$

for all $p \in U$. In other words, X is defined by a function, $f\colon U \to \mathbb{R}^n$ (namely, $f(p) = u$).

This corresponds to the "old" definition of a vector field in the more basic case where the manifold, M, is just $\mathbb{R}^n$.

For any vector field $X \in \Gamma^{(k)}(U, T(M))$ and for any $p \in U$, we have $X(p) = (p, v)$ for some $v \in T_p(M)$, and it is convenient to denote the vector v by X_p so that $X(p) = (p, X_p)$.

In fact, in most situations it is convenient to identify $X(p)$ with $X_p \in T_p(M)$, and we will do so from now on.

This amounts to identifying the isomorphic vector spaces $\{p\} \times T_p(M)$ and $T_p(M)$.

Let us illustrate the advantage of this convention with the next definition.

Given any C^k-function, $f \in \mathcal{C}^k(U)$, and a vector field, $X \in \Gamma^{(k)}(U, T(M))$, we define the vector field, fX, by

$$(fX)_p = f(p)X_p, \quad p \in U.$$

Obviously, $fX \in \Gamma^{(k)}(U, T(M))$, which shows that $\Gamma^{(k)}(U, T(M))$ *is also a* $\mathcal{C}^k(U)$*-module*.

For any chart, (U, φ), on M it is easy to check that the map

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p, \quad p \in U,$$

is a C^k-vector field on U (with $1 \leq i \leq n$). This vector field is denoted $\left(\frac{\partial}{\partial x_i}\right)$ or $\frac{\partial}{\partial x_i}$.

Definition 3.18. Let M be a C^{k+1} manifold and let X be a C^k vector field on M. If U is any open subset of M and f is any function in $\mathcal{C}^k(U)$, then the *Lie derivative of f with respect to X*, denoted $X(f)$ or $L_X f$, is the function on U given by

$$X(f)(p) = X_p(f) = X_p(\mathbf{f}), \qquad p \in U.$$

Observe that

$$X(f)(p) = df_p(X_p),$$

where df_p is identified with the linear form in $T_p^*(M)$ defined by

$$df_p(v) = v(\mathbf{f}), \qquad v \in T_p M,$$

by identifying $T_{t_0}\mathbb{R}$ with $\mathbb{R}$ (see the discussion following Proposition 3.14).

The Lie derivative, $L_X f$, is also denoted $X[f]$.

As a special case, when (U, φ) is a chart on M, the vector field, $\frac{\partial}{\partial x_i}$, just defined above induces the function

$$p \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f, \quad f \in U,$$

denoted $\frac{\partial}{\partial x_i}(f)$ or $\left(\frac{\partial}{\partial x_i}\right) f$.

It is easy to check that $X(f) \in \mathcal{C}^{k-1}(U)$.

As a consequence, every vector field $X \in \Gamma^{(k)}(U, T(M))$ induces a linear map,

$$L_X \colon \mathcal{C}^k(U) \longrightarrow \mathcal{C}^{k-1}(U),$$

given by $f \mapsto X(f)$.

It is immediate to check that L_X has the Leibniz property, i.e.,

$$L_X(fg) = L_X(f)g + fL_X(g).$$

Linear maps with this property are called *derivations*.

Thus, we see that every vector field induces some kind of differential operator, namely, a derivation.

Unfortunately, not every derivation of the above type arises from a vector field, although this turns out to be true in the smooth case i.e., when $k = \infty$ (for a proof, see Gallot, Hulin and Lafontaine [17] or Lafontaine [21]).

In the rest of this section, unless stated otherwise, we assume that $k \geq 1$. The following easy proposition holds (c.f. Warner [33]):

Proposition 3.17. *Let X be a vector field on the C^{k+1}-manifold, M, of dimension n. Then, the following are equivalent:*

(a) X is C^k.

(b) If (U, φ) is a chart on M and if $f_1, \ldots, f_n$ are the functions on U uniquely defined by

$$X \restriction U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

then each f_i is a C^k-map.

(c) Whenever U is open in M and $f \in \mathcal{C}^k(U)$, then $X(f) \in \mathcal{C}^{k-1}(U)$.

Given any two C^k-vector field, X, Y, on M, for any function, $f \in \mathcal{C}^k(M)$, we defined above the function $X(f)$ and $Y(f)$.

Thus, we can form $X(Y(f))$ (resp. $Y(X(f))$), which are in $\mathcal{C}^{k-2}(M)$.

Unfortunately, even in the smooth case, there is generally no vector field, Z, such that

$$Z(f) = X(Y(f)), \quad \text{for all } f \in \mathcal{C}^k(M).$$

This is because $X(Y(f))$ (and $Y(X(f))$) involve second-order derivatives.

However, if we consider $X(Y(f)) - Y(X(f))$, then second-order derivatives cancel out and there is a unique vector field inducing the above differential operator.

Intuitively, $XY - YX$ measures the "failure of X and Y to commute."

Proposition 3.18. *Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, there is a unique C^{k-1}-vector field, $[X, Y]$, such that*

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \textit{ for all } f \in \mathcal{C}^{k-1}(M).$$

Definition 3.19. Given any C^{k+1}-manifold, M, of dimension n, for any two C^k-vector fields, X, Y, on M, the *Lie bracket*, $[X, Y]$, of X and Y, is the C^{k-1} vector field defined so that

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \text{for all} \quad f \in \mathcal{C}^{k-1}(M).$$

We also have the following simple proposition whose proof is left as an exercise (or, see Do Carmo [12]):

Proposition 3.19. *Given any C^{k+1}-manifold, M, of dimension n, for any C^k-vector fields, X, Y, Z, on M, for all $f, g \in \mathcal{C}^k(M)$, we have:*

(a) $[[X, Y], Z] + [[Y, Z], X] + [Z, X], Y] = 0$ *(Jacobi identity).*

(b) $[X, X] = 0$.

(c) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

(d) $[-, -]$ *is bilinear.*

Consequently, for smooth manifolds ($k = \infty$), the space of vector fields, $\Gamma^{(\infty)}(T(M))$, is a vector space equipped with a bilinear operation, $[-,-]$, that satisfies the Jacobi identity.

This makes $\Gamma^{(\infty)}(T(M))$ a *Lie algebra*.

Let $h\colon M \to N$ be a diffeomorphism between two manifolds. Then, vector fields can be transported from N to M and conversely.

Definition 3.20. Let $h \colon M \to N$ be a diffeomorphism between two C^{k+1} manifolds. For every C^k vector field, Y, on N, the *pull-back of Y along h* is the vector field, h^*Y, on M, given by

$$(h^*Y)_p = dh^{-1}_{h(p)}(Y_{h(p)}), \qquad p \in M.$$

For every C^k vector field, X, on M, the *push-forward of X along h* is the vector field, h_*X, on N, given by

$$h_*X = (h^{-1})^*X,$$

that is, for every $p \in M$,

$$(h_*X)_{h(p)} = dh_p(X_p),$$

or equivalently,

$$(h_*X)_q = dh_{h^{-1}(q)}(X_{h^{-1}(q)}), \qquad q \in N.$$

It is not hard to check that

$$L_{h_*X}f = L_X(f \circ h) \circ h^{-1},$$

for any function $f \in C^k(N)$.

One more notion will be needed to when we deal with Lie algebras.

Definition 3.21. Let $h\colon M \to N$ be a C^{k+1}-map of manifolds. If X is a C^k vector field on M and Y is a C^k vector field on N, we say that *X and Y are h-related* iff

$$dh \circ X = Y \circ h.$$

Proposition 3.20. *Let $h\colon M \to N$ be a C^{k+1}-map of manifolds, let X and Y be C^k vector fields on M and let X_1, Y_1 be C^k vector fields on N. If X is h-related to X_1 and Y is h-related to Y_1, then $[X, Y]$ is h-related to $[X_1, Y_1]$.*

3.4 Submanifolds, Immersions, Embeddings

Although the notion of submanifold is intuitively rather clear, technically, it is a bit tricky.

In fact, the reader may have noticed that many different definitions appear in books and that it is not obvious at first glance that these definitions are equivalent.

What is important is that a submanifold, N, of a given manifold, M, not only have the topology induced M but also that the charts of N be somewhow induced by those of M.

(Recall that if X is a topological space and Y is a subset of X, then the *subspace topology on Y* or *topology induced by X on Y*, has for its open sets all subsets of the form $Y \cap U$, where U is an arbitary open subset of X.).

Given m, n, with $0 \leq m \leq n$, we can view $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ using the inclusion

$$\mathbb{R}^m \cong \mathbb{R}^m \times \{\underbrace{(0, \ldots, 0)}_{n-m}\} \hookrightarrow \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n,$$

given by

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{n-m}).$$

Definition 3.22. Given a C^k-manifold, M, of dimension n, a subset, N, of M is an *m-dimensional submanifold of M* (where $0 \leq m \leq n$) iff for every point, $p \in N$, there is a chart (U, φ) of M (in the maximal atlas for M), with $p \in U$, so that

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\}).$$

(We write $0_{n-m} = \underbrace{(0, \ldots, 0)}_{n-m}$.)

The subset, $U \cap N$, of Definition 3.22 is sometimes called a *slice* of (U, φ) and we say that (U, φ) is *adapted to N* (See O'Neill [30] or Warner [33]).

Other authors, including Warner [33], use the term submanifold in a broader sense than us and they use the word *embedded submanifold* for what is defined in Definition 3.22.

The following proposition has an almost trivial proof but it justifies the use of the word submanifold:

Proposition 3.21. *Given a C^k-manifold, M, of dimension n, for any submanifold, N, of M of dimension $m \leq n$, the family of pairs $(U \cap N, \varphi \restriction U \cap N)$, where (U, φ) ranges over the charts over any atlas for M, is an atlas for N, where N is given the subspace topology. Therefore, N inherits the structure of a C^k-manifold.*

In fact, every chart on N arises from a chart on M in the following precise sense:

Proposition 3.22. *Given a C^k-manifold, M, of dimension n and a submanifold, N, of M of dimension $m \leq n$, for any $p \in N$ and any chart, (W, η), of N at p, there is some chart, (U, φ), of M at p so that*

$$\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^m \times \{0_{n-m}\})$$

and

$$\varphi \restriction U \cap N = \eta \restriction U \cap N,$$

where $p \in U \cap N \subseteq W$.

It is also useful to define more general kinds of "submanifolds."

Definition 3.23. Let $h\colon N \to M$ be a C^k-map of manifolds.

(a) The map h is an *immersion* of N into M iff dh_p is injective for all $p \in N$.

(b) The set $h(N)$ is an *immersed submanifold* of M iff h is an injective immersion.

(c) The map h is an *embedding* of N into M iff h is an injective immersion such that the induced map, $N \longrightarrow h(N)$, is a homeomorphism, where $h(N)$ is given the subspace topology (equivalently, h is an open map from N into $h(N)$ with the subspace topology). We say that $h(N)$ (with the subspace topology) is an *embedded submanifold* of M.

(d) The map h is a *submersion* of N into M iff dh_p is surjective for all $p \in N$.

Again, we warn our readers that certain authors (such as Warner [33]) call $h(N)$, in (b), a submanifold of M! We prefer the terminology *immersed submanifold*.

The notion of immersed submanifold arises naturally in the framework of Lie groups.

Indeed, the fundamental correspondence between Lie groups and Lie algebras involves Lie subgroups that are not necessarily closed.

But, as we will see later, subgroups of Lie groups that are also submanifolds are always closed.

It is thus necessary to have a more inclusive notion of submanifold for Lie groups and the concept of immersed submanifold is just what's needed.

Immersions of $\mathbb{R}$ into $\mathbb{R}^3$ are parametric curves and immersions of $\mathbb{R}^2$ into $\mathbb{R}^3$ are parametric surfaces. These have been extensively studied, for example, see DoCarmo [11], Berger and Gostiaux [4] or Gallier [16].

Immersions (i.e., subsets of the form $h(N)$, where N is an immersion) are generally neither injective immersions (i.e., subsets of the form $h(N)$, where N is an injective immersion) nor embeddings (or submanifolds).

For example, immersions can have self-intersections, as the plane curve (nodal cubic): $x = t^2 - 1; y = t(t^2 - 1)$.

Injective immersions are generally not embeddings (or submanifolds) because $h(N)$ may not be homeomorphic to N.

An example is given by the Lemniscate of Benoulli, an injective immersion of $\mathbb{R}$ into $\mathbb{R}^2$:

$$\begin{aligned} x &= \frac{t(1+t^2)}{1+t^4}, \\ y &= \frac{t(1-t^2)}{1+t^4}. \end{aligned}$$

Another interesting example is the immersion of $\mathbb{R}$ into the 2-torus, $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$, given by

$$t \mapsto (\cos t, \sin t, \cos ct, \sin ct),$$

where $c \in \mathbb{R}$.

One can show that the image of $\mathbb{R}$ under this immersion is closed in T^2 iff c is rational. Moreover, the image of this immersion is dense in T^2 but not closed iff c is irrational.

The above example can be adapted to the torus in $\mathbb{R}^3$: One can show that the immersion given by

$$t \mapsto ((2 + \cos t)\cos(\sqrt{2}\,t), (2 + \cos t)\sin(\sqrt{2}\,t), \sin t),$$

is dense but not closed in the torus (in $\mathbb{R}^3$) given by

$$(s, t) \mapsto ((2 + \cos s)\cos t, (2 + \cos s)\sin t, \sin s),$$

where $s, t \in \mathbb{R}$.

There is, however, a close relationship between submanifolds and embeddings.

Proposition 3.23. *If N is a submanifold of M, then the inclusion map, $j\colon N \to M$, is an embedding. Conversely, if $h\colon N \to M$ is an embedding, then $h(N)$ with the subspace topology is a submanifold of M and h is a diffeomorphism between N and $h(N)$.*

In summary, embedded submanifolds and (our) submanifolds coincide.

Some authors refer to spaces of the form $h(N)$, where h is an injective immersion, as *immersed submanifolds*.

However, in general, an immersed submanifold is *not* a submanifold.

One case where this holds is when N is compact, since then, a bijective continuous map is a homeomorphism.

3.5 Integral Curves, Flow of a Vector Field, One-Parameter Groups of Diffeomorphisms

We begin with integral curves and (local) flows of vector fields on a manifold.

Definition 3.24. Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. An *integral curve (or trajectory) for X with initial condition p_0* is a C^{k-1}-curve, $\gamma\colon I \to M$, so that

$$\dot{\gamma}(t) = X_{\gamma(t)}, \quad \text{for all } t \in I \quad \text{and} \quad \gamma(0) = p_0,$$

where $I =]a, b[\subseteq \mathbb{R}$ is an open interval containing 0.

What definition 3.24 says is that an integral curve, γ, with initial condition p_0 is a curve on the manifold M passing through p_0 and such that, for every point $p = \gamma(t)$ on this curve, the tangent vector to this curve at p, i.e., $\dot{\gamma}(t)$, coincides with the value, X_p, of the vector field X at p.

Given a vector field, X, as above, and a point $p_0 \in M$, is there an integral curve through p_0? Is such a curve unique? If so, how large is the open interval I?

We provide some answers to the above questions below.

Definition 3.25. Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. A *local flow for X at p_0* is a map,

$$\varphi\colon J \times U \to M,$$

where $J \subseteq \mathbb{R}$ is an open interval containing 0 and U is an open subset of M containing p_0, so that for every $p \in U$, the curve $t \mapsto \varphi(t, p)$ is an integral curve of X with initial condition p.

Thus, a local flow for X is a family of integral curves for all points in some small open set around p_0 such that these curves all have the same domain, J, independently of the initial condition, $p \in U$.

The following theorem is the main existence theorem of local flows.

This is a promoted version of a similar theorem in the classical theory of ODE's in the case where M is an open subset of $\mathbb{R}^n$.

Theorem 3.24. *(Existence of a local flow) Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. There is an open interval $J \subseteq \mathbb{R}$ containing 0 and an open subset $U \subseteq M$ containing p_0, so that there is a unique local flow $\varphi\colon J \times U \to M$ for X at p_0.*

What this means is that if $\varphi_1\colon J \times U \to M$ and $\varphi_2\colon J \times U \to M$ are both local flows with domain $J \times U$, then $\varphi_1 = \varphi_2$. Furthermore, φ is C^{k-1}.

Theorem 3.24 holds under more general hypotheses, namely, when the vector field satisfies some *Lipschitz* condition, see Lang [23] or Berger and Gostiaux [4].

Now, we know that for any initial condition, p_0, there is some integral curve through p_0.

However, there could be two (or more) integral curves $\gamma_1 \colon I_1 \to M$ and $\gamma_2 \colon I_2 \to M$ with initial condition p_0.

This leads to the natural question: How do γ_1 and γ_2 differ on $I_1 \cap I_2$? The next proposition shows they don't!

Proposition 3.25. *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. If $\gamma_1 \colon I_1 \to M$ and $\gamma_2 \colon I_2 \to M$ are any two integral curves both with initial condition p_0, then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$.*

Proposition 3.25 implies the important fact that there is a *unique maximal* integral curve with initial condition p.

Indeed, if $\{\gamma_j\colon I_j \to M\}_{j\in K}$ is the family of all integral curves with initial condition p (for some big index set, K), if we let $I(p) = \bigcup_{j\in K} I_j$, we can define a curve, $\gamma_p\colon I(p) \to M$, so that

$$\gamma_p(t) = \gamma_j(t), \quad \text{if} \quad t \in I_j.$$

Since γ_j and γ_l agree on $I_j \cap I_l$ for all $j, l \in K$, the curve γ_p is indeed well defined and it is clearly an integral curve with initial condition p with the largest possible domain (the open interval, $I(p)$).

The curve γ_p is called the *maximal integral curve with initial condition p* and it is also denoted by $\gamma(p, t)$.

Note that Proposition 3.25 implies that any two distinct integral curves are disjoint, i.e., do not intersect each other.

Consider the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

If we write $\gamma(t) = (x(t), y(t))$, the differential equation, $\dot{\gamma}(t) = X(\gamma(t))$, is expressed by

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t), \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we write $X = \binom{x}{y}$ and $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the above equation is written as

$$X' = AX.$$

Now, as

$$e^{tA} = I + \frac{A}{1!}\,t + \frac{A^2}{2!}\,t^2 + \cdots + \frac{A^n}{n!}\,t^n + \cdots\ ,$$

we get

$$\frac{d}{dt}(e^{tA}) = A + \frac{A^2}{1!}\,t + \frac{A^3}{2!}\,t^2 + \cdots + \frac{A^n}{(n-1)!}\,t^{n-1} + \cdots = Ae^{tA},$$

so we see that $e^{tA}p$ is a solution of the ODE $X' = AX$ with initial condition $X = p$, and by uniqueness, $X = e^{tA}p$ is *the* solution of our ODE starting at $X = p$.

Thus, our integral curve, γ_p, through $p = \binom{x_0}{y_0}$ is the circle given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Observe that $I(p) = \mathbb{R}$, for every $p \in \mathbb{R}^2$.

The following interesting question now arises: Given any $p_0 \in M$, if $\gamma_{p_0} \colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0 and, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$, then there is a maximal integral curve, $\gamma_{p_1} \colon I(p_1) \to M$, with initial condition p_1;

What is the relationship between γ_{p_0} and γ_{p_1}, if any?

The answer is given by

Proposition 3.26. *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$ and let p_0 be a point on M. If $\gamma_{p_0} \colon I(p_0) \to M$ is the maximal integral curve with initial condition p_0, for any $t_1 \in I(p_0)$, if $p_1 = \gamma_{p_0}(t_1) \in M$ and $\gamma_{p_1} \colon I(p_1) \to M$ is the maximal integral curve with initial condition p_1, then*

$$I(p_1) = I(p_0) - t_1 \quad \textit{and} \quad \gamma_{p_1}(t) = \gamma_{\gamma_{p_0}(t_1)}(t) = \gamma_{p_0}(t + t_1),$$

for all $t \in I(p_0) - t_1$

Proposition 3.26 says that the traces $\gamma_{p_0}(I(p_0))$ and $\gamma_{p_1}(I(p_1))$ in M of the maximal integral curves γ_{p_0} and γ_{p_1} are identical; they only differ by a simple reparametrization ($u = t + t_1$).

It is useful to restate Proposition 3.26 by changing point of view.

So far, we have been focusing on integral curves, i.e., given any $p_0 \in M$, we let t vary in $I(p_0)$ and get an integral curve, γ_{p_0}, with domain $I(p_0)$.

Instead of holding $p_0 \in M$ fixed, we can hold $t \in \mathbb{R}$ fixed and consider the set

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\},$$

i.e., the set of points such that it is possible to "travel for t units of time from p" along the maximal integral curve, γ_p, with initial condition p (It is possible that $\mathcal{D}_t(X) = \emptyset$).

By definition, if $\mathcal{D}_t(X) \neq \emptyset$, the point $\gamma_p(t)$ is well defined, and so, we obtain a map, $\Phi_t^X \colon \mathcal{D}_t(X) \to M$, with domain $\mathcal{D}_t(X)$, given by

$$\Phi_t^X(p) = \gamma_p(t).$$

Definition 3.26. Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. For any $t \in \mathbb{R}$, let

$$\mathcal{D}_t(X) = \{p \in M \mid t \in I(p)\}$$

and

$$\mathcal{D}(X) = \{(t, p) \in \mathbb{R} \times M \mid t \in I(p)\}$$

and let $\Phi^X \colon \mathcal{D}(X) \to M$ be the map given by

$$\Phi^X(t, p) = \gamma_p(t).$$

The map Φ^X is called the *(global) flow of X* and $\mathcal{D}(X)$ is called its *domain of definition*.

For any $t \in \mathbb{R}$ such that $\mathcal{D}_t(X) \neq \emptyset$, the map, $p \in \mathcal{D}_t(X) \mapsto \Phi^X(t, p) = \gamma_p(t)$, is denoted by Φ_t^X (i.e.,

$$\Phi_t^X(p) = \Phi^X(t, p) = \gamma_p(t)).$$

Observe that

$$\mathcal{D}(X) = \bigcup_{p \in M} (I(p) \times \{p\}).$$

Also, using the Φ_t^X notation, the property of Proposition 3.26 reads

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X, \tag{$*$}$$

whenever both sides of the equation make sense.

Indeed, the above says

$$\Phi_s^X(\Phi_t^X(p)) = \Phi_s^X(\gamma_p(t)) = \gamma_{\gamma_p(t)}(s) = \gamma_p(s+t) = \Phi_{s+t}^X(p).$$

Using the above property, we can easily show that the Φ_t^X are invertible. In fact, the inverse of Φ_t^X is Φ_{-t}^X.

Theorem 3.27. *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. The following properties hold:*

(a) For every $t \in \mathbb{R}$, if $\mathcal{D}_t(X) \neq \emptyset$, then $\mathcal{D}_t(X)$ is open (this is trivially true if $\mathcal{D}_t(X) = \emptyset$).

(b) The domain, $\mathcal{D}(X)$, of the flow, Φ^X, is open and the flow is a C^{k-1} map, $\Phi^X \colon \mathcal{D}(X) \to M$.

(c) Each $\Phi_t^X \colon \mathcal{D}_t(X) \to \mathcal{D}_{-t}(X)$ is a C^{k-1}-diffeomorphism with inverse Φ_{-t}^X.

(d) For all $s, t \in \mathbb{R}$, the domain of definition of $\Phi_s^X \circ \Phi_t^X$ is contained but generally not equal to $\mathcal{D}_{s+t}(X)$. However, $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X) = \mathcal{D}_{s+t}(X)$ if s and t have the same sign. Moreover, on $\mathrm{dom}(\Phi_s^X \circ \Phi_t^X)$, we have

$$\Phi_s^X \circ \Phi_t^X = \Phi_{s+t}^X.$$

We may omit the superscript, X, and write Φ instead of Φ^X if no confusion arises.

The reason for using the terminology flow in referring to the map Φ^X can be clarified as follows:

For any t such that $\mathcal{D}_t(X) \neq \emptyset$, every integral curve, γ_p, with initial condition $p \in \mathcal{D}_t(X)$, is defined on some open interval containing $[0, t]$, and we can picture these curves as "flow lines" along which the points p flow (travel) for a time interval t.

Then, $\Phi^X(t, p)$ is the point reached by "flowing" for the amount of time t on the integral curve γ_p (through p) starting from p.

Intuitively, we can imagine the flow of a fluid through M, and the vector field X is the field of velocities of the flowing particles.

Given a vector field, X, as above, it may happen that $\mathcal{D}_t(X) = M$, for all $t \in \mathbb{R}$.

In this case, namely, when $\mathcal{D}(X) = \mathbb{R} \times M$, we say that the vector field X is *complete*.

Then, the Φ_t^X are diffeomorphisms of M and they form a group.

The family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ a called a *1-parameter group of X*.

In this case, Φ^X induces a group homomorphism, $(\mathbb{R}, +) \longrightarrow \mathrm{Diff}(M)$, from the additive group $\mathbb{R}$ to the group of C^{k-1}-diffeomorphisms of M.

By abuse of language, even when it is **not** the case that $\mathcal{D}_t(X) = M$ for all t, the family $\{\Phi_t^X\}_{t\in\mathbb{R}}$ is called a *local 1-parameter group of X*, even though it is *not* a group, because the composition $\Phi_s^X \circ \Phi_t^X$ may not be defined.

If we go back to the vector field in $\mathbb{R}^2$ given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

since the integral curve, $\gamma_p(t)$, through $p = \binom{x_0}{x_0}$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

the global flow associated with X is given by

$$\Phi^X(t, p) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} p,$$

and each diffeomorphism, Φ^X_t, is the rotation,

$$\Phi^X_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The 1-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is the group of rotations in the plane, $\mathbf{SO}(2)$.

More generally, if B is an $n\times n$ invertible matrix that has a real logarithm A (that is, if $e^A = B$), then the matrix A defines a vector field, X, in $\mathbb{R}^n$, with

$$X = \sum_{i,j=1}^{n} (a_{ij}x_j)\frac{\partial}{\partial x_i},$$

whose integral curves are of the form,

$$\gamma_p(t) = e^{tA}p,$$

and we have

$$\gamma_p(1) = Bp.$$

The one-parameter group, $\{\Phi_t^X\}_{t\in\mathbb{R}}$, generated by X is given by $\{e^{tA}\}_{t\in\mathbb{R}}$.

When M is compact, it turns out that every vector field is complete, a nice and useful fact.

Proposition 3.28. *Let X be a C^{k-1} vector field on a C^k-manifold, M, $(k \geq 2)$. If M is compact, then X is complete, i.e., $\mathcal{D}(X) = \mathbb{R} \times M$. Moreover, the map $t \mapsto \Phi_t^X$ is a homomorphism from the additive group $\mathbb{R}$ to the group, $\mathrm{Diff}(M)$, of (C^{k-1}) diffeomorphisms of M.*

Remark: The proof of Proposition 3.28 also applies when X is a vector field with compact support (this means that the closure of the set $\{p \in M \mid X(p) \neq 0\}$ is compact).

If $h\colon M \to N$ is a diffeomorphism and X is a vector field on M, it can be shown that the local 1-parameter group associated with the vector field, h_*X, is

$$\{h \circ \Phi_t^X \circ h^{-1}\}_{t \in \mathbb{R}}.$$

A point $p \in M$ where a vector field vanishes, i.e., $X(p) = 0$, is called a *critical point of X*.

Critical points play a major role in the study of vector fields, in differential topology (e.g., the celebrated Poincaré–Hopf index theorem) and especially in Morse theory, but we won't go into this here.

Another famous theorem about vector fields says that every smooth vector field on a sphere of even dimension (S^{2n}) must vanish in at least one point (the so-called "hairy-ball theorem."

On S^2, it says that you can't comb your hair without having a singularity somewhere. Try it, it's true!).

Let us just observe that if an integral curve, γ, passes through a critical point, p, then γ is reduced to the point p, i.e., $\gamma(t) = p$, for all t.

Then, we see that if a maximal integral curve is defined on the whole of $\mathbb{R}$, either it is injective (it has no self-intersection), or it is simply periodic (i.e., there is some $T > 0$ so that $\gamma(t + T) = \gamma(t)$, for all $t \in \mathbb{R}$ and γ is injective on $[0, T[\,$), or it is reduced to a single point.

We conclude this section with the definition of the Lie derivative of a vector field with respect to another vector field.

Say we have two vector fields X and Y on M. For any $p \in M$, we can flow along the integral curve of X with initial condition p to $\Phi_t(p)$ (for t small enough) and then evaluate Y there, getting $Y(\Phi_t(p))$.

Now, this vector belongs to the tangent space $T_{\Phi_t(p)}(M)$, but $Y(p) \in T_p(M)$.

So to "compare" $Y(\Phi_t(p))$ and $Y(p)$, we bring back $Y(\Phi_t(p))$ to $T_p(M)$ by applying the tangent map, $d\Phi_{-t}$, at $\Phi_t(p)$, to $Y(\Phi_t(p))$. (Note that to alleviate the notation, we use the slight abuse of notation $d\Phi_{-t}$ instead of $d(\Phi_{-t})_{\Phi_t(p)}$.)

Then, we can form the difference $d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)$, divide by t and consider the limit as t goes to 0.

Definition 3.27. Let M be a C^{k+1} manifold. Given any two C^k vector fields, X and Y on M, for every $p \in M$, the *Lie derivative of Y with respect to X at p*, denoted $(L_X\, Y)_p$, is given by

$$\begin{aligned}
(L_X\, Y)_p &= \lim_{t \longrightarrow 0} \frac{d\Phi_{-t}(Y(\Phi_t(p))) - Y(p)}{t} \\
&= \frac{d}{dt}\left(d\Phi_{-t}(Y(\Phi_t(p)))\right)\bigg|_{t=0}.
\end{aligned}$$

It can be shown that $(L_X\, Y)_p$ is our old friend, the Lie bracket, i.e.,

$$(L_X\, Y)_p = [X, Y]_p.$$

(For a proof, see Warner [33] or O'Neill [30]).

In terms of Definition 3.20, observe that

$$\begin{aligned}
(L_X\, Y)_p &= \lim_{t \longrightarrow 0} \frac{\big((\Phi_{-t})_* Y\big)(p) - Y(p)}{t} \\
&= \lim_{t \longrightarrow 0} \frac{\big(\Phi_t^* Y\big)(p) - Y(p)}{t} \\
&= \frac{d}{dt} \big(\Phi_t^* Y\big)(p)\bigg|_{t=0},
\end{aligned}$$

since $(\Phi_{-t})^{-1} = \Phi_t$.

3.6 Partitions of Unity

To study manifolds, it is often necessary to construct various objects such as functions, vector fields, Riemannian metrics, volume forms, etc., by gluing together items constructed on the domains of charts.

Partitions of unity are a crucial technical tool in this gluing process.

The first step is to define "bump functions" (also called plateau functions). For any $r > 0$, we denote by $B(r)$ the open ball

$$B(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < r\},$$

and by $\overline{B(r)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq r\}$, its closure.

Given a topological space, X, for any function, $f \colon X \to \mathbb{R}$, the *support of f*, denoted $\operatorname{supp} f$, is the closed set

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Proposition 3.29. *There is a smooth function, $b\colon \mathbb{R}^n \to \mathbb{R}$, so that*

$$b(x) = \begin{cases} 1 & \text{if } x \in \overline{B(1)} \\ 0 & \text{if } x \in \mathbb{R}^n - B(2). \end{cases}$$

Proposition 3.29 yields the following useful technical result:

Proposition 3.30. *Let M be a smooth manifold. For any open subset, $U \subseteq M$, any $p \in U$ and any smooth function, $f\colon U \to \mathbb{R}$, there exist an open subset, V, with $p \in V$ and a smooth function, $\widetilde{f}\colon M \to \mathbb{R}$, defined on the whole of M, so that $\overline{V}$ is compact,*

$$\overline{V} \subseteq U, \qquad \operatorname{supp} \widetilde{f} \subseteq U$$

and

$$\widetilde{f}(q) = f(q), \qquad \text{for all } q \in \overline{V}.$$

If X is a (Hausdorff) topological space, a family, $\{U_\alpha\}_{\alpha \in I}$, of subsets U_α of X is a *cover* (or *covering*) of X iff $X = \bigcup_{\alpha \in I} U_\alpha$.

A cover, $\{U_\alpha\}_{\alpha \in I}$, such that each U_α is open is an *open cover*.

If $\{U_\alpha\}_{\alpha \in I}$ is a cover of X, for any subset, $J \subseteq I$, the subfamily $\{U_\alpha\}_{\alpha \in J}$ is a *subcover* of $\{U_\alpha\}_{\alpha \in I}$ if $X = \bigcup_{\alpha \in J} U_\alpha$, i.e., $\{U_\alpha\}_{\alpha \in J}$ is still a cover of X.

Given a cover $\{V_\beta\}_{\beta \in J}$, we say that a family $\{U_\alpha\}_{\alpha \in I}$ is a *refinement* of $\{V_\beta\}_{\beta \in J}$ if it is a cover and if there is a function, $h \colon I \to J$, so that $U_\alpha \subseteq V_{h(\alpha)}$, for all $\alpha \in I$.

A family $\{U_\alpha\}_{\alpha \in I}$ of subsets of X is *locally finite* iff for every point, $p \in X$, there is some open subset, U, with $p \in U$, so that $U \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in I$.

A space, X, is *paracompact* iff every open cover has an open locally finite refinement.

Remark: Recall that a space, X, is *compact* iff it is Hausdorff and if every open cover has a *finite* subcover. Thus, the notion of paracompactness (due to Jean Dieudonné) is a generalization of the notion of compactness.

Recall that a topological space, X, is *second-countable* if it has a countable basis, i.e., if there is a countable family of open subsets, $\{U_i\}_{i\geq 1}$, so that every open subset of X is the union of some of the U_i's.

A topological space, X, if *locally compact* iff it is Hausdorff and for every $a \in X$, there is some compact subset, K, and some open subset, U, with $a \in U$ and $U \subseteq K$.

As we will see shortly, every locally compact and second-countable topological space is paracompact.

It is important to observe that every manifold (even not second-countable) is locally compact.

Definition 3.28. Let M be a (smooth) manifold. A *partition of unity on M* is a family, $\{f_i\}_{i\in I}$, of smooth functions on M (the index set I may be uncountable) such that

(a) The family of supports, $\{\mathrm{supp}\, f_i\}_{i\in I}$, is locally finite.

(b) For all $i \in I$ and all $p \in M$, we have $0 \leq f_i(p) \leq 1$, and
$$\sum_{i\in I} f_i(p) = 1, \quad \text{for every } p \in M.$$

Note that condition (b) implies that $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a cover of M. If $\{U_\alpha\}_{\alpha\in J}$ is a cover of M, we say that the partition of unity $\{f_i\}_{i\in I}$ is *subordinate* to the cover $\{U_\alpha\}_{\alpha\in J}$ if $\{\mathrm{supp}\, f_i\}_{i\in I}$ is a refinement of $\{U_\alpha\}_{\alpha\in J}$.

When $I = J$ and $\mathrm{supp}\, f_i \subseteq U_i$, we say that $\{f_i\}_{i\in I}$ is *subordinate* to $\{U_\alpha\}_{\alpha\in I}$ *with the same index set as the partition of unity*.

In Definition 3.28, by (a), for every $p \in M$, there is some open set, U, with $p \in U$ and U meets only finitely many of the supports, supp f_i.

So, $f_i(p) \neq 0$ for only finitely many $i \in I$ and the infinite sum $\sum_{i\in I} f_i(p)$ is well defined.

Proposition 3.31. *Let X be a topological space which is second-countable and locally compact (thus, also Hausdorff). Then, X is paracompact. Moreover, every open cover has a countable, locally finite refinement consisting of open sets with compact closures.*

Remarks:

1. Proposition 3.31 implies that a second-countable, locally compact (Hausdorff) topological space is the union of countably many compact subsets. Thus, X is *countable at infinity*, a notion that we already encountered in Proposition 2.19 and Theorem 2.20.

2. A manifold that is countable at infinity has a countable open cover by domains of charts. It follows that M is second-countable. Thus, for manifolds, second-countable is equivalent to countable at infinity.

Recall that we are assuming that our manifolds are Hausdorff and second-countable.

Theorem 3.32. *Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for M. Then, there is a countable partition of unity, $\{f_i\}_{i \geq 1}$, subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ and the support,* $\mathrm{supp}\, f_i$, *of each f_i is compact.*

If one does not require compact supports, then there is a partition of unity, $\{f_\alpha\}_{\alpha \in I}$, subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ with at most countably many of the f_α not identically zero. (In the second case, $\mathrm{supp}\, f_\alpha \subseteq U_\alpha$.*)*

We close this section by stating a famous theorem of Whitney whose proof uses partitions of unity.

Theorem 3.33. *(Whitney, 1935) Any smooth manifold (Hausdorff and second-countable), M, of dimension n is diffeomorphic to a closed submanifold of $\mathbb{R}^{2n+1}$.*

For a proof, see Hirsch [19], Chapter 2, Section 2, Theorem 2.14.

3.7 Manifolds With Boundary

Up to now, we have defined manifolds locally diffeomorphic to an open subset of $\mathbb{R}^m$.

This excludes many natural spaces such as a closed disk, whose boundary is a circle, a closed ball, $\overline{B(1)}$, whose boundary is the sphere, S^{m-1}, a compact cylinder, $S^1 \times [0,1]$, whose boundary consist of two circles, a Möbius strip, etc.

These spaces fail to be manifolds because they have a boundary, that is, neighborhoods of points on their boundaries are not diffeomorphic to open sets in $\mathbb{R}^m$.

Perhaps the simplest example is the (closed) upper half space,

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

Under the natural embedding $\mathbb{R}^{m-1} \cong \mathbb{R}^{m-1} \times \{0\} \hookrightarrow \mathbb{R}^m$, the subset $\partial\mathbb{H}^m$ of $\mathbb{H}^m$ defined by

$$\partial\mathbb{H}^m = \{x \in \mathbb{H}^m \mid x_m = 0\}$$

is isomorphic to $\mathbb{R}^{m-1}$ and is called the *boundary of* $\mathbb{H}^m$. We also define the *interior* of $\mathbb{H}^m$ as

$$\mathrm{Int}(\mathbb{H}^m) = \mathbb{H}^m - \partial\mathbb{H}^m.$$

Now, if U and V are open subsets of $\mathbb{H}^m$, where $\mathbb{H}^m \subseteq \mathbb{R}^m$ has the subset topology, and if $f \colon U \to V$ is a continuous function, we need to explain what we mean by f being smooth.

We say that $f \colon U \to V$, as above, is *smooth* if it has an extension, $\widetilde{f} \colon \widetilde{U} \to \widetilde{V}$, where $\widetilde{U}$ and $\widetilde{V}$ are open subsets of $\mathbb{R}^m$ with $U \subseteq \widetilde{U}$ and $V \subseteq \widetilde{V}$ and with $\widetilde{f}$ a smooth function.

We say that f is a (smooth) *diffeomorphism* iff f^{-1} exists and if both f and f^{-1} are smooth, as just defined.

To define a *manifold with boundary*, we replace everywhere $\mathbb{R}$ by $\mathbb{H}$ in Definition 3.1 and Definition 3.2.

So, for instance, given a topological space, M, a *chart* is now pair, (U, φ), where U is an open subset of M and $\varphi\colon U \to \Omega$ is a homeomorphism onto an open subset, $\Omega = \varphi(U)$, of $\mathbb{H}^{n_\varphi}$ (for some $n_\varphi \geq 1$), etc.

Definition 3.29. Given some integer $n \geq 1$ and given some k such that k is either an integer $k \geq 1$ or $k = \infty$, a *C^k-manifold of dimension n with boundary* consists of a topological space, M, together with an equivalence class, $\overline{\mathcal{A}}$, of C^k n-atlases, on M (where the charts are now defined in terms of open subsets of $\mathbb{H}^n$).

Any atlas, $\mathcal{A}$, in the equivalence class $\overline{\mathcal{A}}$ is called a *differentiable structure of class C^k (and dimension n) on M*. We say that M is *modeled on $\mathbb{H}^n$*.

When $k = \infty$, we say that M is a *smooth manifold with boundary*.

It remains to define what is the boundary of a manifold with boundary!

By definition, the *boundary*, ∂M, of a manifold (with boundary), M, is the set of all points, $p \in M$, such that there is some chart, $(U_\alpha, \varphi_\alpha)$, with $p \in U_\alpha$ and $\varphi_\alpha(p) \in \partial \mathbb{H}^n$.

We also let $\mathrm{Int}(M) = M - \partial M$ and call it the *interior* of M.

Do not confuse the boundary ∂M and the interior $\mathrm{Int}(M)$ of a manifold with boundary embedded in $\mathbb{R}^N$ with the topological notions of boundary and interior of M as a topological space. In general, they are different.

Note that manifolds as defined earlier (In Definition 3.3) are also manifolds with boundary: their boundary is just empty.

We shall still reserve the word "manifold" for these, but for emphasis, we will sometimes call them "boundary-less."

The definition of tangent spaces, tangent maps, etc., are easily extended to manifolds with boundary.

The reader should note that if M is a manifold with boundary of dimension n, the tangent space, T_pM, is defined for all $p \in M$ and has dimension n, *even* for boundary points, $p \in \partial M$.

The only notion that requires more care is that of a sub-manifold. For more on this, see Hirsch [19], Chapter 1, Section 4.

One should also beware that the product of two manifolds with boundary is generally **not** a manifold with boundary (consider the product $[0, 1] \times [0, 1]$ of two line segments).

There is a generalization of the notion of a manifold with boundary called *manifold with corners* and such manifolds are closed under products (see Hirsch [19], Chapter 1, Section 4, Exercise 12).

If M is a manifold with boundary, we see that $\mathrm{Int}(M)$ is a manifold without boundary. What about ∂M?

Interestingly, the boundary, ∂M, of a manifold with boundary, M, of dimension n, is a manifold of dimension $n-1$.

Proposition 3.34. *If M is a manifold with boundary of dimension n, for any $p \in \partial M$ on the boundary on M, for any chart, (U, φ), with $p \in M$, we have $\varphi(p) \in \partial \mathbb{H}^n$.*

Using Proposition 3.34, we immediately derive the fact that ∂M is a manifold of dimension $n - 1$.

3.8 Orientation of Manifolds

Although the notion of orientation of a manifold is quite intuitive, it is technically rather subtle.

We restrict our discussion to smooth manifolds (although the notion of orientation can also be defined for topological manifolds, but more work is involved).

Intuitively, a manifold, M, is orientable if it is possible to give a consistent orientation to its tangent space, T_pM, at every point, $p \in M$.

So, if we go around a closed curve starting at $p \in M$, when we come back to p, the orientation of T_pM should be the same as when we started.

For exampe, if we travel on a Möbius strip (a manifold with boundary) dragging a coin with us, we will come back to our point of departure with the coin flipped. Try it!

To be rigorous, we have to say what it means to orient T_pM (a vector space) and what consistency of orientation means.

We begin by quickly reviewing the notion of orientation of a vector space.

Let E be a vector space of dimension n. If $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases of E, a basic and crucial fact of linear algebra says that there is a unique linear map, g, mapping each u_i to the corresponding v_i (i.e., $g(u_i) = v_i$, $i = 1, \ldots, n$).

Then, look at the determinant, $\det(g)$, of this map. We know that $\det(g) = \det(P)$, where P is the matrix whose j-th columns consist of the coordinates of v_j over the basis $u_1, \ldots, u_n$.

Either $\det(g)$ is negative or it is positive.

Thus, we define an equivalence relation on bases by saying that two bases have the *same orientation* iff the determinant of the linear map sending the first basis to the second has positive determinant.

An *orientation* of E is the choice of one of the two equivalence classes, which amounts to picking some basis as an orientation frame.

The above definition is perfectly fine but it turns out that it is more convenient, in the long term, to use a definition of orientation in terms of alternating multilinear maps (in particular, to define the notion of integration on a manifold).

Recall that a function, $h\colon E^k \to \mathbb{R}$, is *alternating multilinear* (or *alternating k-linear*) iff it is linear in each of its arguments (holding the others fixed) and if

$$h(\ldots, x, \ldots, x, \ldots) = 0,$$

that is, h vanishes whenever two of its arguments are identical.

Using multilinearity, we immediately deduce that h vanishes for all k-tuples of arguments, $u_1, \ldots, u_k$, that are linearly dependent and that h is *skew-symmetric*, i.e.,

$$h(\ldots, y, \ldots, x, \ldots) = -h(\ldots, x, \ldots, y, \ldots).$$

In particular, for $k = n$, it is easy to see that if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are two bases, then

$$h(v_1, \ldots, v_n) = \det(g)h(u_1, \ldots, u_n),$$

where g is the unique linear map sending each u_i to v_i.

This shows that any alternating n-linear function is a multiple of the determinant function and that the space of alternating n-linear maps is a one-dimensional vector space that we will denote $\bigwedge^n E^*$.

We also call an alternating n-linear map on E an *n-form on E*. But then, observe that two bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ have the same orientation iff

$\omega(u_1, \ldots, u_n)$ and $\omega(v_1, \ldots, v_n)$ have the same sign for all $\omega \in \bigwedge^n E^* - \{0\}$

(where 0 denotes the zero n-form).

As $\bigwedge^n E^*$ is one-dimensional, *picking an orientation of E is equivalent to picking a generator (a one-element basis), ω, of $\bigwedge^n E^*$, and to say that $u_1, \ldots, u_n$ has positive orientation iff $\omega(u_1, \ldots, u_n) > 0$.*

Given an orientation (say, given by $\omega \in \bigwedge^n E^*$) of E, a linear map, $f\colon E \to E$, is *orientation preserving* iff $\omega(f(u_1), \ldots, f(u_n)) > 0$ whenever $\omega(u_1, \ldots, u_n) > 0$ (or equivalently, iff $\det(f) > 0$).

Now, to define the orientation of an n-dimensional manifold, M, we use charts.

Given any $p \in M$, for any chart, (U, φ), at p, the tangent map, $d\varphi^{-1}_{\varphi(p)}\colon \mathbb{R}^n \to T_pM$ makes sense.

If $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$, as it gives an orientation to $\mathbb{R}^n$, we can orient T_pM by giving it the orientation induced by the basis $d\varphi^{-1}_{\varphi(p)}(e_1), \ldots, d\varphi^{-1}_{\varphi(p)}(e_n)$.

Then, the consistency of orientations of the T_pM's is given by the overlapping of charts.

We require that the Jacobian determinants of all $\varphi_j \circ \varphi_i^{-1}$ have the same sign, whenever (U_i, φ_i) and (U_j, φ_j) are any two overlapping charts.

Thus, we are led to the definition below. All definitions and results stated in the rest of this section apply to manifolds with or without boundary.

Definition 3.30. Given a smooth manifold, M, of dimension n, an *orientation atlas* of M is any atlas so that the transition maps, $\varphi_i^j = \varphi_j \circ \varphi_i^{-1}$, (from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$) all have a positive Jacobian determinant for every point in $\varphi_i(U_i \cap U_j)$.

A manifold is *orientable* iff its has some orientation atlas.

Definition 3.30 can be hard to check in practice and there is an equivalent criterion is terms of n-forms which is often more convenient.

The idea is that a manifold of dimension n is orientable iff there is a map, $p \mapsto \omega_p$, assigning to every point, $p \in M$, a nonzero n-form, $\omega_p \in \bigwedge^n T_p^* M$, so that this map is smooth.

In order to explain rigorously what it means for such a map to be smooth, we can define the *exterior n-bundle*, $\bigwedge^n T^*M$ (also denoted $\bigwedge_n^* M$) in much the same way that we defined the bundles TM and T^*M.

There is an obvious smooth projection map, $\pi\colon \bigwedge^n T^*M \to M$.

Then, leaving the details of the fact that $\bigwedge^n T^*M$ can be made into a smooth manifold (of dimension n) as an exercise, a smooth map, $p \mapsto \omega_p$, is simply a smooth section of the bundle $\bigwedge^n T^*M$, i.e., a smooth map, $\omega\colon M \to \bigwedge^n T^*M$, so that $\pi \circ \omega = \mathrm{id}$.

Definition 3.31. If M is an n-dimensional manifold, a smooth section, $\omega \in \Gamma(M, \bigwedge^n T^*M)$, is called a (smooth) *$n$-form*. The set of n-forms, $\Gamma(M, \bigwedge^n T^*M)$, is also denoted $\mathcal{A}^n(M)$.

An n-form, ω, is a *nowhere-vanishing n-form on M* or *volume form on M* iff ω_p is a nonzero form for every $p \in M$.

This is equivalent to saying that $\omega_p(u_1, \ldots, u_n) \neq 0$, for all $p \in M$ and all bases, $u_1, \ldots, u_n$, of T_pM.

The determinant function, $(u_1, \ldots, u_n) \mapsto \det(u_1, \ldots, u_n)$, where the u_i are expressed over the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, is a volume form on $\mathbb{R}^n$. We will denote this volume form by ω_0.

Observe the justification for the term volume form: the quantity $\det(u_1, \ldots, u_n)$ is indeed the (signed) volume of the parallelepiped

$$\{\lambda_1 u_1 + \cdots + \lambda_n u_n \mid 0 \leq \lambda_i \leq 1,\ 1 \leq i \leq n\}.$$

A volume form on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is obtained as follows:

$$\omega_p(u_1, \ldots u_n) = \det(p, u_1, \ldots u_n),$$

where $p \in S^n$ and $u_1, \ldots u_n \in T_pS^n$. As the u_i are orthogonal to p, this is indeed a volume form.

Observe that if f is a smooth function on M and ω is any n-form, then $f\omega$ is also an n-form.

Definition 3.32. Let $h\colon M \to N$ be a smooth map of manifolds of the same dimension, n, and let $\omega \in \mathcal{A}^n(N)$ be an n-form on N. The *pull-back, $h^*\omega$, of ω to M* is the n-form on M given by

$$h^*\omega_p(u_1, \ldots, u_n) = \omega_{h(p)}(dh_p(u_1), \ldots, dh_p(u_n)),$$

for all $p \in M$ and all $u_1, \ldots, u_n \in T_pM$.

One checks immediately that $h^*\omega$ is indeed an n-form on M. More interesting is the following Proposition:

Proposition 3.35. *(a) If $h\colon M \to N$ is a local diffeomorphism of manifolds, where $\dim M = \dim N = n$, and $\omega \in \mathcal{A}^n(N)$ is a volume form on N, then $h^*\omega$ is a volume form on M. (b) Assume M has a volume form, ω. Then, for every n-form, $\eta \in \mathcal{A}^n(M)$, there is a unique smooth function, $f \in C^\infty(M)$, so that $\eta = f\omega$. If η is a volume form, then $f(p) \neq 0$ for all $p \in M$.*

Remark: If h_1 and h_2 are smooth maps of manifolds, it is easy to prove that

$$(h_2 \circ h_1)^* = h_1^* \circ h_2^*$$

and that for any smooth map $h\colon M \to N$,

$$h^*(f\omega) = (f \circ h)h^*\omega,$$

where f is any smooth function on N and ω is any n-form on N.

The connection between Definition 3.30 and volume forms is given by the following important theorem whose proof contains a wonderful use of partitions of unity.

Theorem 3.36. *A smooth manifold (Hausdorff and second-countable) is orientable iff it possesses a volume form.*

Since we showed that there is a volume form on the sphere, S^n, by Theorem 3.36, the sphere S^n is orientable.

It can be shown that the projective spaces, $\mathbb{RP}^n$, are non-orientable iff n is even and thus, orientable iff n is odd. In particular, $\mathbb{RP}^2$ is not orientable.

Also, even though M may not be orientable, its tangent bundle, $T(M)$, is always orientable! (Prove it).

It is also easy to show that if $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth submersion, then $M = f^{-1}(0)$ is a smooth orientable manifold.

Another nice fact is that every Lie group is orientable.

By Proposition 3.35 (b), given any two volume forms, ω_1 and ω_2 on a manifold, M, there is a function, $f\colon M \to \mathbb{R}$, never 0 on M such that $\omega_2 = f\omega_1$.

This fact suggests the following definition:

Definition 3.33. Given an orientable manifold, M, two volume forms, ω_1 and ω_2, on M are *equivalent* iff $\omega_2 = f\omega_1$ for some smooth function, $f\colon M \to \mathbb{R}$, such that $f(p) > 0$ for all $p \in M$.

An *orientation of M* is the choice of some equivalence class of volume forms on M and an *oriented manifold* is a manifold together with a choice of orientation.

If M is a manifold oriented by the volume form, ω, for every $p \in M$, a basis, $(b_1, \ldots, b_n)$ of T_pM is *posively oriented* iff $\omega_p(b_1, \ldots, b_n) > 0$, else it is *negatively oriented* (where $n = \dim(M)$).

A connected orientable manifold has two orientations.

We will also need the notion of orientation-preserving diffeomorphism.

Definition 3.34. Let $h\colon M \to N$ be a diffeomorphism of oriented manifolds, M and N, of dimension n and say the orientation on M is given by the volume form ω_1 while the orientation on N is given by the volume form ω_2. We say that h is *orientation preserving* iff $h^*\omega_2$ determines the same orientation of M as ω_1.

Using Definition 3.34 we can define the notion of a positive atlas.

Definition 3.35. If M is a manifold oriented by the volume form, ω, an atlas for M is *positive* iff for every chart, (U, φ), the diffeomorphism, $\varphi\colon U \to \varphi(U)$, is orientation preserving, where U has the orientation induced by M and $\varphi(U) \subseteq \mathbb{R}^n$ has the orientation induced by the standard orientation on $\mathbb{R}^n$ (with $\dim(M) = n$).

The proof of Theorem 3.36 shows

Proposition 3.37. *If a manifold, M, has an orientation atlas, then there is a uniquely determined orientation on M such that this atlas is positive.*

3.9 Covering Maps and Universal Covering Manifolds

Covering maps are an important technical tool in algebraic topology and more generally in geometry.

We begin with covering maps.

Definition 3.36. A map, $\pi\colon M \to N$, between two smooth manifolds is a *covering map* (or *cover*) iff

(1) The map π is smooth and surjective.

(2) For any $q \in N$, there is some open subset, $V \subseteq N$, so that $q \in V$ and
$$\pi^{-1}(V) = \bigcup_{i\in I} U_i,$$
where the U_i are pairwise disjoint open subsets, $U_i \subseteq M$, and $\pi\colon U_i \to V$ is a diffeomorphism for every $i \in I$. We say that V is *evenly covered*.

The manifold, M, is called a *covering manifold* of N.

A *homomorphism* of coverings, $\pi_1 \colon M_1 \to N$ and $\pi_2 \colon M_2 \to N$, is a smooth map, $\phi \colon M_1 \to M_2$, so that

$$\pi_1 = \pi_2 \circ \phi,$$

that is, the following diagram commutes:

$$\begin{tikzcd} M_1 \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & M_2 \arrow[dl, "\pi_2"] \\ & N & \end{tikzcd}.$$

We say that the coverings $\pi_1 \colon M_1 \to N$ and $\pi_2 \colon M_2 \to N$ are *equivalent* iff there is a homomorphism, $\phi \colon M_1 \to M_2$, between the two coverings and ϕ is a diffeomorphism.

As usual, the inverse image, $\pi^{-1}(q)$, of any element $q \in N$ is called the *fibre over q*, the space N is called the *base* and M is called the *covering space*.

As π is a covering map, each fibre is a discrete space.

Note that a homomorphism maps each fibre $\pi_1^{-1}(q)$ in M_1 to the fibre $\pi_2^{-1}(\phi(q))$ in M_2, for every $q \in M_1$.

Proposition 3.38. *Let $\pi\colon M \to N$ be a covering map. If N is connected, then all fibres, $\pi^{-1}(q)$, have the same cardinality for all $q \in N$. Furthermore, if $\pi^{-1}(q)$ is not finite then it is countably infinite.*

When the common cardinality of fibres is finite it is called the *multiplicity* of the covering (or the number of *sheets*).

For any integer, $n > 0$, the map, $z \mapsto z^n$, from the unit circle $S^1 = \mathbf{U}(1)$ to itself is a covering with n sheets. The map,

$$t\colon \mapsto (\cos(2\pi t), \sin(2\pi t)),$$

is a covering, $\mathbb{R} \to S^1$, with infinitely many sheets.

It is also useful to note that a covering map, $\pi\colon M \to N$, is a local diffeomorphism (which means that $d\pi_p\colon T_pM \to T_{\pi(p)}N$ is a bijective linear map for every $p \in M$).

The crucial property of covering manifolds is that curves in N can be lifted to M, in a unique way. For any map, $\phi\colon P \to N$, a *lift of ϕ through π* is a map, $\widetilde{\phi}\colon P \to M$, so that

$$\phi = \pi \circ \widetilde{\phi},$$

as in the following commutative diagram:

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \end{tikzcd}$$

We state without proof the following results:

Proposition 3.39. *If $\pi\colon M \to N$ is a covering map, then for every smooth curve, $\alpha\colon I \to N$, in N (with $0 \in I$) and for any point, $q \in M$, such that $\pi(q) = \alpha(0)$, there is a unique smooth curve, $\widetilde{\alpha}\colon I \to M$, lifting α through π such that $\widetilde{\alpha}(0) = q$.*

Proposition 3.40. *Let $\pi\colon M \to N$ be a covering map and let $\phi\colon P \to N$ be a smooth map. For any $p_0 \in P$, any $q_0 \in M$ and any $r_0 \in N$ with $\pi(q_0) = \phi(p_0) = r_0$, the following properties hold:*

(1) If P is connected then there is at most one lift, $\widetilde{\phi}\colon P \to M$, of ϕ through π such that $\widetilde{\phi}(p_0) = q_0$.

(2) If P is simply connected, then such a lift exists.

$$\begin{tikzcd} & M \ni q_0 \arrow[d, "\pi"] \\ p_0 \in P \arrow[ur, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \ni r_0 \end{tikzcd}$$

Theorem 3.41. *Every connected manifold, M, possesses a simply connected covering map, $\pi\colon \widetilde{M} \to M$, that is, with $\widetilde{M}$ simply connected. Any two simply connected coverings of N are equivalent.*

In view of Theorem 3.41, it is legitimate to speak of *the* simply connected cover, $\widetilde{M}$, of M, also called *universal covering* (or *cover*) of M.

Given any point, $p \in M$, let $\pi_1(M, p)$ denote the fundamental group of M with basepoint p.

If $\phi\colon M \to N$ is a smooth map, for any $p \in M$, if we write $q = \phi(p)$, then we have an induced group homomorphism

$$\phi_*\colon \pi_1(M, p) \to \pi_1(N, q).$$

Proposition 3.42. *If $\pi\colon M \to N$ is a covering map, for every $p \in M$, if $q = \pi(p)$, then the induced homomorphism, $\pi_*\colon \pi_1(M, p) \to \pi_1(N, q)$, is injective.*

Proposition 3.43. *Let $\pi\colon M \to N$ be a covering map and let $\phi\colon P \to N$ be a smooth map. For any $p_0 \in P$, any $q_0 \in M$ and any $r_0 \in N$ with $\pi(q_0) = \phi(p_0) = r_0$, if P is connected, then a lift, $\widetilde{\phi}\colon P \to M$, of ϕ such that $\widetilde{\phi}(p_0) = q_0$ exists iff*

$$\phi_*(\pi_1(P, p_0)) \subseteq \pi_*(\pi_1(M, q_0)),$$

as illustrated in the diagram below

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\widetilde{\phi}"] \arrow[r, "\phi"'] & N \end{tikzcd} \qquad \text{iff} \qquad \begin{tikzcd} & \pi_1(M, q_0) \arrow[d, "\pi_*"] \\ \pi_1(P, p_0) \arrow[ur, dashed] \arrow[r, "\phi_*"'] & \pi_1(N, r_0) \end{tikzcd}$$

Basic Assumption: For any covering, $\pi\colon M \to N$, if N is connected then we also assume that M is connected.

Using Proposition 3.42, we get

Proposition 3.44. *If* $\pi\colon M \to N$ *is a covering map and* N *is simply connected, then* π *is a diffeomorphism (recall that* M *is connected); thus,* M *is diffeomorphic to the universal cover,* $\widetilde{N}$*, of* N*.*

The following proposition shows that the universal covering of a space covers every other covering of that space. This justifies the terminology "*universal covering*."

Proposition 3.45. *Say* $\pi_1\colon M_1 \to N$ *and* $\pi_2\colon M_2 \to N$ *are two coverings of* N*, with* N *connected. Every homomorphism,* $\phi\colon M_1 \to M_2$*, between these two coverings is a covering map. As a consequence, if* $\pi\colon \widetilde{N} \to N$ *is a universal covering of* N*, then for every covering,* $\pi'\colon M \to N$*, of* N*, there is a covering,* $\phi\colon \widetilde{N} \to M$*, of* M*.*

The notion of deck-transformation group of a covering is also useful because it yields a way to compute the fundamental group of the base space.

Definition 3.37. If $\pi\colon M \to N$ is a covering map, a *deck-transformation* is any diffeomorphism, $\phi\colon M \to M$, such that $\pi = \pi \circ \phi$, that is, the following diagram commutes:

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi"] \\ & N & \end{tikzcd}.$$

Note that deck-transformations are just automorphisms of the covering map.

The commutative diagram of Definition 3.37 means that a deck transformation permutes every fibre. It is immediately verified that the set of deck transformations of a covering map is a group denoted Γ_π (or simply, Γ), called the *deck-transformation group* of the covering.

Observe that any deck transformation, ϕ, is a lift of π through π. Consequently, if M is connected, by Proposition 3.40 (1), every deck-transformation is determined by its value at a single point.

So, the deck-transformations are determined by their action on each point of any fixed fibre, $\pi^{-1}(q)$, with $q \in N$.

Since the fibre $\pi^{-1}(q)$ is countable, Γ is also countable, that is, a discrete Lie group.

Moreover, if M is compact, as each fibre, $\pi^{-1}(q)$, is compact and discrete, it must be finite and so, the deck-transformation group is also finite.

The following proposition gives a useful method for determining the fundamental group of a manifold.

Proposition 3.46. *If $\pi \colon \widetilde{M} \to M$ is the universal covering of a connected manifold, M, then the deck-transformation group, $\widetilde{\Gamma}$, is isomorphic to the fundamental group, $\pi_1(M)$, of M.*

Remark: When $\pi\colon \widetilde{M} \to M$ is the universal covering of M, it can be shown that the group $\widetilde{\Gamma}$ acts simply and transitively on every fibre, $\pi^{-1}(q)$.

This means that for any two elements, $x, y \in \pi^{-1}(q)$, there is a unique deck-transformation, $\phi \in \widetilde{\Gamma}$ such that $\phi(x) = y$.

So, there is a bijection between $\pi_1(M) \cong \widetilde{\Gamma}$ and the fibre $\pi^{-1}(q)$.

Proposition 3.41 together with previous observations implies that if the universal cover of a connected (compact) manifold is compact, then M has a finite fundamental group.

We will use this fact later, in particular, in the proof of Myers' Theorem.