Mathematics 622
Assignment 5 (Shatz)
Due Thursday, December 1, 2003

1 Part A

AI. Say X is a complex variety and assume X is non-singular. Prove X is Z-connected $\Longleftrightarrow$ X irreducible. If X is again a cx. variety and X_0 is its non-singular locus, prove X is Z-connected $\Longleftrightarrow$ X_0 is norm connected.

AII. If Θ is a function on a covering, X, of Y, and Θ satisfies an equation

$$\Theta^n + \pi^*(a_1)\Theta^{n-1} + \ldots + \pi^*(a_n) = 0$$

where $\pi : X \longrightarrow Y$ and $a_j \in \Gamma(Y, \mathcal{O}_Y)$, prove the estimate used in lecture:

$$(\forall \xi \in X)(\|\Theta(\xi)\| < 1 + \max\{\|a_1(\pi(\xi))\|, \ldots, \|a_n(\pi(\xi))\|\}).$$

AIII. (*Eine kleine Garben Theorie*)
Here $\mathcal{F}, \mathcal{S}, \mathcal{F}', \mathcal{F}''$, etc., are sheaves of $\mathcal{O}_X$-modules on the cx. variety X. Of course, $\mathcal{F}$ is *locally free of rank* r iff

$$(\forall x \in X)(\exists \, \text{open}\, U \ni x)(\mathcal{F}|U \overset{\varphi_U}{\underset{\sim}{\longrightarrow}} \mathcal{O}_X^r|U).$$

The maps need not patch on the overlaps. We make the $\mathcal{O}_X$-algebra, $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F})$, where

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}) = \coprod_{n \geq 0} (\underbrace{\mathcal{F} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}}_{n})/\mathcal{I}$$

and $\mathcal{I}$ is the sheaf of two-sided ideals generated by $\alpha \otimes \beta - \beta \otimes \alpha$. Write $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}) = \coprod_{n \geq 0} S^n(\mathcal{F})$.

a) Show $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F})$ is a f.g. $\mathcal{O}_X$-algebra if $\mathcal{F}$ is a f.g. $\mathcal{O}_X$-module (cf. appendix A) and locally looks like $A[Z_1, \ldots, Z_m]/\mathcal{J}$, where $A = \Gamma(U, \mathcal{O}_X|U)$ and U is affine open in X. Then $\mathrm{Spec}\, A[Z_1, \ldots, Z_m]/\mathcal{J} = V_U$ and the cx. affine varieties V_U glue together on overlaps where U meets $\widetilde{U}$, (another affine open). This gives a cx. variety V and it has a morphism $\pi : V \longrightarrow X$. Call this variety $V(\mathcal{F})$. Show that $V(\mathcal{F})$ is the total space of a vector bundle of rank r iff $\mathcal{F}$ is locally free of rank r as $\mathcal{O}_X$-module. Prove: $\mathcal{F} \rightsquigarrow V(\mathcal{F}^D)$ is an equivalence of categories: locally free finite rank sheaves and finite rank vector bundles over X.

b) If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of loc. free sheaves, then

(b1) $(\forall r)(S^r(\mathcal{F}))$ has a filtration

$$S^r(\mathcal{F}) = \mathcal{F}^{(0)} \supseteq \mathcal{F}^{(1)} \supseteq \cdots \supseteq \mathcal{F}^{(r+1)} = (0)$$

and

$$\mathcal{F}^{(l)}/\mathcal{F}^{(l+1)} \cong S^l(\mathcal{F}') \otimes S^{r-1}(\mathcal{F}'')$$

(b2) $(\forall r)(\bigwedge^r(\mathcal{F})$ has a filtration

$$\bigwedge^r(\mathcal{F}) = \mathcal{F}^{(0)} \supseteq \mathcal{F}^{(1)} \supseteq \cdots \supseteq \mathcal{F}^{(r+1)} = (0)$$

and

$$\mathcal{F}^{(l)}/\mathcal{F}^{(l+1)} \cong \bigwedge^l(\mathcal{F}') \otimes \bigwedge^{r-l}(\mathcal{F}'')).$$

AIV. Prove on $\mathbb{R}^{2n}$ the collection of cx. structures is the homogeneous space

$$\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$$

via class (= orbit) of $A \rightsquigarrow A^{-1}JA$, where J is the usual cx. structure. Make a suitable generalisation for the tangent bundle, T_X, of a $2n$-dimensional $\mathbb{R}$-manifold which is almost complex (given $J\colon T_X \longrightarrow T_X$).

AV. Prove: A sheaf, $\mathcal{F}$, is fine iff $\mathcal{H}om(\mathcal{F},\mathcal{F})$ is soft. Prove further that softness is local on X; that is, $\mathcal{F}$ is soft iff each $x \in X$ has a neighbourhood, U, s.t. $\forall S \subseteq U$ with S closed in X, the map $\mathcal{F}(U) \longrightarrow \mathcal{F}(S)$ is surjective.

2 Part B

BI. a) Prove the **Basic Extension Theorem**. *If X is locally compact and of our type as a topological space underlying a cx. variety (more generally, X is paracompact) and if S is a norm-closed subspace, then for all sheaves, $\mathcal{F}$,*
$(\forall s \in \mathcal{F}(S)(=\Gamma(S,\mathcal{F})))(\exists$ *neighborhood* $U \supseteq S)$ *and a prolongation of s to a section $\sigma \in \mathcal{F}(U)$).*

b) Show each flasque sheaf is soft.

c) Hypotheses as in a), then $\forall S-$closed,

$$\mathcal{F}(S) = \Gamma(S,\mathcal{F}) = \varinjlim_{U \supseteq S} \mathcal{F}(U), \quad U\text{-open}.$$

d) If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, and $\mathcal{F}', \mathcal{F}$ are soft, so is $\mathcal{F}''$.

e) If $\mathcal{F}$ is a fine sheaf and $\mathcal{S}$ is any sheaf of abelian grps, then $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{S}$ is a fine sheaf (thus, also soft).

BII. (*Euler Sequence and consequences*)
Here $X = \mathbb{P}^n$. Have the homogeneous coordinate ring, $S = \mathbb{C}[Z_0,\ldots,Z_n]$. For each graded S-module, M, we define the *k-fold Serre twist*, $M(k)$, of M, *via* $M = \coprod_{d\in\mathbb{Z}} M_d$, then $M(k)_d = M_{k+d}$. Each graded S-module, M, gives an $\mathcal{O}_X$-module, $M^\sharp$, as follows:

$$\Gamma(U, M^\sharp) = \left\{ \sigma : U \longrightarrow \coprod_{x\in U} M_{(x)} \;\middle|\; \begin{array}{l} 1)\ \sigma(x) \in M_{(x)} \\ 2)\ (\forall x \in U)(\exists \text{ open } V \ni x, V \subseteq U), \text{ so that} \\ 3)\ (\exists m \in M_d, \text{ and } \exists f \in S_d, \text{ with} \\ \quad \frac{m}{f} \longrightarrow \sigma(y), \forall y \in V) \end{array} \right\}$$

(Of course, here $M_{(x)}$ = localisation at x in homog. sense, i.e., things of the form $\frac{m}{f}$, $\deg(m) = \deg(f)$.) Then, we find

$$M(1)^\sharp = M^\sharp \otimes \mathcal{O}_X(1), \quad \text{where} \quad \mathcal{O}_X(1) = S(1)^\sharp$$

and more generally,

$$M(k)^\sharp = M^\sharp \otimes \mathcal{O}_X(k).$$

Let $\Omega^1_X = \mathrm{Ker}\,(\bar\partial : \bigwedge^{1,0} \longrightarrow \bigwedge^{1,1})$; this is the sheaf of germs of holomorphic 1-forms on X.

a) Establish *the Euler sequence*:

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \coprod_{(n+1) \text{ times}} \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

(Begin with $S = \mathbb{C}[Z_0, \dots, Z_n]$. Look at $\coprod_{(n+1)\text{ times}} S(-1)$; it has basis $e_0, e_1, \dots, e_n$ as graded S-module. Map each e_j to Z_j; this gives a graded (deg 0) S-module map

$$\coprod_{n+1} S(-1) \longrightarrow S,$$

write K for its kernel. Show $K^\sharp = \Omega^1_X = \Omega^1_{\mathbb{P}^1}$ and from $0 \longrightarrow K \longrightarrow \coprod_{n+1} S(-1) \longrightarrow S$ show the Euler sequence.)

b) Use the Euler sequence, induction and *Eine kleine Garben Theorie* (prob. AIII) to compute:

$$H^{p,q} = H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}).$$

Note: *This* is the proof I want, not any of the others in books.

c) Compute as well

$$H^q(\mathbb{P}^n, S^p\Omega^1_{\mathbb{P}^n}).$$

BIII. Consider a morphism $X \xrightarrow{\pi} S$ of cx. varieties. On X, we have $\mathcal{O}_X^n$ as $\mathcal{O}_X$-module. For any variety, T, over S (i.e., morphism $\varphi\colon T \longrightarrow S$) we make the set

$$\mathrm{Grass}_{X/S}(k,n)(T) = \left\{ E \;\middle|\; \begin{array}{ll} 1)\ E & \text{is a loc. free, rank } k \text{ sheaf on } X \prod_S T \\ 2)\ \exists & \text{surjection} \quad \mathcal{O}_X^n \otimes_S \mathcal{O}_T \longrightarrow E \longrightarrow 0 \end{array} \right\}.$$

The association $T \rightsquigarrow \mathrm{Grass}_{X/S}(k,n)(T)$ is a cofunctor from cx. varieties over S to *Sets*—is it representable is what we want to know here.

a) Take $S = X = \operatorname{Spec}\mathbb{C}$, show $G(n,k)$, the cx. Grassmanian, represents $\mathrm{Grass}_{\mathbb{C}/\mathbb{C}}(k,n)$.

b) Prove: If $\mathrm{Grass}_{X/S}(k,n)$ is representable when S is affine, it is always representable, thus we may assume S is affine.

c) Take $X = S =$ affine, prove $\mathrm{Grass}_{S/S}(k,n)$ is representable.

d) Do the same if X is finite and flat over S.

The existence of $\mathrm{Grass}_{X/S}(k,n)$ depends vitally on the structure of X over S.