Mathematics 622
Assignment 4 (Shatz)
Due Thursday, November 20, 2003

1 Part A

AI. Consider 3 non-collinear points in $\mathbb{P}^2$. They form a reducible complex variety, call it X. Show $\mathcal{I}(X)$ *cannot* be generated by two polynomials. Nevertheless, show there exists an ideal, $\mathfrak{A}$, so that $X = V(\mathfrak{A})$ and $\mathfrak{A}$ *can* be generated by two polys. In fact, prove more: If X is any finite set $\subset \mathbb{P}^2$, there exists an ideal, $\mathfrak{A}$, with $X = V(\mathfrak{A})$ and $\mathfrak{A}$ can be generated by two polynomials. (Of course, $\mathcal{I}(X)$ may need more polys to generate it.) An old open question is: If C is an irred. curve in $\mathbb{P}^3$, does there exist an ideal, $\mathfrak{A}$, with $V(\mathfrak{A}) = C$ and C generated by 2 polys? (Of course, $\mathcal{I}(C)$ cannot be expected to work—try the twisted cubic as a counter-example.)

AII. Let X now be an affine *curve* over $\mathbb{C}$, say $X \subset \mathbb{C}^n$. Write $A = \Gamma(X, \mathcal{O}_X)$ = ring of global holomorphic functions on X. If X is irred., then A = domain, and $\mathcal{M}er(X)$ (the mero fcns on X) is just $\mathrm{Frac}(A)$. Assume X is irred., let $\widetilde{A}$ = integral closure of A in $\mathcal{M}er(X)$. Show there exists a cx affine variety $\widetilde{X} \subset \mathbb{C}^N$, so that $\widetilde{A} = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ and that there exists a finite morphism $\widetilde{X} \longrightarrow X$. Show this morphism is surjective, 1–1 on a Z-dense open set (in $\widetilde{X}$ and X) and $\widetilde{X}$ is a non-singular curve. The curve $\widetilde{X}$ is *the normalization* of X.

AIII. a) Carry out the scheme and results from AII for *any* dim'l affine variety, X. Here, $\widetilde{X}$ is not non-singular, it is merely normal ($\mathcal{O}_{\widetilde{X},\widetilde{x}}$ is integrally closed in $\mathcal{M}er(X)$.)

b) Do a), this time for an arbitrary irred. cx. variety—not necessarily affine. Get $\widetilde{X}$, the normalisation.

AIV. a) Prove the

Theorem *Every nonsingular cx. curve, proper over $\mathbb{C}$, admits a closed embedding into $\mathbb{P}^d$, some d.*

(Hints: A Chow's lemma kind of argument will work. Cover X, your curve, by affines, X_α. Each $X_\alpha \subset \mathbb{C}^{n_\alpha} \subset \mathbb{P}^{n_\alpha}$; show the locally closed immersion, $\varphi_\alpha \colon X_\alpha \longrightarrow \mathbb{P}^{n_\alpha}$, extends to a morphism, call it Φ_α, taking X to $\mathbb{P}^{n_\alpha}$. Say Y_α = image $\Phi_\alpha(X)$ so Y_α is closed $\subset \mathbb{P}^{n_\alpha}$. Consider the diagonal map and Segre:

$$X \xrightarrow[\Delta]{} \prod_\alpha Y_\alpha \hookrightarrow \prod_\alpha \mathbb{P}^{n_\alpha} \xhookrightarrow[\mathit{Segre}]{} \mathbb{P}^M .$$

Prove this is the desired embedding.)

b) Is a) true even if X is singular?

c) What goes wrong if $\dim X \geq 2$, even in the nonsingular case?

2 Part B

BI. a) Say, $f \colon X \longrightarrow Y$ is a morphism and is a birational isomorphism. Pick $y \in Y$, assume y is a non-singular pt. of Y, and further assume f^{-1} is not defined at y. Show there exists a subvariety, Z, of X, so that for $x \in X$ with $f(x) = y$:

(i) Z is a divisor of X (this means $\mathrm{codim}(Z) = 1$ in X) and Z passes through x (you may allow Z to depend on x).

(ii) $f(Z)$ has codimension ≥ 2 in Y.

Deduce that if X, Y are smooth projective surfaces and f^{-1} is not defined on y (where $f(x) = y$), then there exists a curve, C, in X, with $f(C) = y$.

The latter result is a main step in the proof of *Zariski's Factorization Theorem*: Every birational isomorphism of projective surfaces factors as a finite product of blowing-ups of pts and their inverses.

(At some point or points of your argument, you may need to assume X is normal. Do not hesitate to do so.)

b) Use Zariski's Factorisation Theorem to factor explicitly the following two maps:

(i) $X =$ smooth quadric $\subset \mathbb{P}^3$, $Y = \mathbb{P}^2$, $\xi =$ pt. of X and f is projection from $\mathbb{P}^3 - \xi$ to $\mathbb{P}^2$ restricted to X.

(ii) In $\mathbb{P}^2$, take an affine patch and let x, y be (inhomogeneous) coordinates there. Write $f(x, y) = (x, y+x^2)$. Then f extends to a birational automorphism of $\mathbb{P}^2$—factor it.

BII. Say X, Y are complex varieties, and $\pi\colon X \longrightarrow Y$ is a morphism. We call π an *étale* morphism $\iff$ it has two further properties:

(a) It is flat (i.e., $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module, $\forall x \in X$).

(b) $\Omega^1_{X/Y}$–the sheaf of relative Kähler differentials–is (0).

The word *étale* comes from the phrase "*la mer étale*" which translates as "slack water" in the sailor's vernacular—however, "slack" is not used as a translation of étale. Rather, one says "flat and unramified" or just uses "étale". Prove the

Theorem *Suppose $\pi : X \longrightarrow Y$ is a morphism of cx. varieties, with Y connected. Then the following are equivalent:*

(1) π *is* étale

(2) The Jacobian condition of last assignment

(3) The unique lifting condition of last assignment

(4) $\widehat{\mathcal{O}}_{X,x} \xleftarrow{\sim} \widehat{\mathcal{O}}_{Y,\ \pi(x)}$, *all* $x \in X$.

(5) X is a covering space, finite over Y in the sense of differential topology, i.e., all fibres are finite of the same cardinality and π is a local (smooth = C^∞) diffeomorphism.

(6) $\mathcal{O}_{X^{\mathrm{an}},x} \xleftarrow{\sim} \mathcal{O}_{Y^{\mathrm{an}},\pi(x)}$.

Remark. In the last time's assignment, 2) $\iff$ 3) $\iff$ 4) were proved. Do NOT reprove here.

BIII. Investigate pairs of cx. algebraic varieties Z, X so that there exists a morphism $\pi\colon Z \longrightarrow X$ satisfying:

(α) π is a finite morphism, X is irreducible and $\deg \pi = 2$.

(β) π is flat ($\mathcal{O}_{Z,z}$ is a flat $\mathcal{O}_{X,\pi(z)}$ module, all z.)

We refer to Z as a *two-sided branched covering* of X.

a) We can consider the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_X$ module. Prove there exists a line bundle, $\mathcal{L}$, on X, canonically determined by π, so that $\mathcal{O}_Z = \mathcal{O}_X \coprod \mathcal{L}$ as $\mathcal{O}_X$ modules. Describe in terms of $\mathcal{L}$ the data needed in order that the module $\mathcal{O}_X \coprod \mathcal{L}$ be the $\mathcal{O}_X$-algebra, $\mathcal{O}_Z$, of some twofold branched cover of X.

b) Examples are important. Take $X = \mathbb{P}^1$, any cx. curve, $\mathbb{P}^2, \mathbb{P}^n$ and make at least two distinct examples in each case. Make your examples so that Z is connected.

c) Answer the following. If the answer is "no", give an example, then further hypotheses so that it is "yes" . If the answer is "yes", give a proof:

(i) If X is affine, is Z affine? Same with "projective" replacing affine.

(ii) If Z is affine (resp. projective) is X the same?

(iii) If Z is singular, must X be singular? Same question interchanging Z and X.

(iv) If X is smooth (= nonsingular) is it a rule or exception that Z is smooth?

(v) Can non-isomorphic line bundles give isomorphic Z's? Is the isomorphism an isomorphism over X? Try for a classification in terms of being given X.

d) Let $\mathcal{B}$ be the branch locus in X, i.e.,

$$\mathcal{B} = \{x \in X \mid \pi^{-1}(x) = \text{ one pt.}\}$$

Then $\mathcal{B}$ is Z-closed. Relate $\mathcal{B}$ to Z. If Z is also irred., is $\mathcal{B}$ irreducible? If not, can it have components of several dimensions? What is $\operatorname{codim}_X \mathcal{B}$ $(= \dim X - \dim \mathcal{B})$?

e) What is the relation of some invariants of X and Z? E.g., $\dim(X) = \dim(Z)$; what about T_X (tangent bundle) and T_Z, Ω_X and $\Omega^1_Z, K_X (= \bigwedge^{\bullet} \Omega^1_X)$ and $K_Z (= \bigwedge^{\bullet} \Omega^1_Z)$ (here $\bigwedge^{\bullet}$ being the highest wedge). Any connection between $\mathcal{B}$ and the Jacobian of $\pi\colon Z \longrightarrow X$? If you know about Chern classes and Euler numbers, discuss these.

f) More examples: Take $X = \mathbb{P}^1 \prod \mathbb{P}^1$ or $X = C \prod \mathbb{P}^1$ with C a curve. Find several (connected) Z's. View Z as a parametrized family of coverings over one of the factors—discuss. If Z is ruled (birational to $C \prod \mathbb{P}^1$, C a curve) is X necessarily ruled?

g) Your turn to ask (and answer) some (hopefully) nontrivial questions.