Mathematics 622
Assignment 1 (Shatz)
Due Thursday, October 2, 2003

1 Part A (Easier problems—not for discussion)

AI. If V, W are affine varieties, we have defined a morphism from V to W (here, the sheaves of functions $\mathcal{O}_V$, $\mathcal{O}_W$ are being suppressed in the notation) in terms of a pair: a continuous map of topological spaces, $\varphi\colon V \longrightarrow W$, and a local morphism, $\mathcal{O}_W \longrightarrow \varphi^*\mathcal{O}_V$. Suppose V, W are contained respectively in $\mathbb{C}^N$ and $\mathbb{C}^M$. Prove that a morphism $\varphi\colon V \longrightarrow W$ is just a set-theoretic map, $V \longrightarrow W$, so that for every $\xi \in V$, there is some open, U_ξ, of V with $\xi \in U_\xi$ and there exist $f_1, \ldots, f_M;\ g_1, \ldots, g_M \in A(W)$ in such a manner that

(i) $\prod_{j=1}^{M} g_j$ is never zero on U_ξ and

(ii) For every $x \in U_\xi \subseteq V$, we have

$$\varphi(x) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_1(x)}{g_1(x)}\right).$$

Here, $A(W)$ is the ring $\mathbb{C}[T_1, \ldots, T_M]/\sqrt{(P_1, \ldots, P_r)}$, where $P_1 = 0, \ldots, P_r = 0$ define $W \subseteq \mathbb{C}^M$. Prove the converse.

AII. Consider the open, U, of $\mathbb{C}^n$ given by

$$(Z_1 \neq 0) \cap (Z_2 \neq 0) \cap \cdots \cap (Z_n \neq 0).$$

We give this the structure of affine variety by noticing that the LRS $(U, \mathcal{O}_{\mathbb{C}^n}|U)$ is isomorphic to the standard affine of $\mathbb{C}^{2n}$ whose equations are

$$Z_1W_1 = \cdots = Z_nW_n = 1.$$

There is a notation for this U as affine variety: $\mathbb{G}_m^n$.

(a) Compute $\mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m)$ (where $\mathrm{Hom}(-,-)$ = morphisms of varieties.)

(b) Prove that $\mathrm{Aut}(\mathbb{G}_m^n)$ is a group extension of $\mathrm{GL}(n, \mathbb{Z})$ by $(\mathbb{C}^*)^n$.

AIII. Suppose X, Y are varieties. The closed subvarieties of a variety have all the properties of closed sets in a crude topology—this is the *Zariski topology* (much weaker and cruder than the usual topology—yet useful to describe where properties hold or don't hold). We'll write Z-closed, Z-open, Z-dense, etc. for these concepts in the Zariski top. If $f\colon X \to Y$, we call f a *dominating* morphism provided the image of f is Z-dense in Y.

(a) Now suppose $f\colon X \to Y$ is a dominating morphism of affine varieties, let $r = \dim X - \dim Y$; we assume X and Y are irreducible. Show that there is a dominating morphism $\psi\colon X \to \mathbb{C}^r \prod Y$ so that the diagram

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "f"'] & & \mathbb{C}^r \prod Y \arrow[dl, "pr_2"] \\ & Y & \end{tikzcd}$$

commutes.

(b) Under the hypotheses of (a), suppose there is a Z-dense subset, T, of Y for which $f^{-1}(t)$ is discrete for all $t \in T$. Prove that $r = 0$.

2 Part B (Problems to be discussed at our evening session)

BI. Look at $\mathbb{C}^n$ as a variety and pick $t \geq 1$ points, $P_1, \dots, P_t \in \mathbb{C}^n$. There is a condition on the integer n which is necessary and sufficient that the set $\mathbb{C}^n - \{P_1, \dots, P_t\}$, always a variety, be an *affine* variety. Find this condition and prove the theorem.

NB. This is meant to be done with your bare hands, so if you use someting we've not yet done be prepared to prove it from scratch as part of your proof. If you are sophisticated and know a proof by cohomology, forget it and find a simpler proof.

BII. The notion of product (notation, $\prod$) in a category should be known to you. Affine varieties are a category, this category possesses products (supply proof). Now, affine varieties are a subcategory of all varieties, show that the product you constructed before is still a product in the larger category of all varieties. Lastly, use these two facts and the fact that every variety is locally affine to prove that the category of varieties possesses products.

BIII. (a) Consider the varieties $\mathbb{P}^n, \mathbb{P}^m$. There is a canonical map (the Segre morphism)

$$\mathbb{P}^n \prod \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1},$$

we simply send $\langle (X_0, \dots, X_n), (Y_0, \dots, Y_m) \rangle$ to the tuple consisting of all pairwise products $\langle \cdots, X_i Y_j, \cdots \rangle$ (in some order). Show that the image is actually a subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$; i.e., find (homogeneous) equations describing it. After doing this prove that the latter variety is isomorphic to $\mathbb{P}^n \prod \mathbb{P}^m$ where the product is a variety *via* Problem BII. You should be able to write your description of the image of the Segre morphism as the locus of points of $\mathbb{P}^{(n+1)(m+1)-1}$ where a certain matrix has rank one. What is the matrix? and write the equations in matrix form.

(b) In $\mathbb{P}^N$, consisting as it does of either lines through 0 in $\mathbb{A}^{N+1}$ or hyperplanes (through 0) of $\mathbb{A}^{N+1}$, the linear group

$$\mathrm{PGL}(N, \mathbb{C}) = \mathrm{GL}(N+1, \mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ is embedded as diagonal matrices, acts as automorphisms of $\mathbb{P}^N$. (In fact, these are **all** the automorphisms of $\mathbb{P}^N$.) Let's agree to call two subvarieties V, W of $\mathbb{P}^N$ *conjugate* iff there is some $\sigma \in \mathrm{PGL}(N, \mathbb{C})$ with $\sigma(V) = W$. We have the image of $\mathbb{P}^n \prod \mathbb{P}^m$ in $\mathbb{P}^N$ (here, $N = (n+1)(m+1) - 1$, and we call any conjugate of this image a *Segre variety*, $\Sigma(n, m)$, of $\mathbb{P}^N$. Pick three disjoint lines X, Y, Z of $\mathbb{P}^3$ and consider $S(X, Y, Z)$ the union of all lines in $\mathbb{P}^3$ which meet all three of X, Y, Z. Show that $S(X, Y, Z)$ is a $\Sigma(1, 1)$, and in fact is the unique $\Sigma(1, 1)$ which contains X, Y, Z.

(c) Continue (b) by investigating $\mathbb{P}^{t-1} \prod \mathbb{P}^1$ in $\mathbb{P}^{2t-1}$. Show that if X, Y, Z are three pairwise disjoint $(t-1)$-planes in $\mathbb{P}^{2t-1}$, the union of all lines meeting X, Y, Z is a $\Sigma(t-1, 1)$ and in fact the *unique* $\Sigma(t-1, 1)$ containing X, Y, Z. What is the situation for the general case $\mathbb{P}^{t-1} \prod \mathbb{P}^{s-1}$? Formulate and prove.

By the way, the Segre morphism has a coordinate-free description. If A, B are $\mathbb{C}$-vector spaces, we make $\mathbb{P}(A)$, $\mathbb{P}(B)$ (the spaces of lines or hyperplanes through 0 in A or B). Then the Segre morphism is just

$$\mathbb{P}(A) \prod \mathbb{P}(B) \mapsto \mathbb{P}(A \otimes B),$$

via $\langle [v], [w] \rangle \mapsto [v \otimes w]$.

BIV. (a) In class we asked to classify the k-planes in $\mathbb{C}^n$. The cases $k = 0, n-1, n$ were discussed. Say Z is a k-plane in $\mathbb{C}^n$, pick a basis $z_1, \dots, z_k$ of Z. Then, $z_1 \wedge \cdots \wedge z_k$ is a vector in $\bigwedge^k \mathbb{C}^n$; show that up to multiplication by non-zero constants, this vector depends on Z alone. Therefore, $z_1 \wedge \cdots \wedge z_k$ defines a point $[z_1 \wedge \cdots \wedge z_k]$ of $\mathbb{P}\big(\bigwedge^k \mathbb{C}^n\big) = \mathbb{P}^{\binom{n}{k}-1}$ space and we consider the map

$$\mathrm{Pl}\colon Z \mapsto [z_1 \wedge \cdots \wedge z_k] \in \mathbb{P}^{\binom{n}{k}-1}.$$

Show that Pl is a 1–1 map, continuous, of $G(k,n)$ to $\mathbb{P}^{\binom{n}{k}-1}$. Show further that the image of Pl is closed as follows: Pick $\xi \in \mathbb{P}^{\binom{n}{k}-1}$, represent it by a vector ξ_0 of $\mathbb{C}^{\binom{n}{k}}$. We have a map, $\mathbb{C}^n \longrightarrow \bigwedge^{k+1} \mathbb{C}^n$, namely $v \mapsto v \wedge \xi_0$. Prove $\xi \in \operatorname{Im} \mathrm{Pl}$ iff $\operatorname{rk}(v \mapsto v \wedge \xi_0) \leq n-k$. Show that the entires of the matrix of $(v \mapsto v \wedge \xi_0)$ are the homogeneous coordinates on $\mathbb{P}^{\binom{n}{k}-1}$; now finish up using the $(n-k+1) \times (n-k+1)$ minors of our matrix. We give $G(k,n)$ the structure of variety it inherits from Im Pl—therefore it is a projective variety.

(b) The equations we got in (a) defining $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are not the best we can do. Show in fact there are quadratic equations defining $G(k,n)$ in $\mathbb{P}^{\binom{n}{k}-1}$; these are the *Plücker relations*.

(c) Look at the Segre variety $\Sigma(1,k) \subseteq \mathbb{P}^{2k+1}$. As $\Sigma(1,k) \subseteq \mathbb{P}^1 \prod \mathbb{P}^k$, we have two projections, $pr_1 \colon \Sigma(1,k) \to \mathbb{P}^1$ and $pr_2 \colon \Sigma(1,k) \to \mathbb{P}^k$. If $p \in \mathbb{P}^1$ look at $pr_1^{-1}(p)$. Prove that $pr_1^{-1}(p)$ is a k-plane in $\mathbb{P}^{2k+1}$. This gives a map $p \mapsto pr_1^{-1}(p)$ from $\mathbb{P}^1$ to $G(k+1, 2k+2)$, since $\mathbb{C}^{2k+2} - \{0\} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{P}^{2k+1}$. Usually, the k-planes in $\mathbb{P}^l$ are denoted $\mathbb{G}(k,l)$ (of course, $\mathbb{G}(k,l)$ is $G(k+1, l+1)$), so we get

$$\mathbb{P}^1 \longrightarrow \mathbb{G}(k, 2k+1).$$

Prove that this is a morphism, the image is a curve and prove that this image lies in a $(k+1)$-plane of $\mathbb{P}^{\binom{2k+2}{k+1}-1}$. Can the image curve lie in a smaller dimensional plane of $\mathbb{P}^{\binom{2k+2}{k+1}-1}$?