2.4 Topological Groups

Since Lie groups are topological groups (and manifolds), it is useful to gather a few basic facts about topological groups.

Definition 2.11. A set, G, is a *topological group* iff

(a) G is a Hausdorff topological space;

(b) G is a group (with identity 1);

(c) Multiplication, $\cdot\colon G \times G \to G$, and the inverse operation, $G \longrightarrow G\colon g \mapsto g^{-1}$, are continuous, where $G \times G$ has the product topology.

It is easy to see that the two requirements of condition (c) are equivalent to

(c′) The map $G \times G \longrightarrow G\colon (g, h) \mapsto gh^{-1}$ is continuous.

Given a topological group G, for every $a \in G$ we define *left translation* as the map, $L_a\colon G \to G$, such that $L_a(b) = ab$, for all $b \in G$, and *right translation* as the map, $R_a\colon G \to G$, such that $R_a(b) = ba$, for all $b \in G$.

Observe that $L_{a^{-1}}$ is the inverse of L_a and similarly, $R_{a^{-1}}$ is the inverse of R_a. As multiplication is continuous, we see that L_a and R_a are continuous.

Moreover, since they have a continuous inverse, they are homeomorphisms.

As a consequence, if U is an open subset of G, then so is $gU = L_g(U)$ (resp. $Ug = R_gU$), for all $g \in G$.

Therefore, the topology of a topological group (i.e., its family of open sets) is *determined* by the knowledge of the open subsets containing the identity, 1.

Given any subset, $S \subseteq G$, let $S^{-1} = \{s^{-1} \mid s \in S\}$; let $S^0 = \{1\}$ and $S^{n+1} = S^n S$, for all $n \geq 0$.

Property (c) of Definition 2.11 has the following useful consequences:

Proposition 2.11. *If G is a topological group and U is any open subset containing 1, then there is some open subset, $V \subseteq U$, with $1 \in V$, so that $V = V^{-1}$ and $V^2 \subseteq U$. Furthermore, $\overline{V} \subseteq U$.*

A subset, U, containing 1 such that $U = U^{-1}$, is called *symmetric*.

Using Proposition 2.11, we can give a very convenient characterization of the Hausdorff separation property in a topological group.

Proposition 2.12. *If G is a topological group, then the following properties are equivalent:*

(1) G is Hausdorff;

(2) The set $\{1\}$ is closed;

(3) The set $\{g\}$ is closed, for every $g \in G$.

If H is a subgroup of G (not necessarily normal), we can form the set of left cosets, G/H and we have the projection, $p\colon G \to G/H$, where $p(g) = gH = \overline{g}$.

If G is a topological group, then G/H can be given the *quotient topology*, where a subset $U \subseteq G/H$ is open iff $p^{-1}(U)$ is open in G.

With this topology, p is continuous.

The trouble is that G/H is not necessarily Hausdorff. However, we can neatly characterize when this happens.

Proposition 2.13. *If G is a topological group and H is a subgroup of G then the following properties hold:*

(1) The map $p\colon G \to G/H$ is an open map, which means that $p(V)$ is open in G/H whenever V is open in G.

(2) The space G/H is Hausdorff iff H is closed in G.

(3) If H is open, then H is closed and G/H has the discrete topology (every subset is open).

(4) The subgroup H is open iff $1 \in \overset{\circ}{H}$ (i.e., there is some open subset, U, so that $1 \in U \subseteq H$).

Proposition 2.14. *If G is a connected topological group, then G is generated by any symmetric neighborhood, V, of 1. In fact,*

$$G = \bigcup_{n \geq 1} V^n.$$

A subgroup, H, of a topological group G is *discrete* iff the induced topology on H is discrete, i.e., for every $h \in H$, there is some open subset, U, of G so that $U \cap H = \{h\}$.

Proposition 2.15. *If G is a topological group and H is discrete subgroup of G, then H is closed.*

Proposition 2.16. *If G is a topological group and H is any subgroup of G, then the closure, $\overline{H}$, of H is a subgroup of G.*

Proposition 2.17. *Let G be a topological group and H be any subgroup of G. If H and G/H are connected, then G is connected.*

Proposition 2.18. *Let G be a topological group and let V be any connected symmetric open subset containing 1. Then, if G_0 is the connected component of the identity, we have*

$$G_0 = \bigcup_{n\geq 1} V^n$$

and G_0 is a normal subgroup of G. Moreover, the group G/G_0 is discrete.

A topological space, X is *locally compact* iff for every point $p \in X$, there is a compact neighborhood, C of p, i.e., there is a compact, C, and an open, U, with $p \in U \subseteq C$.

For example, manifolds are locally compact.

Proposition 2.19. *Let G be a topological group and assume that G is connected and locally compact. Then, G is countable at infinity, which means that G is the union of a countable family of compact subsets. In fact, if V is any symmetric compact neighborhood of* 1*, then*

$$G = \bigcup_{n\geq 1} V^n.$$

If a topological group, G acts on a topological space, X, and the action $\cdot \colon G \times X \to X$ is continous, we say that G acts *continuously on X*.

The following theorem gives sufficient conditions for the quotient space, G/G_x, to be homeomorphic to X.

Theorem 2.20. *Let G be a topological group which is locally compact and countable at infinity, X a locally compact Hausdorff topological space and assume that G acts transitively and continuously on X. Then, for any $x \in X$, the map $\varphi \colon G/G_x \to X$ is a homeomorphism.*

Proof. A proof can be found in Mneimné and Testard [17] (Chapter 2). □

Remark: If a topological group acts continuously and transitively on a Hausdorff topological space, then for every $x \in X$, the stabilizer, G_x, is a closed subgroup of G.

This is because, as the action is continuous, the projection $\pi \colon G \longrightarrow X \colon g \mapsto g \cdot x$ is continuous, and $G_x = \pi^{-1}(\{x\})$, with $\{x\}$ closed.

As an application of Theorem 2.20 and Proposition 2.17, we show that the Lorentz group $\mathbf{SO}_0(n,1)$ is connected.

Firstly, it is easy to check that $\mathbf{SO}_0(n,1)$ and $\mathcal{H}_n^+(1)$ satisfy the assumptions of Theorem 2.20 because they are both manifolds.

Also, we saw at the end of Section 2.3 that the action $\cdot\colon \mathbf{SO}_0(n,1) \times \mathcal{H}_n^+(1) \longrightarrow \mathcal{H}_n^+(1)$ of $\mathbf{SO}_0(n,1)$ on $\mathcal{H}_n^+(1)$ is transitive, so that, as topological spaces

$$\mathbf{SO}_0(n,1)/\mathbf{SO}(n) \cong \mathcal{H}_n^+(1).$$

Now, we already showed that $\mathcal{H}_n^+(1)$ is connected so, by Proposition 2.17, the connectivity of $\mathbf{SO}_0(n,1)$ follows from the connectivity of $\mathbf{SO}(n)$ for $n \geq 1$.

The connectivity of $\mathbf{SO}(n)$ is a consequence of the surjectivity of the exponential map (see Theorem 1.11) but we can also give a quick proof using Proposition 2.17.

Indeed, $\mathbf{SO}(n+1)$ and S^n are both manifolds and we saw in Section 2.2 that

$$\mathbf{SO}(n+1)/\mathbf{SO}(n) \cong S^n.$$

Now, S^n is connected for $n \geq 1$ and $\mathbf{SO}(1) \cong S^1$ is connected. We finish the proof by induction on n.

Corollary 2.21. *The Lorentz group* $\mathbf{SO}_0(n,1)$ *is connected; it is the component of the identity in* $\mathbf{O}(n,1)$.