Mathematics 622

Assignment 3 (Shatz)

Due Thursday, November 6, 2003

1 Part A

AI. Prove the assertion made in class: An integral morphism is a closed (Z-top) map. (Hint: Reduce to affines, use the Lying Over Theorem of Cohen-Seidenberg.)

AII. This looks like linear algebra, but it's really alg. geometry: Say $\theta\colon V \times W \longrightarrow Z$ is a bilinear map of $\mathbb{C}-$vector spaces (all non-zero). Assume θ is non-degenerate, i.e.,

$$(\forall \alpha \in V)(\forall \beta \in W)(\alpha \neq 0 \quad \text{and} \quad \beta \neq 0 \Longrightarrow \theta(\alpha, \beta) \neq 0).$$

Show that $\dim Z \geq \dim V + \dim W - 1$.

AIII. Look at $\mathbb{P}^n$ and recall that a "k-plane" ("k-dimensional linear subspace") of $\mathbb{P}^n$ is just a $k+1$ dimensional linear subspace of the corresponding $\mathbb{C}^{n+1}$. The $k-$planes in $\mathbb{P}^n$ are denoted $\mathbb{G}(k, n)$ and the latter is the projective variety $G(k+1, n+1)$. So, $\dim \mathbb{G}(k, n) = (k+1)(n-k)$. Compute the following dimensions:

a) All $2-$planes of $\mathbb{P}^4$ through a fixed point.

b) All $2-$planes of $\mathbb{P}^4$ containing a given line.

c) All $k-$planes of $\mathbb{P}^n$ through a fixed pt.

d) All $k-$planes of $\mathbb{P}^n$ containing a given $l-$plane, $0 \leq l \leq k$.

e) If V is a non-singular non-degenerate d-dimensional subvariety of $\mathbb{P}^n$ (say, d is sufficiently small), what is the dimension of all $k-$planes of $\mathbb{P}^n$ somewhere *tangent* to V?

2 Part B

BI. (Rational maps)

a) Say, $\varphi\colon V \longrightarrow Z$ is a rational map of complex varieties. Assume V is a normal variety (i.e., all the local rings $\mathcal{O}_{V,v}$ are integrally closed integral domains) and Z is a quasi-projective variety (i.e., a possibly open subvariety of a projective variety). Prove that

$$\{v \in V \mid \varphi \quad \text{cannot be defined at} \quad v\}$$

has codimension ≥ 2 in V. (**Caution**: We don't know about "points of codimension 1" and their local rings—we just know about complex (= closed) points; so an argument about 1-dimensional local rings and normality, etc., won't work.)

b) An immediate corollary of a) is that a rational map between a non-singular curve and a quasi-projective curve is automatically a morphism. Hence, two non-singular projective curves are birationally equivalent iff they are isomorphic. It's only for surfaces and higher-dimentional varieties that birational isomorphism is coarser (so more useful) equivalence relation than isomorphism between normal varieties. Prove: If $\varphi\colon V \longrightarrow Z$ is a rational map, V is normal and Z is a quasi-affine variety, and if $\{v \mid \varphi \quad \text{not defined at} \quad v\}$

has
codim ≥ 2 in V, then φ is actually a morphism.

BII. (Fano Varieties and continuation of Prob BIV of last time)

a) Warm-up. Consider $\mathbb{G}(k,n)$ and in $\mathbb{G}(k,n)\prod\mathbb{P}^n$ the *incidence variety*, Γ:

$$\Gamma = \{(\xi,x) \mid \xi \in \mathbb{G}(k,n),\ x\in\mathbb{P}^n,\ x\in\xi\}.$$

Show that Γ, (which is just the "universal $k-$plane" when considered over $\mathbb{G}$ *via* pr_1) is actually a subvariety of the product $\mathbb{G}\prod\mathbb{P}^n$. Further, if Y is any subvariety of $\mathbb{G}(k,n)$, prove that

$$X = \{x \mid x\in\mathbb{P}^n;\ x\in\xi,\ \text{some}\ \xi\in Y\}$$

is a subvariety of $\mathbb{P}^n$. Dually, if X is a subvariety of $\mathbb{P}^n$, write $I_k(X)$ for the collection of $k-$planes meeting X. Show $I_k(X)$ is a subvariety of $\mathbb{G}(k,n)$.

b) Let X be a subvariety of $\mathbb{P}^n$. The *Fano Variety*, $F_k(X)$, is the collection of all $k-$planes of $\mathbb{P}^n$ which are contained in X;

$$F_k(X) = \{\xi\in\mathbb{G}(k,n) \mid \xi\subset X\}.$$

Show that $F_k(X)$ is actually a cx subvariety of $\mathbb{G}(k,n)$ as follows:

i) Say first it is OK when X is a hypersurface given by $f=0$ (some form f of degree d). Then prove $F_k(X)=\bigcup_{i=1}^t F_k(X_i)$, where X_i are hypersurfaces.

ii) Fix the degree, d, of the hypersurface X. For a $k-$plane, ξ, there is its ideal $\mathcal{I}(\xi)$ defining it as a subspace of $\mathbb{P}^n$. This ideal is the sum of its graded pieces $\mathcal{I}(\xi)_m$, which pieces are subspaces of the vector space S_m of *all* forms of degree m in the variables $Z_0,\dots Z_n$ of $\mathbb{P}^n$. We send ξ to $\mathcal{I}(\xi)_m$ and get a map

$$\mathbb{G}(k,n)\xrightarrow{\theta}G(l,N)$$

where $l=\dim\mathcal{I}(\xi)_m$ (independent of ξ!) and $N=\dim S_m$. Show this is a morphism of cx. varieties.

iii) Now fix $m=d$, the degree of X, then $f\in S_d$, where $f=0$ is the eqn of X. Show that the subspaces of S_d, containing the fixed point are a subvariety, T, of $G(l,N)$. Relate $F_k(X)$ to T by θ.

c) Continue the investigation of Problem IV: Again $\mathrm{Hyp}(\mathbb{P}^n,d)=\deg d$ hypersurfaces of $\mathbb{P}^n$, forming a $\mathbb{P}^N$, where $N=\binom{n+d}{d}-1$. But, take $\mathbb{G}(k,n)$ now, and form the incidence variety, $\Gamma(d,n,k)$,

$$\Gamma(d,n,k)=\{(H,\xi)\mid H\in\mathrm{Hyp}(\mathbb{P}^n,d),\ \xi\in\mathbb{G}(k,n),\ \xi\subset H\}.$$

Show that $\Gamma(d,n,k)$ is $Z-$closed in $\mathrm{Hyp}(\mathbb{P}^n,n)\prod\mathbb{G}(k,n)$. Prove that $pr_2\colon\Gamma(d,n,k)\longrightarrow\mathbb{G}(k,n)$ is surjective for all $k\leq n-1$. Examine the fibre pr_2^{-1} when $k=2$. Is it linear, as it was when $k=1$? Is it of constant dimension? Is $\Gamma(d,n,2)$ irreducible?

d) Do as much of c) as you can for general k, with $k\leq n-1$. When $k=2$, examine the existence of a critical degree, d. Especially, make computations for $n=4, n=5$.

e) Now take the middle dimension $k=[\frac{n}{2}]$ = largest integer $\leq\frac{n}{2}$. For $n=4,5$ you have $k=2$; look at $n=6,7$. Is there a critical degree; what is is? What about general n when $k=[\frac{n}{2}]$?

BIII. Consider a morphism $\varphi\colon Y\longrightarrow X$ of cx. alg. varieties. Choose $x\in X$, and take some affine open X_0 around x. Then for $y\in Y$ with $\varphi(y)=x$, there is an affine open $Y_0\subset\varphi^{-1}(X_0)$, with $y\in Y_0$. So, $\varphi\colon Y_0\longrightarrow X_0$, $y\in Y_0$, $x\in X_0$, and $\varphi(y)=x$. Write A_0 = coord ring of X_0, B_0 = coord ring of Y_0; so then we have a map $A_0\longrightarrow B_0$ corresponding to φ and it makes B_0 an A_0-algebra. Consider two properties of φ:

(a) *Jacobian condition* on φ: $\forall x, y$ with $\varphi(y) = x$, there exist X_0, Y_0, as above, so that

$$B_0 = A_0[T_1, \ldots, T_p]/(f_1, \ldots, f_p)$$

and $J(f) = \det\left(\frac{\partial f_i}{\partial T_j}\right)(y) = \text{unit of } B_0$.

(b) *Unique lifting condition* on φ. Again B_0, A_0 as above but no explicit description of B_0 as A_0-algebra and no $J(f)$. Instead for all exact sequences of $\mathbb{C}$-algebras

$$0 \longrightarrow I \longrightarrow \widetilde{A} \longrightarrow \widetilde{B} \longrightarrow 0$$

where

(i) $\widetilde{A}, \widetilde{B}$ are f.g. as vector spaces $/\mathbb{C}$ and are $\mathbb{C}$-algebras and are local rings,

(ii) The ideal I has square $= (0)$ we assume given a commutative diagram

$$\begin{tikzcd} B_0 \arrow[r, "\psi'"'] \arrow[dr, dashed, "\theta"'] & \widetilde{B} \\ A_0 \arrow[u] \arrow[r, "\psi"'] & \widetilde{A} \arrow[u] \end{tikzcd}$$

$\psi(\mathcal{O}_{X_0,x}) \subset \mathcal{M}_{\widetilde{A}}$ and $\psi'(\mathcal{O}_{Y_0,y}) \subset \mathcal{M}_{\widetilde{B}}$.

Condition on φ: Given the above for x, y (i.e. X_0, Y_0 exist and when exact sequence with $\widetilde{A}, \widetilde{B}$ and commutative diagram given), there exists a unique lifting homomorphism $\theta\colon B_0 \longrightarrow \widetilde{A}$ (shown dotted above), rendering all subdiagrams commutative.

Prove. Given $x, y \in X, Y, \varphi(y) = x$, condition $a)$ holds for $x, y \Leftrightarrow$ condition b) holds for $x, y \Leftrightarrow$ the completions $\widehat{\mathcal{O}}_{X,x_0}$ and $\widehat{\mathcal{O}}_{Y,y_0}$ are isomorphic via the map induced by $A_0 \longrightarrow B_0$. Is it true that these conditions are equivalent to the isomorphism of the subrings of $\widehat{\mathcal{O}}_{X,x_0}$ and $\widehat{\mathcal{O}}_{Y,y_0}$ consisting of convergent power series near x_0, resp. y_0?

BIV. (Some singularities) Pick an integer $r \geq 2$ and integers $0 < i_1 < i_2 < \ldots < i_{n-1}$, make the cx curve

$$X(r; i_1, \ldots, i_{r-1}) \subset \mathbb{C}^n$$

whose parametric equations are:

$$Z_1 = t^r, Z_2 = t^{r+i_1}, \ldots, Z_n = t^{r+i_{n-1}}.$$

a) Show this curve is singular. Compute its normalisation (as subvar of $\mathbb{C}^N$, some $N >> 0$, give equations for the normalisation, which is just the curve whose coordinate ring is the integral closure of the ring of $X(r; i_1, \ldots, i_{n-1})$.)

b) If $\widetilde{X}$ is the normalisation of X, it lies in $\mathbb{P}^N$, so its closure is in $\mathbb{P}^N$. Is this closure non-singular? If so, Y (the closure of $\widetilde{X}$) may be actually embedded in $\mathbb{P}^3$ by the argument of class. Can $X(r; i_1, \ldots, i_{n-1})$ be embedded in $\mathbb{C}^{\text{small}}$? How small? For example, examine the case $X(2; 1, 3, 5, 7, 11)$.