1 Part A

AI. Consider 3 non-collinear points in $\mathbb{P}^2$. They form a reducible complex variety, call it $X$. Show $I(X)$ cannot be generated by two polynomials. Nevertheless, show there exists an ideal, $\mathfrak{A}$, so that $X = V(\mathfrak{A})$ and $\mathfrak{A}$ can be generated by two polys. In fact, prove more: If $X$ is any finite set $\subset \mathbb{P}^2$, there exists an ideal, $\mathfrak{A}$, with $X = V(\mathfrak{A})$ and $\mathfrak{A}$ can be generated by two polys. (Of course, $I(X)$ may need more polys to generate it.) An old open question is: If $C$ is an irreducible curve in $\mathbb{P}^3$, does there exist an ideal, $\mathfrak{A}$, with $V(\mathfrak{A}) = C$ and $C$ generated by 2 polys? (Of course, $I(C)$ cannot be expected to work—try the twisted cubic as a counter-example.)

AII. Let $X$ now be an affine curve over $\mathbb{C}$, say $X \subset \mathbb{C}^n$. Write $A = \Gamma(X, \mathcal{O}_X) =$ ring of global holomorphic functions on $X$. If $X$ is irreducible, then $A = \text{domain}$, and $\text{Mer}(X)$ (the meromorphic functions on $X$) is just $\text{Frac}(A)$. Assume $X$ is irreducible, let $\overline{A} =$ integral closure of $A$ in $\text{Mer}(X)$. Show there exists a curve affine variety $\overline{X} \subset \mathbb{C}^N$, so that $\overline{A} =$ integral closure of $A$ in $\text{Mer}(X)$. Show this morphism is surjective, 1–1 on a Z-dense open set (in $\overline{X}$ and $X$) and $\overline{X}$ is a non-singular curve. The curve $\overline{X}$ is the normalization of $X$.

AIII. a) Carry out the scheme and results from AII for any dim’l affine variety, $X$. Here, $\overline{X}$ is not non-singular, it is merely normal ($\mathcal{O}_{\overline{X}, \overline{x}}$ is integrally closed in $\text{Mer}(X)$.)

b) Do a), this time for an arbitrary irreducible complex variety—not necessarily affine. Get $\overline{X}$, the normalisation.

AIV. a) Prove the

**Theorem** Every nonsingular complex curve, proper over $\mathbb{C}$, admits a closed embedding into $\mathbb{P}^d$, some $d$.

(Hints: A Chow’s lemma kind of argument will work. Cover $X$, your curve, by affines, $X_\alpha$. Each $X_\alpha \subset \mathbb{C}^{n_\alpha} \subset \mathbb{P}^{n_\alpha}$; show the locally closed immersion, $\varphi \colon X_\alpha \rightarrow \mathbb{P}^{n_\alpha}$, extends to a morphism, call it $\Phi_\alpha$, taking $X$ to $\mathbb{P}^{n_\alpha}$. Say $Y_\alpha = \text{image} \Phi_\alpha(X)$ so $Y_\alpha$ is closed $\subset \mathbb{P}^{n_\alpha}$. Consider the diagonal map and Segre:

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & \prod_{\alpha} Y_\alpha \\
\downarrow \phi & & \downarrow \text{Segre} \\
\prod_{\alpha} \mathbb{P}^{n_\alpha} & \subset & \mathbb{P}^M.
\end{array}
\]

Prove this is the desired embedding.)

b) Is a) true even if $X$ is singular?

c) What goes wrong if dim $X \geq 2$, even in the nonsingular case?

2 Part B

BI. a) Say, $f : X \rightarrow Y$ is a morphism and is a birational isomorphism. Pick $y \in Y$, assume $y$ is a non-singular pt. of $Y$, and further assume $f^{-1}$ is not defined at $y$. Show there exists a subvariety, $Z$, of $X$, so that for $x \in X$ with $f(x) = y$:

(i) $Z$ is a divisor of $X$ (this means codim($Z$) = 1 in $X$) and $Z$ passes through $x$ (you may allow $Z$ to depend on $x$).
(ii) $f(Z)$ has codimension $\geq 2$ in $Y$.

Deduce that if $X, Y$ are smooth projective surfaces and $f^{-1}$ is not defined on $y$ (where $f(x) = y$), then there exists a curve, $C$, in $X$, with $f(C) = y$.

The latter result is a main step in the proof of Zariski’s Factorization Theorem: Every birational isomorphism of projective surfaces factors as a finite product of blowing-ups of pts and their inverses.

(At some point or points of your argument, you may need to assume $X$ is normal. Do not hesitate to do so.)

b) Use Zariski’s Factorisation Theorem to factor explicitly the following two maps:

(i) $X = \text{smooth quadric} \subset \mathbb{P}^3, Y = \mathbb{P}^2, \xi = \text{pt. of } X \text{ and } f \text{ is projection from } \mathbb{P}^3 - \xi \text{ to } \mathbb{P}^2 \text{ restricted to } X$.

(ii) In $\mathbb{P}^2$, take an affine patch and let $x, y$ be (inhomogeneous) coordinates there. Write $f(x, y) = (x, y + x^2)$. Then $f$ extends to a birational automorphism of $\mathbb{P}^2$—factor it.

BII. Say $X, Y$ are complex varieties, and $\pi : X \longrightarrow Y$ is a morphism. We call $\pi$ an étale morphism $\iff$ it has two further properties:

(a) It is flat (i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y, \pi(x)}$-module, $\forall x \in X$).

(b) $\Omega^1_{X/Y}$—the sheaf of relative Kähler differentials—is (0).

The word étale comes from the phrase “la mer étale” which translates as “slack water” in the sailor’s vernacular—however, “slack” is not used as a translation of étale. Rather, one says “flat and unramified” or just uses “étale”. Prove the

**Theorem** Suppose $\pi : X \longrightarrow Y$ is a morphism of cx. varieties, with $Y$ connected. Then the following are equivalent:

(1) $\pi$ is étale

(2) The Jacobian condition of last assignment

(3) The unique lifting condition of last assignment

(4) $\mathcal{O}_{X,x} \overset{\sim}{\longrightarrow} \mathcal{O}_{Y, \pi(x)}, \text{ all } x \in X$.

(5) $X$ is a covering space, finite over $Y$ in the sense of differential topology, i.e., all fibres are finite of the same cardinality and $\pi$ is a local (smooth $= C^\infty$) diffeomorphism.

(6) $\mathcal{O}_{X^{\text{an}}, x} \overset{\sim}{\longrightarrow} \mathcal{O}_{Y^{\text{an}}, \pi(x)}$.

**Remark.** In the last time’s assignment, 2) $\iff$ 3) $\iff$ 4) were proved. Do NOT reprove here.

BIII. Investigate pairs of cx. algebraic varieties $Z, X$ so that there exists a morphism $\pi : Z \longrightarrow X$ satisfying:

(a) $\pi$ is a finite morphism, $X$ is irreducible and $\deg \pi = 2$.

(b) $\pi$ is flat ($\mathcal{O}_{Z,z}$ is a flat $\mathcal{O}_{X, \pi(z)}$ module, all $z$.)
We refer to $Z$ as a \textit{two-sided branched covering} of $X$.

a) We can consider the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_X$ module. Prove there exists a line bundle, $\mathcal{L}$, on $X$, canonically determined by $\pi$, so that $\mathcal{O}_Z = \mathcal{O}_X \bigoplus \mathcal{L}$ as $\mathcal{O}_X$ modules. Describe in terms of $\mathcal{L}$ the data needed in order that the module $\mathcal{O}_X \bigoplus \mathcal{L}$ be the $\mathcal{O}_X$-algebra, $\mathcal{O}_Z$, of some twofold branched cover of $X$.

b) Examples are important. Take $X = \mathbb{P}^1$, any cx. curve, $\mathbb{P}^2, \mathbb{P}^n$ and make at least two distinct examples in each case. Make your examples so that $Z$ is connected.

c) Answer the following. If the answer is “no”, give an example, then further hypotheses so that it is “yes”. If the answer is “yes”, give a proof:

(i) If $X$ is affine, is $Z$ affine? Same with “projective” replacing affine.

(ii) If $Z$ is affine (resp. projective) is $X$ the same?

(iii) If $Z$ is singular, must $X$ be singular? Same question interchanging $Z$ and $X$.

(iv) If $X$ is smooth (= nonsingular) is it a rule or exception that $Z$ is smooth?

(v) Can non-isomorphic line bundles give isomorphic $Z$’s? Is the isomorphism an isomorphism over $X$? Try for a classification in terms of being given $X$.

d) Let $\mathcal{B}$ be the branch locus in $X$, i.e.,

\[ \mathcal{B} = \{ x \in X \mid \pi^{-1}(x) = \text{one pt.} \} \]

Then $\mathcal{B}$ is $Z$-closed. Relate $\mathcal{B}$ to $Z$. If $Z$ is also irreducible, is $\mathcal{B}$ irreducible? If not, can it have components of several dimensions? What is $\text{codim}_X \mathcal{B} (= \dim X - \dim \mathcal{B})$?

e) What is the relation of some invariants of $X$ and $Z$? E.g., $\dim(X) = \dim(Z)$; what about $T_X$ (tangent bundle) and $T_Z, \Omega_X$ and $\Omega_Z^2, K_X (= \wedge^* \Omega_X)$ and $K_Z (= \wedge^* \Omega_Z^2)$ (here $\wedge^*$ being the highest wedge). Any connection between $\mathcal{B}$ and the Jacobian of $\pi: Z \to X$? If you know about Chern classes and Euler numbers, discuss these.

f) More examples: Take $X = \mathbb{P}^1 \prod \mathbb{P}^1$ or $X = C \prod \mathbb{P}^1$ with $C$ a curve. Find several (connected) $Z$’s. View $Z$ as a parametrized family of coverings over one of the factors—discuss. If $Z$ is ruled (birational to $C \prod \mathbb{P}^1$, $C$ a curve) is $X$ necessarily ruled?

g) Your turn to ask (and answer) some (hopefully) nontrivial questions.