Mathematics 622
Assignment 2 (Shatz)
Due Thursday, October 23, 2003

1 Part A

AI. Continue the problem from the last HW about removing points from $\mathbb{C}^n$. Consider a subvariety, V, of $\mathbb{C}^n$, not necessarily irred., and form the Z-open set $\mathbb{C}^n - V$. Show if $\operatorname{codim} V \geq 2$, then $\mathbb{C}^n - V$ is never an affine variety. Is it true that if $\operatorname{codim} V = 1$, then $\mathbb{C}^n - V$ is always affine?

AII. Let $z_1, \ldots, z_n$ be coords in $\mathbb{C}^n$ and let Φ denote the morphism $\mathbb{C}^n \to \mathbb{C}^n$ given by

$$\Phi(z_1, \ldots, z_n) = (z_1, z_1 z_2, z_1 z_2 z_3, \ldots, z_1 \cdots z_n)$$

a) Describe the image of Φ. In particular, is it Z-open, Z-closed, neither, Z-dense, all of $\mathbb{C}^n$?

b) Examine the map

$$\delta : \mathbb{C} \xrightarrow[\Delta]{} \mathbb{C}^n \xrightarrow[\Phi]{} \mathbb{C}^n ,$$

where $\Delta =$ diagonal map. Show that the image is a Z-closed curve in $\mathbb{C}^n$. We have $\mathbb{C}^n \subset \mathbb{P}^n$; so, Φ gives a morphism $\mathbb{C}^n \to \mathbb{P}^n$. Does this extend to a morphism $\mathbb{P}^n \to \mathbb{P}^n$? If not, how far beyond $\mathbb{C}^n$ does it extend? Answer the same question for δ and its domain $\mathbb{C}^1$ and any extensions $\mathbb{P}^1 \to \mathbb{P}^n$?

c) Show that the image of δ (and any extension you can make) does NOT lie in any hyperplane of $\mathbb{C}^n$ (resp. $\mathbb{P}^n$). Such an object is *non-degenerate* in $\mathbb{C}^n$ (resp. $\mathbb{P}^n$).

AIII. In $\mathbb{P}^2$, with coords $(X : Y : Z)$, look at the curve, $E(A, B)$, given by

$$Y^2 Z = X^3 + AXZ^2 + BZ^2.$$

Prove that $E(A, B)$ is irreducible. Cover $E(A, B)$ by affines (in the standard way)—call these E_X, E_Y, E_Z (each an irred. plane curve) and let $\mathbb{C}[E_X]$, etc., denote their coordinate rings. By irreducibility, each ring $\mathbb{C}[E_X]$, etc., is a domain. Prove that

$$\text{Frac } \mathbb{C}[E_X] = \text{Frac } \mathbb{C}[E_Y] = \text{Frac } \mathbb{C}[E_Z].$$

The common value of these fraction fields is called *the field of meromorphic functions* on $E(A, B)$; denote it by $\mathcal{M}er(E(A, B))$. Prove that

$$(\exists T \in \mathcal{M}er(E(A, B)))(\mathcal{M}er(E(A, B)) = \mathbb{C}(T)) \quad \text{iff}$$

the polynomial $X^3 + AX + B$ has a multiple root.

AIV. A *rational map* $V \to W$ is just a morphism defined (perhaps only) on a Z-open, Z-dense subset, U, of W. As an example, the map

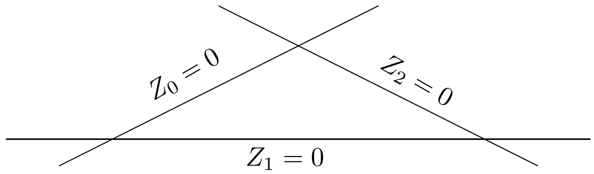
$$c(Z_0, Z_1, Z_2) = (Z_0 Z_1, Z_0 Z_2, Z_1 Z_2)$$

is a rational map of $\mathbb{P}^2$ to itself.

a) Find the maximal open $U \subset \mathbb{P}^2$ where c is defined.

b) Show that c is its own inverse and therefore there exists an open $\widetilde{V} \subset \mathbb{P}^2$ so that, $c \colon U \xrightarrow{\sim} \widetilde{V}$. Find $\widetilde{V}$. Such a map is called a *birational map*, it is an isomorphism of a Z-open, Z-dense open of V to a corresponding Z-dense, Z-open of W.

c) Examine the fundamental triangle



and explain what happens to it under the rational map c.

d) Let Q be a quadric, (i.e. $\deg Q = 2$) curve passing through the vertices of our triangle. What is $c(Q)$?

e) Prove that there exists a variety, X, and *morphisms* $\pi_1 \colon X \to \mathbb{P}^2$, $\pi_2 \colon X \to \mathbb{P}^2$, so that

(i)

$$\begin{tikzcd} X \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^2 \end{tikzcd}$$

commutes and

(ii) π_1, π_2 are birational morphisms (inverses are just rational maps).

Revisit d) in light of e).

2 Part B

BI. Consider the map Φ of problem AII) above as a rational map, $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$.

a) Look at the cases $n = 2, n = 3$, a prove there exists a variety X as in part e) of prob. AIV) with a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n \arrow[r, dashed] & \mathbb{P}^n \end{tikzcd}$$

in which π_1, π_2 are *morphisms* and are birational. Identify the domains of the inverses of π_1 and π_2 precisely.

b) Let n be general, do part a) and find as well as you can the domains of the rational maps inverting the $\pi_i, i = 1, 2$.
Please, use bare hands–no machinery we haven't done yet, no fancy theorems you quote.

BII. (The degree of a curve). Suppose $C \subset \mathbb{P}^n$ is an *irreducible* curve. If H is any hyperplane of $\mathbb{P}^n$ with $C \not\subseteq H$, consider $C \cap H$. Write $\#(C \cap H)$ for the number of points of $C \cap H$ (just counting, no multiplicities, no fanciness). Note: $C \cap H$ is Z-closed in $C \Rightarrow C \cap H$ has dim $= 0$ (explain why) and so is finite.

a) Prove:

$$(\exists d > 0)(\forall H, \text{ hyperplane of } \quad \mathbb{P}^n)(C \not\subseteq H \Rightarrow \#(C \cap H) \leq d).$$

b) Choose the min positive integer, d, from a) which works for a); call this min integer the *degree* of C. Now the hyperplanes of $\mathbb{P}^n$ form another $\mathbb{P}^n$ (the *dual* of $\mathbb{P}^n$), namely, we associate to H the homogeneous coords $(a_0 : \cdots : a_n)$ where the linear form $\sum_{i=0}^n a_i Z_i$ cuts out H from $\mathbb{P}^n$. Prove:

$$\{H \mid \#(H \cap C) < \deg C \quad \text{or} \quad C \subset H\}$$

is a Z-closed set in $\mathbb{P}^n$(dual). What is its dimension?

c) Now suppose given a curve of degree d in $\mathbb{P}^n$, our curve being irred. and we assume $d < n$. Show C is degenerate (cf. Prob AII part c)); in fact, show there exists a $\mathbb{P}^d \subset \mathbb{P}^n$ and $C \subset \mathbb{P}^d$. (Hint: Try $d = 2, n = 3$ and use the example of skew-lines to show we cannot omit the irreducibility of C as hypothesis.)

d) Suppose now C is an irreducible, non-degenerate curve in $\mathbb{P}^n$, so, by part a), $\deg C \geq n$. Let $P_1, \ldots, P_{n-1}$ be $n-1$ points of $C \cap H$, then prove: the linear span of the $P_j (1 \leq j \leq n-1)$ forms a hyperplane of H (i.e., a $\mathbb{P}^{n-2}$). Prove further, no other point of C is on this linear span. These statements are the "tri-secant lemma"—explain the name.

BIII. (Rational Varieties) Say V is an irreducible cx. variety, show by a variant of the argument of problem AIII) that the field of meromorphic functions $\mathcal{M}er(V)$ is well-defined. When $\mathcal{M}er(V)$ is a pure transcendental extension of transcendence degree $= \dim V$ over $\mathbb{C}$, the variety V is called *rational*. This is an old concept, historically important as it arose naturally in the cases $\dim V = 1, 2$ (first 75 years of algebraic geometry after 1850). It is, however, not a good concept if $\dim V \geq 3$; be this as it may, consider the surface in $\mathbb{C}^3$ defined by

$$X^2 + Y^3 + Z^5 = 0.$$

This surface, Σ, is irred. (very easy) but singular exactly at one point $(0, 0, 0)$.

a) Show that Σ is a normal surface.

b) Prove that Σ is a rational surface.

c) Suppose C is a curve and there exists a rational map (NOT necessarily birational)

$$\mathbb{P}^1 \dashrightarrow C$$

which is dominating (AIII of HW I). Show that C is a rational curve. (If you use an algebra theorem to prove this—be prepared to prove that.) This result is also true for surfaces, but the proof is *much* harder. It is false in every dimension 3 and above.

d) Here we consider the Fermat cubic hypersurface in $\mathbb{P}^3$: $X^3 + Y^3 + Z^3 + W^3 = 0$; call it F. Now F contains at least two lines:
l_1 given by $X + Y = Z + W = 0$
l_2 given by $X + \zeta Y = Z + \zeta W = 0$, ζ—a non-trivial root of 1.
By varying this prescription, how many curves can you find in F?

e) Continue d): Prove F is a manifold ($\operatorname{Sing} F = \varnothing$) and that $l_1 \cap l_2 = \varnothing$. Pick a plane, Π, in $\mathbb{P}^3$ not containing either l_1 or l_2 and choose any $P \in F$ not on either l_1 or l_2. Show there exists one line, l_P, with

(i) $P \in l_P$

(ii) $l_1 \cap l_p \neq \varnothing$.

Now define a map:

$$\psi_{l_1,l_2} \colon F - l_1 - l_2 \longrightarrow \Pi = \mathbb{P}^2$$

via $\psi_{l_1,l_2}(P) =$ point where l_P cuts Π.
Prove that this is a morphism $\psi_{l_1,l_2} \colon F - l_1 - l_2 \to \mathbb{P}^2$ and gives a birational equivalence $F \dashleftarrow\dashrightarrow \mathbb{P}^2$. Therefore, F is a rational surface. Describe the image of ψ_{l_1,l_2} in $\Pi = \mathbb{P}^2$ and the opens where we have an isomorphism. What happens to all the lines you found in part d) under ψ_{l_1,l_2}?

f) Consider $G(k, n)$, the Grasmannian of k-planes in n-space. Pick a special $L \in G(k, n)$; namely, if a basis is given in $\mathbb{C}^n$, let L be defined by $e_{k+1} = \cdots = e_m = 0$. Write $\mathrm{GL}(k, n-k)$ for the subgroup of

$\mathrm{GL}(n,\mathbb{C})$ stabilizing L, i.e., mapping L to itself. Show that, as sets, $G(k,n)$ and the homogeneous space $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(k,n-k)$ are the same. Explain how you deduce:

$$\dim G(k,n) = k(n-k).$$

If $V \in G(k,n)$ write W for $\mathbb{C}^n/V$. In terms of V, W alone, compute $T_{G(k,n),V}$. (Hint : $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(k,n-k)$ is naturally a cx. anal. space, show it's isomorphic to $G(k,n)^{\mathrm{an}}$, then all is easy.) You deduce $G(k,n)$ is a non-singular variety, then show that Z-near each pt of $G(k,n)$ it is isomorphic to $\mathbb{C}^{k(n-k)}$ and this implies that $G(k,n)$ is an irreducible, nonsingular rat'l variety.

BIV. (Lines on hypersurfaces) The hypersurfaces of $\deg d$ in $\mathbb{P}^n$ are given as the zeros of single homogeneous polynomials of degree d; $F=0$. View the coeffs of F as homogenous coords, then the hypersurfaces of deg d in $\mathbb{P}^n$ form a variety, $\mathrm{Hyp}(\mathbb{P}^n,d)$ which is in fact $\mathbb{P}^{N(n,d)}$, $N(n,d)=\binom{n+d}{d}-1$.

a) Consider $G(2,n+1)=\mathbb{G}(1,n)=$ Grasmannian of lines in $\mathbb{P}^n$ and $\mathrm{Hyp}(\mathbb{P}^n,d)$. Show that the sets

$$Z(l) = \{H \in \mathrm{Hyp}(\mathbb{P}^n,d) \mid l \subset H\}$$

and

$$Z(H) = \{l \in \mathbb{G}(1,n) \mid l \subset H\}$$

are Z-closed in $\mathrm{Hyp}(\mathbb{P}^n,d)$, resp. $\mathbb{G}(1,n)$. Write

$$\Gamma(d,n) = \{(H,l) \mid 1)\ H \in \mathrm{Hyp}(\mathbb{P}^n,d); 2)\ l \in \mathbb{G}(1,n); 3)\ l \subset H\}$$

and show $\Gamma(d,n)$ is Z-closed in $\mathrm{Hyp}(\mathbb{P}^n,d)\prod\mathbb{G}(1,n)$.

b) Examine the fundamental diagram

$$\begin{tikzcd} \Gamma(d,n) \arrow[d, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Hyp}(\mathbb{P}^n,d) & \mathbb{G}(1,n) \end{tikzcd}$$

Show pr_2 is surjective. Pick a line l and choose coords of $\mathbb{P}^n$ so that l is given by $Z_0=\cdots=Z_{n-2}=0$. Prove that $pr_2^{-1}(l)=$ all H's whose eqn: $F=0$ has the form

$$F = Z_0F_0 + \ldots + Z_{n-2}F_{n-2},$$

each F_j has deg $(d-1)$. Show that all such H's form a *linear* subspace of $\mathbb{P}^{N(n,d)}$ and its dimension is just a function of n and d (independent of l), say $\delta(n,d)$.

c) Prove $\Gamma(d,n)$ is irreducible and compute its dimension $\gamma(n,d)$ explicitly as a fcn of n and d. Explain why the image of pr_1 is Z-closed in $\mathrm{Hyp}(\mathbb{P}^n,d)$ and compare $\gamma(n,d)$ and $N(n,d)$ to prove the

Theorem For every $n\geq 2$, there is an integer D, so that

(i) If $d>D$, then for a Z-open subset of $\mathrm{Hyp}(\mathbb{P}^n,d)$ no H in this Z-open contains *any* line.

(ii) If $d<D$, then every hypersurface of degree d contains infinitely many lines.

d) The integer, D, is called the *critical degree*. Find the crit. degree for $n=3$ (surface $\subset \mathbb{P}^3$), $n=4$ (3-folds $\subset \mathbb{P}^4$), $n=5$ (4-folds in $\mathbb{P}^5$). Suppose for some H of degree $=D$, we can show there exist but finitely many lines on H. Does it follow that pr_1 is onto, i.e., that on *each* hypersurface of crit. degree there is a line? Does this depend on n? If pr_1 is onto, prove there exists a Z-open subset of the crit. degree hypersurfaces on which there are but finitely many lines.

When $n=3$ look at the Fermat surface (Prob BIII) or at the surface $Z_0Z_1Z_2=Z_3^3$. Try $n=4$. (In fact, the Z-open is the family of non-singular surfaces and by analysing the differential of pr_1, one shows each such has exactly 27 lines (case $n=3$).)